DIFFERENTIAL AND INTEGRAL CALCULUS

Ca

Differential and Integral Calculus

HAROLD MAILE BACON, PH.D.

Professor of Mathematics
Stanford University

12395

SECOND EDITION

INTERNATIONAL STUDENT EDITION

McGRAW-HILL BOOK COMPANY, INC.
New York Toronto London
KŌGAKUSHA COMPANY, LTD.
Tokyo

DIFFERENTIAL AND INTEGRAL CALCULUS

INTERNATIONAL STUDENT EDITION

II

Library of Congress Catalog Card Number 54-9690

TOSHO PRINTING CO., LTD., TOKYO, JAPAN

PREFACE

This text is planned to give the student a clear understanding of the basic principles of the calculus as well as a practical working knowledge of the subject. It is recognized, however, that a completely rigorous treatment of the subject cannot ordinarily be given in a first course, and certain proofs are of necessity omitted. Where this is done, attention is called to the fact. Even in the discussion and proofs that are included, considerable latitude is given the instructor. For instance, since the concept of a *limit* is of fundamental importance, considerable attention is given to it. This material is, however, so presented that the instructor may omit class discussion of as much of it as he desires. Throughout the book the explanations are sufficiently detailed to permit the student to grasp the ideas with a minimum of assistance from the instructor. Although a number of sections have been rewritten to provide a more concise exposition, this characteristic feature of the first edition has not been sacrificed. Illustrative examples are worked out in some detail in the hope that the reader will be able to follow easily all steps in the argument.

The wide and effective use of the calculus is amply demonstrated in the many geometrical and physical applications that are included in the text. A large number of exercises furnishes the student with the opportunity of testing his understanding of the subject.

The author is indebted to a number of his colleagues, students, and friends as well as to the many users of the first edition whose helpful suggestions find expression in many ways throughout the second edition.

HAROLD M. BACON

CONTENTS

CHAPTER 1

INTRODUCTION

In our study of the world about us, we are continually observing relationships among the things which we see and with which we have to deal. For example, we notice that old pine trees are tall and young ones short; that people with large incomes pay high taxes and people with small incomes pay lower taxes; that the postage required on a letter is determined by the weight of the letter; that a baseball thrown into the air rises to a height which depends upon the velocity with which it leaves the ground; that the price of an article affects the volume of sales; that the tangent of an angle depends upon the size of the angle. We might say that the height of a pine tree depends upon or is a *function* of its age, that the amount of tax is a function of the size of the income, that the postage required is a function of the weight of the letter, that the maximum height attained by the baseball is a function of its initial velocity, that the volume of sales is a function of the selling price, and that the tangent of an angle is a function of the size of the angle. To study functions such as these and their variation is the purpose of calculus. By its aid, many problems of great practical and theoretical importance can be solved.

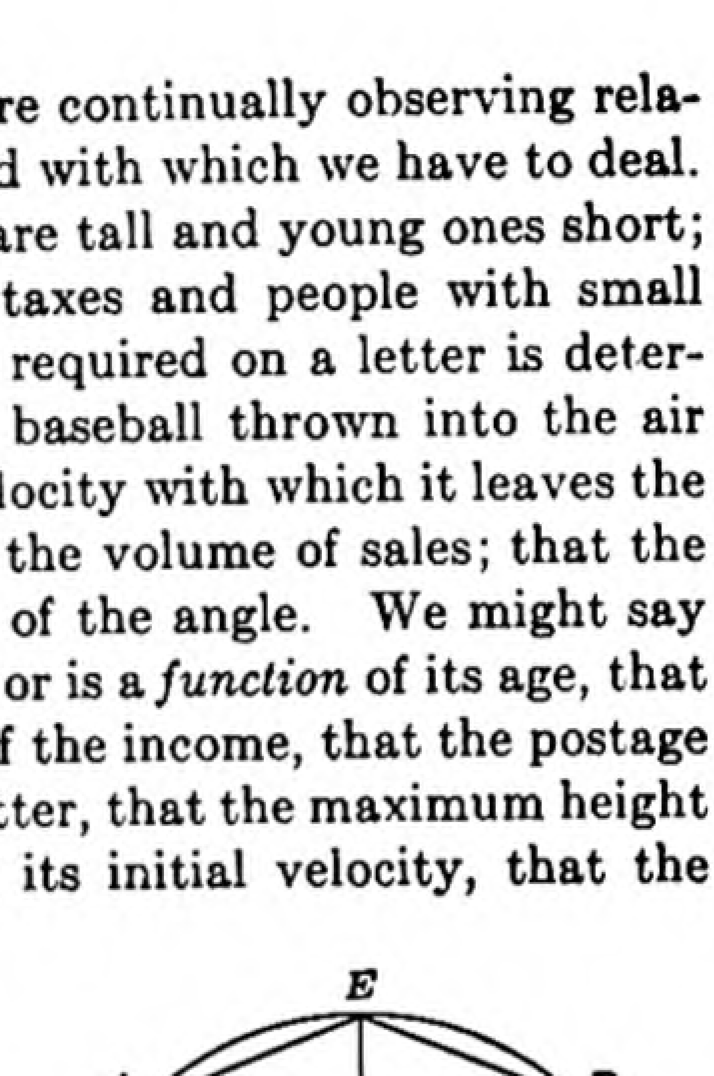


FIG. 1.

This subject that we are about to study has had a long and fascinating history. Attempts to solve two problems eventually led to the formulation and development of what today we call the "calculus." The older of these two problems is concerned with the determination of the area bounded by particular curves; the other is concerned with the method of finding the line tangent to a curve at a given point upon the curve.

The first problem was attacked by the Greeks, who sought, among other things, a means of finding the area of a circle of given radius. Antiphon (*ca.* 420 B.C.) was one of the first whose use of the "method of exhaustion" to solve this problem is known to us. His procedure was essentially as follows: Inscribe a square $ABCD$ in the circle (Fig. 1);

draw radii through the mid-points of the sides of this square. These radii meet the circle in points E, F, G, H. Join these points to the vertices of the square. The resulting figure is an inscribed regular octagon. Draw radii through the mid-points of the sides of the octagon. Join the points where these meet the circle to the vertices of the octagon to form an inscribed regular polygon of 16 sides. Draw radii to the mid-points of the sides of this figure, and proceed as before. Continue the process until a regular inscribed polygon is obtained whose sides "coincide with the circumference of the circle. And, as we can make a square equal to any polygon . . . we shall be in a position to make a square equal to any circle."* Of course, we see at once that no matter how numerous (and therefore minute) the sides of the polygon, it is quite incorrect to say that they "coincide with the circumference of the circle." But we are probably willing to admit that we can make the area of the polygon as close to the area of the circle as we please by increasing sufficiently the number of sides. In fact, we might say that "the area of the polygon approaches the area of the circle as a limit" without as yet trying to give a precise meaning to this statement. As indicated, the desire was to find a *square* equal in area to the area bounded by the given curve (in this case, the circle). For this reason the problem has been called the *problem of quadrature*.

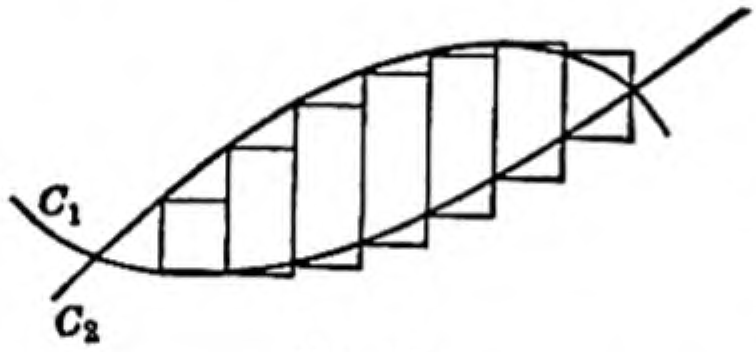


FIG. 2.

It is impossible to give here an account of the use by the Greeks of this *method of exhaustion*, but it must be noted that it contained the essential ideas of the method which we shall use for finding areas bounded by curves. For example, we shall find the area bounded by the curves C_1 and C_2 (Fig. 2) by constructing rectangles as indicated. Evidently the sum of areas of these rectangles is approximately the area sought. The error committed arises from the fact that the bases of the rectangles do not coincide with the curves. The larger the number of rectangles used for the given area, the smaller this error will be. We shall follow the Greeks in "exhausting" this error by increasing indefinitely the number of rectangles and finding the *limiting value* of their sum. The problem of finding this limiting value is solved by the *integral calculus*.

The question of finding the line tangent to a curve at a given point upon the curve requires a different treatment. Finding the tangent is equivalent to determining its slope; that is, if the slope can be found, then the line can be drawn—or, to use the language of analytic geometry, its equation can be written. In Fig. 3, let C be a curve and P a given point upon

* From a fragment of Eudemus (*ca.* 335 B.C.) restored by G. J. Allman, *Greek Geometry from Thales to Euclid*, pp. 65–66, Hodges, Figgis and Company, Dublin, 1889.

it. Let the scales on the x and y axes be the same. Let Q be a nearby point upon the curve, and draw the secant PQ. Draw PR and RQ parallel to the x and y axes, respectively. The ratio RQ/PR is the slope of PQ. Now, if Q is allowed to move along the curve toward P, the secant will come closer and closer to the position of the tangent line at P. In fact, we shall later define this tangent to be the *limiting position* of the secant PQ as Q approaches coincidence with P. The limiting value of the ratio RQ/PR is the slope of the tangent line at P. The evaluation of the *limit* of this ratio is the fundamental problem of the *differential calculus*. Although the problem of quadrature is a very old one, it was not until comparatively recent times that the problem of tangents received wide attention. The development of analytic geometry by Descartes (1596–1650) provided a very convenient means of attacking the problem of tangents, and its solution was soon achieved for many different curves.

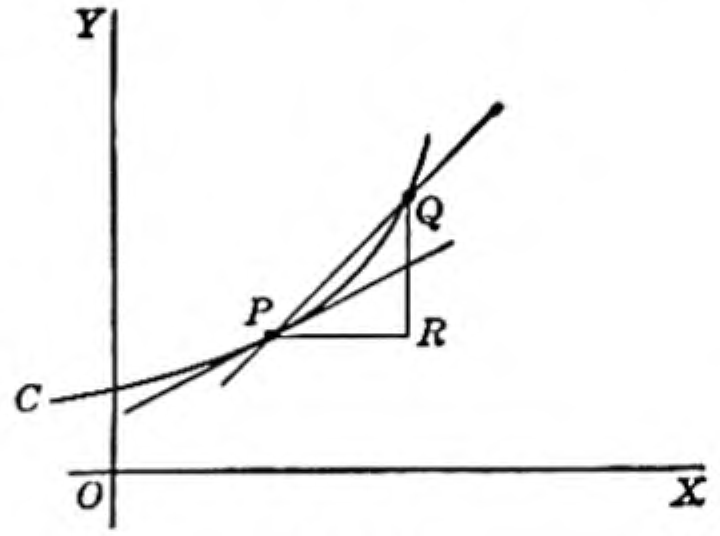


FIG. 3.

Although the problems of quadrature and of tangents had been solved for various specific curves prior to the middle of the seventeenth century and although the methods used closely approached the methods of the calculus, the connection between the two problems was not generally known; no general system of attack and no common principle had been formulated by which all questions of the kind, and even problems not yet thought of, could be solved. It is principally to Sir Isaac Newton (1642–1727) and G. W. Leibnitz (1646–1716) that we owe such a formulation. It is now recognized that these two remarkable men developed the calculus quite independently of one another. The notation devised by Leibnitz has been adopted as being the more convenient and with some modifications is in use today.

Though no attempt will be made at this point to describe the great number of different problems whose solutions depend upon the problems of quadrature and tangents, one or two examples will indicate their diversity. Newton was chiefly interested in mechanics and proposed these problems: (1) The position of a point moving along a given path being continually (that is, at all times) known, to find the velocity of the point at any given instant. (2) The velocity of the point being continually known, to find the length of the path traversed during a given interval of time. The first problem reduces to the problem of tangents, and the second to the problem of quadrature. Again, a modern manufacturer might put this problem: To find the dimensions of a 1-gal. cylindrical tin can requiring the least amount of tin. The solution may be given by solving a proper problem of tangents. In general, a knowledge of the

methods for solving the problems of quadrature and of tangents will greatly aid our study of functions.

Before going ahead, however, we must first come to a clear understanding of exactly what we mean by *function*. Also, we have been talking about "limiting value," "limit," "limiting position," "increasing indefinitely" without giving any explanation of the meaning of these terms. Because the whole structure of the calculus is based upon the idea of *limits*, it is essential that we reach a clear understanding of this idea before we proceed. Otherwise our study of calculus would become a mere memorizing of rules for working exercises, a dreary prospect indeed! Having clearly in mind the notion of a *limit*, we shall be ready to take up the problem of tangents and its implications. This will occupy considerable time, after which we shall be ready to proceed to the problem of quadrature with its many applications.

CHAPTER 2

FUNCTIONS AND LIMITS

1. Variables and Constants. A *variable* is a quantity that may have different values. The values that it takes may be restricted to a certain finite set of numbers; for example, the variable may be the number of oranges of a given size that can be packed in a standard box by different packers. Evidently the values of this variable will be confined to a set of integers, perhaps 59, 60, 61, . . . , 72. Or the values may be restricted to a set of numbers in some other way. For example, the tangent of an angle between 0 and 180 deg. (including 0 and 180 deg. but excluding 90 deg.) is a real number, positive, negative, or zero; the square roots of all real numbers (positive, negative, or zero) are either real or purely imaginary numbers. Hereafter, we shall restrict our variables to *real* values, unless otherwise expressly stated.

A *constant* is a quantity that retains the same value throughout any given problem or discussion. For example, the speed and height of a baseball thrown into the air are variables, but its mass is a constant.

2. Function. We have already noticed some instances of one variable a function of another; but before we can obtain any very extensive information about functions, we must come to an agreement as to precisely what we mean by *function*. It will be convenient to call one of the variables x and the other y; this has the advantage of conciseness, and it will help us to avoid confusing one variable with the other.

We are now ready to make our **definition of a function**: *A variable y is said to be a function of another variable x, if for every one of some set of values of x there corresponds a value, or a set of values, of y.* The variable x is called the *argument* of the function; the set of values of x is called the *domain of definition*. We note that, of course, the function may be undefined for other values of x. For instance, if x is the age in years of a certain pine tree, and if y is its height, then the domain of definition of this function y can consist of only positive values of x; for negative x the function is undefined. On the other hand, if we arbitrarily assign the value zero to y for all negative or zero values of x, we shall have enlarged the domain of definition to include all (real) values of x. Again, if $y = \tan x$, the domain of definition of this function y consists of all angles

whose measure is x, except those whose measure is an odd multiple of 90 deg. for which values of x there are no values of y.

Since the value of y is determined for a given value of x, it is customary to call y the *dependent variable*. In many cases, x may be taken at will to be any value whatever (sometimes with certain restrictions), and under such circumstances x is called the *independent variable*. It may happen, however, that x is in turn a function of a third variable t. In this case a given value of t will determine a value of x. For this value of x, there is determined a value of y so that y is also a function of t, although this time t rather than x is to be regarded as the independent variable. For example, if $y = x^2$, and $x = \sin t$, then y is a function of t, namely, $y = \sin^2 t$.

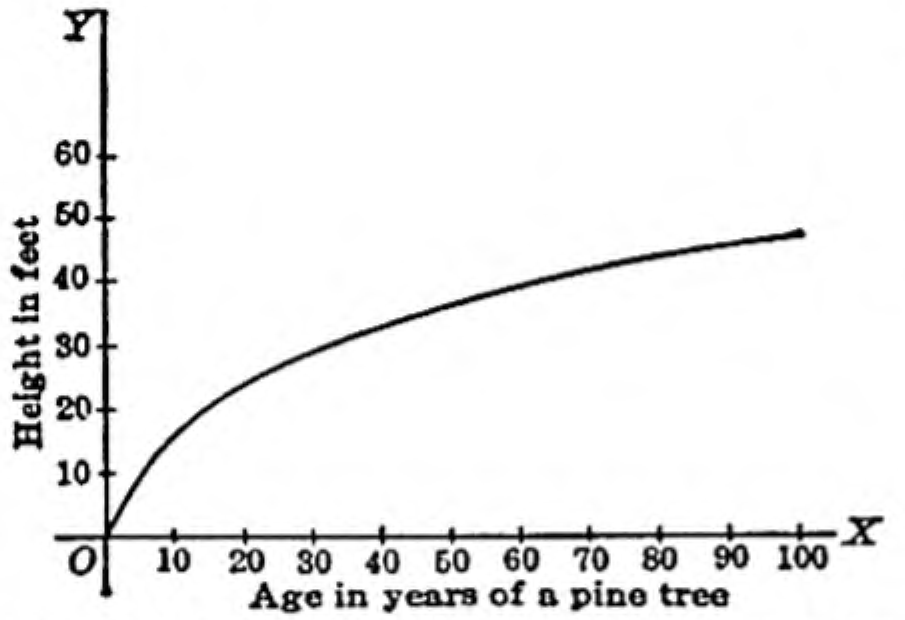


FIG. 4. Height of a pine tree as a function of its age.

3. Graph of a Function. It is convenient to represent functions graphically. As is usually done, we shall let the independent variable x be plotted horizontally as the abscissa, and the corresponding y vertically as the ordinate of a point. The collection of points obtained by assigning various values to x and calculating the corresponding values of

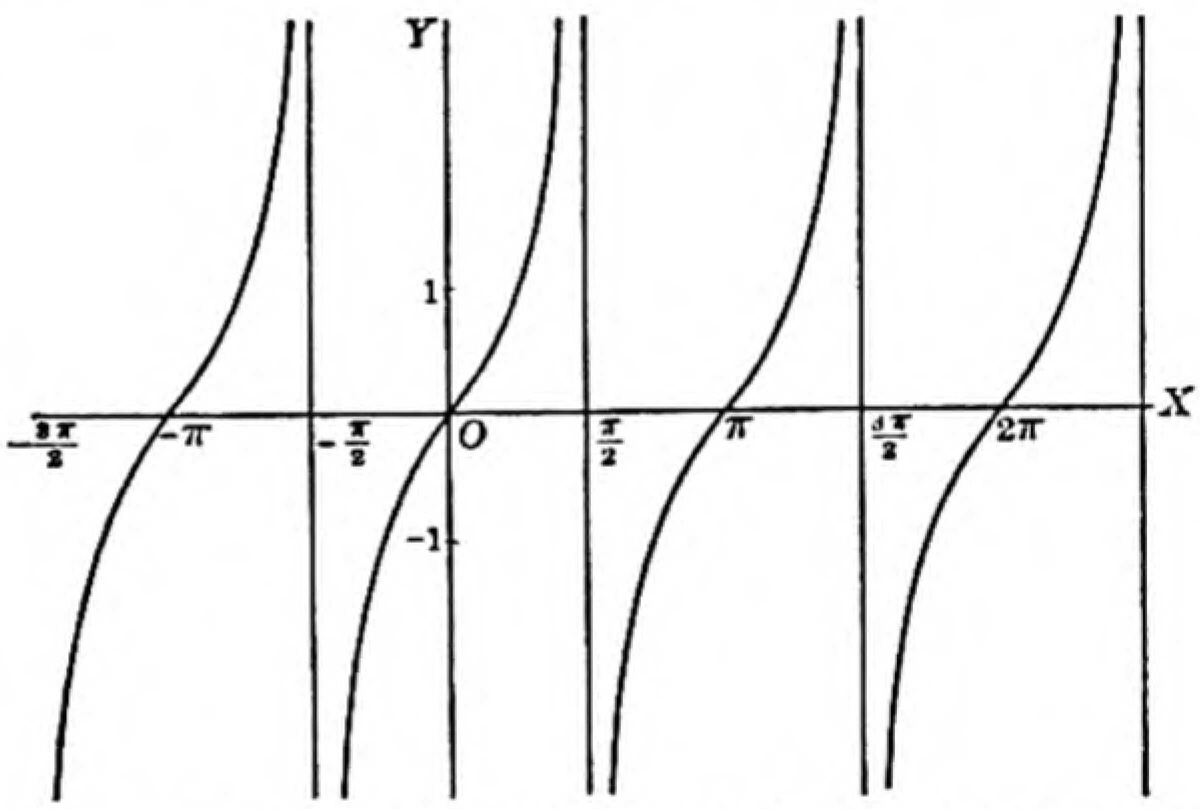


FIG. 5. $y = \tan x$. The tangent of an angle as a function of the angle.

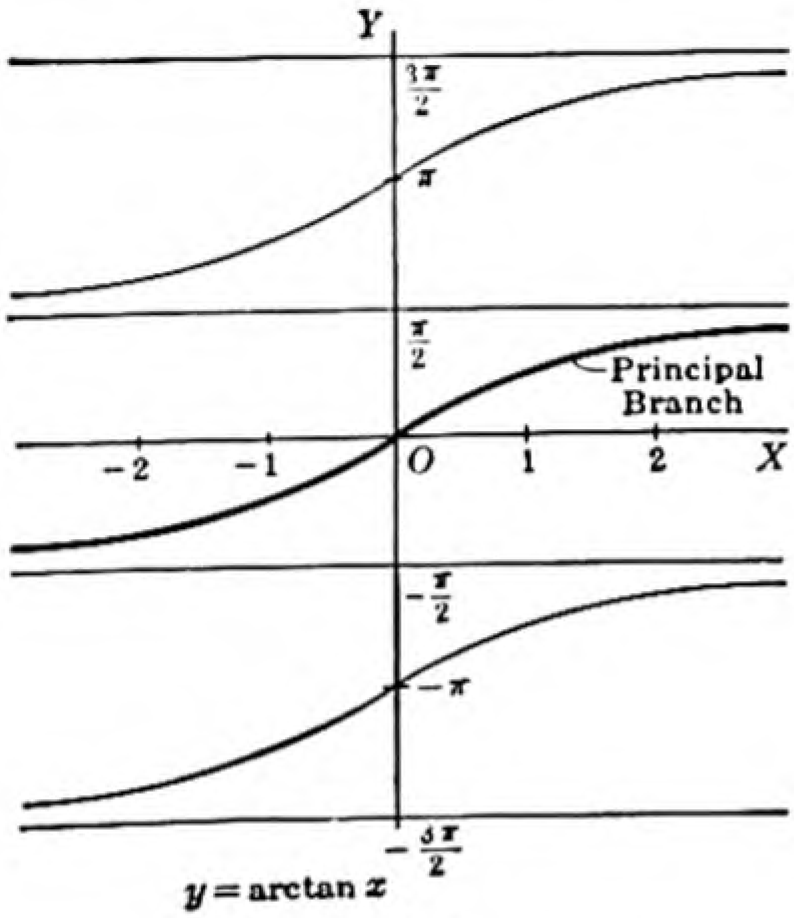
y is called the *graph* of the function. Graphs of two of the functions already discussed are shown in Figs. 4 and 5.

In the definition of a function and in the examples just considered, it is to be understood that we always restrict ourselves to real values of x and y. Furthermore, it will be observed that we have illustrated, in the two examples given, only *single-valued functions*. That is, for each value

of x, there corresponds only one value of y; for a given age the pine tree has just one height; for a given angle the tangent has just one value. Now suppose that x and y are related by the equation $y^2 = x$. Then $y = \pm \sqrt{x}$, so that, for $x = 4$, $y = \pm 2$; similarly, for any given value of x, there are two values of y. Such a function of x is called a *double-valued function*. In this case, we shall regard y as made up of two *branches*, namely, $\sqrt{x}$ and $-\sqrt{x}$, each one of which is single-valued. Note that the symbol $\sqrt{x}$ *always* means the *positive* square root. For example, $\sqrt{(a-b)^2}$ means $a - b$ if $a > b$, but $b - a$ if $a < b$. Or suppose that $y = \arctan x$. Here, if $x = 0$, then $y = 0, \pm\pi, \pm 2\pi, \ldots$; similarly, for each value of x, there are many (even infinitely many) corresponding values of y, and y is called a *many-valued function* of x. Again we shall regard y as made up of many branches, each one of which is a single-valued function of x. In the case of arctan x, it is customary to regard

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$$

as the *principal branch* (shown by the heavy curve in Fig. 6).



$y = \arctan x$

FIG. 6.

Although we have been using x and y as convenient designations for the variables involved in our discussion, there is no special reason to prefer them to any other pair of letters.

4. Functional Notation. A very useful way of abbreviating the statement "y is a function of x" is to write $y = f(x)$ (read "f of x"). This means that y depends upon x in some definite way. In case we wish to speak of another function of x where y depends upon x in a different way, we write $y = g(x)$. Still other functions of x could be indicated by $f_1(x)$, $f_2(x)$, $F(x)$, $G(x)$, $\varphi(x)$, $\Phi(x)$, and so on. If we wish to express the fact "the value of the function $f(x)$ for $x = 2$ is 10," we shall write $f(2) = 10$. For example, if $f(x) = 2x + 6$, then $f(2) = 10$, $f(0) = 6$, $f(-17) = -28$. If $g(x) = \tan x$, then $g(0) = 0$, $g(\pi/4) = 1$, $g(-\pi/4) = -1$.

Example. If the function $f(x)$ has the property that $f(-x) = f(x)$, the function is called an *even* function. If, on the other hand, $f(-x) = -f(x)$, the function is called an *odd* function.

Suppose $\qquad f(x) = x^4 + 2x^2 + 7$

Then $\qquad f(-x) = (-x)^4 + 2(-x)^2 + 7 = x^4 + 2x^2 + 7 = f(x)$

Therefore $f(x)$ is, in this case, an even function.

Suppose
$$g(x) = x^5 - 16x^3 + 11x - \frac{92}{x}$$

Then
$$g(-x) = (-x)^5 - 16(-x)^3 + 11(-x) - \frac{92}{-x}$$
$$= -\left(x^5 - 16x^3 + 11x - \frac{92}{x}\right) = -g(x)$$

Therefore, in this case, $g(x)$ is an odd function.

EXERCISES

Express each of the functions of Exercises 1 to 12 by a formula, draw the graph, and indicate the values of the independent variable for which the formula has a meaning in the particular problem.

1. (a) The area of a square as a function of the length of the side
   (b) The length of the side of a square as a function of the area
2. (a) The volume of a sphere as a function of its radius
   (b) The radius of a sphere as a function of its volume
3. The cost of a carpet as a function of its area in square yards if its price is \$14.50 per square yard
4. The function of Exercise 3 if \$10 must be added for a decorator's fee
5. The time required for an airplane to travel 1000 miles as a function of the average speed
6. The distance traveled in t hr. by a train whose speed is 55 m.p.h.
7. The amount at the end of t years of \$1 at 2 per cent interest compounded annually as a function of t
8. The cost at 25 cents per square foot of a square cement floor as a function of the length of the side
9. The function of Exercise 8 if the floor is surrounded by a curbing costing \$1.25 per linear foot and \$5.00 is added for incidental expenses
10. y as a function of x if y is inversely proportional to the square root of x
11. If the pressure of a gas at constant temperature is inversely proportional to the volume, express the pressure as a function of the volume.
12. The stiffness of a beam of rectangular cross section is proportional to the breadth and the cube of the depth. If the breadth is 10 in., express the stiffness as a function of the depth.

13. Find $f(0), f(1), f(-1), f(2), f(3), f(x+1)$ if $f(x) = x^3 - 4x^2 - x + 4$.
14. Find $\varphi(2), \varphi(1), \varphi(0), \varphi(-1), \varphi(-2)$ if $\varphi(x) = \dfrac{x^2(x-1)}{(x+2)}$.
15. Find $g(0), g(1), g(2), g(3), g(3-x)$ if $g(x) = x^2(x-3)^2$.
16. Find $F(1), F(3), F(4), F(4-y)$ if $F(y) = \dfrac{y^2 - 4y + 5}{16y - 4y^2}$.
17. Find $f(0), f\left(\dfrac{\pi}{4}\right), f\left(\dfrac{3\pi}{4}\right), f(\pi), f\left(-\dfrac{\pi}{4}\right), f(\pi - x), f\left(\dfrac{\pi}{2} - x\right)$, if $f(x) = \tan x$.
18. Find $f\left(\dfrac{\pi}{2} - x\right)$ $f\left(\dfrac{\pi}{2} + x\right), f(\pi - x), f(\pi + x)$

(a) if $f(x) = \sin x$ (b) if $f(x) = \cos x$

19. By considering $f(-x)$, discover which of the following are even functions, which are odd functions, and which are neither even nor odd:

(a) $f(x) = x^4 + 7x^2 + 9$

(b) $f(x) = x^3 + x + \dfrac{1}{x^3}$

(c) $f(x) = \sqrt{x^4 + 16}$

(d) $f(x) = \dfrac{1}{4x^2 + 9}$

(e) $f(x) = \sin x$

(f) $f(x) = \cos x$

(g) $f(x) = \tan x$

(h) $f(x) = \sin x + \csc x$

(i) $f(x) = \tan x + \cos x$

(j) $f(x) = \tan x \sec x$

(k) $f(x) = \tan^2 x$

(l) $f(x) = \tan^3 x$

(m) $f(x) = \sin x + \cos x$

(n) $f(x) = x + \dfrac{\cos x}{x}$

20. If $f(x) = \sin x$ and $g(x) = \cos x$, show that

$$f(2x) = 2f(x)\,g(x)$$
$$f(x+y) = f(x)\,g(y) + f(y)\,g(x)$$
$$g(x+y) = g(x)\,g(y) - f(x)\,f(y)$$

21. If $f_1(x) = \sqrt{x^2 + 4a^2}$ and $f_2(x) = \sqrt{x^2 - 4a^2}$, show that, for $0 < a < k$, $f_1\left(k - \dfrac{a^2}{k}\right) + f_2\left(k + \dfrac{a^2}{k}\right) = 2k.$

22. If $f(x) = \dfrac{2x-5}{2x+5}$, show that $f(x) = \dfrac{1}{f(-x)}$.

5. Implicit Functions. All the examples of functions so far considered have been such that the mere designation of a value of x serves to provide at once a value of y. In other words, $y = f(x)$ expresses y as an *explicit* function of x. It may happen that x and y are connected by some sort of equation; for instance, we may have $x^2 + y^2 - 25 = 0$. Here, evidently, if values are assigned to x, corresponding values of y are determined. For example, if $x = 3$, then $y = \pm 4$. Therefore, y is a double-valued function of x, consisting of two branches, although the equation that defines the function is not solved for y and so does not give y explicitly in terms of x. Under such circumstances, we say that y is an *implicit* function of x. Our notation for functions can be modified for use in this case, and we write $F(x,y) = 0$. It may be possible to solve this equation conveniently for y, say $y = \Phi(x)$, and the result gives y explicitly as a function of x. Here $\Phi(x)$ is the function that is *implicit* in the equation $F(x,y) = 0$. In the example given we have

$$y = \pm\sqrt{25 - x^2} = \Phi(x)$$

One of the branches of this double-valued function is $\sqrt{25 - x^2}$ and the other is $-\sqrt{25 - x^2}$.

Another illustration of y as an implicit function of x is

$$F(x,y) = y^5 + y + x = 0.$$

In this case it can be proved, although the proof is difficult, that it is

impossible to find an explicit algebraic expression for y in terms of x. Nevertheless, this equation does define y as a function of x.*

6. Classification of Functions. It is interesting and useful to classify functions in the following way. The simplest kind of function of x is the *polynomial*, or *integral rational function*,

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

where n is a positive integer and $a_0, a_1, \ldots, a_n$ are any constants.

The next more complicated kind of function is a quotient of two polynomials, and it is called a *rational function*, or *rational fraction*. For example, $3x^5 + 15x^2 - 9x + 1$ and $x^4 - x + 11$ are both polynomials, and their quotient

$$\frac{3x^5 + 15x^2 - 9x + 1}{x^4 - x + 11}$$

is a rational function of x. Any polynomial is, of course, a rational function, since it may be regarded as a rational fraction whose denominator is a constant. A constant is a polynomial in which all the coefficients except the last one are zeros.

The next more complicated kind of function is called an *algebraic* function. Such a function can be obtained by combining rational functions in a way that employs a finite number of root extractions. For example, if

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

then it is easy to verify that

$$y^8 - (4x)y^6 + 2x(3x - 1)y^4 - 4x^2(x - 1)y^2 + (x^4 - 2x^3 + x^2 - x) = 0$$

by repeated squaring of both sides of the equation. Observe that this second equation is a polynomial in y whose coefficients are rational functions of x. In general, y is said to be an algebraic function of x if it is a root of an equation of nth degree in y whose coefficients are rational functions of x. It is clear that rational functions are included in the class of algebraic functions. For instance, the rational function in the example in the middle of this page may be expressed as a root of the following first-degree equation in y:

$$(x^4 - x + 11)y - (3x^5 + 15x^2 - 9x + 1) = 0$$

Any function that is not algebraic is said to be *transcendental*. A negative kind of definition like this naturally includes an immense variety of

* That such equations as this actually, under certain conditions, define y as a function of x requires proof. However, no such proof will be given here, for it is best to leave these details to more advanced courses.

functions of varying degrees of importance and interest. Perhaps the most familiar transcendental functions are the trigonometric and inverse trigonometric functions. Other elementary transcendental functions are the logarithmic, exponential, and hyperbolic functions. Besides these, there are many important transcendental functions whose properties have been studied but which lie beyond the scope of this book; for instance, the elliptic, Bessel's, and gamma functions are of this type.

EXERCISES

1. Express y as an explicit function of x in each case:

(a) $x^3 - y^2 - 8 = 0$ (b) $(x^2 + x - 1)y + 3x - 3 = 0$

(c) $x^2 + 4y^2 = 16$ (d) $x^2 + 3xy + 2y^2 - x + y - 1 = 0$

2. Express y as an explicit function of x, draw the graph, and indicate the branches of the function if

(a) $4x - y^2 = 0$ (b) $9x^2 + 4y^2 = 36$

(c) $9x^2 - 4y^2 = 36$ (d) $xy^2 - 16 = 0$

(e) $x^3 - y^2 = 0$

3. The following functions of x are roots of algebraic equations in y in which the coefficients are rational functions of x. Find these equations.

(a) $y = \sqrt[3]{2x + \sqrt{x}}$ (b) $y = \sqrt{3x + \sqrt{2x + \sqrt{x}}}$

7. Sequences; Limit of a Function. We are now ready to examine the concept of a *limit*—the foundation upon which the whole of our future work will rest. Most of us probably have a rough idea of what is meant when we say, "The variable x approaches the constant a as a limit." Note that we are not talking about x equaling a, for, if we were interested in having x equal to a, we should simply say, "Let x equal a," and settle the matter. We are, instead, concerned with what is going on "close" to a, or "in the neighborhood of a."

Such an idea must be put into more precise and abstract form if we are to use it for the basis of any careful and extensive study of functions. We first define the word **sequence**: *If to each positive integer* 1, 2, 3, . . . , n, . . . *there corresponds a definite real number* x_n, *then the numbers* $x_1, x_2, x_3, \ldots, x_n, \ldots$ *are said to form a sequence.* Such a sequence will be designated by the symbol $\{x_n\}$. A sequence $\{x_n\}$ is said to *converge* to a number a, or to *have limit* a, if the numerical values of the differences

$$a - x_1, a - x_2, a - x_3, \ldots, a - x_n, \ldots$$

eventually become and remain smaller than any preassigned, arbitrarily small positive number. Using $|a - x_n|$ to denote the *numerical value* or *positive value* of $a - x_n$, this can be stated in a formal way as follows: If, for any positive number ϵ, no matter how small, there can be found a positive integer N such that $|a - x_n| < \epsilon$ for every $n > N$, then the sequence

$\{x_n\}$ has limit a. We shall, for convenience, say that x *approaches* a, and write $x \to a$.

Example 1. Let x assume the sequence of values

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{2}{3}, \quad x_3 = \frac{3}{4}, \quad \ldots, \quad x_n = \frac{n}{n+1}, \quad \ldots$$

Points whose abscissas are $x_1, x_2, x_3, \ldots$ can be plotted on the x axis as in Fig. 7. It looks as if $\{x_n\}$ may have limit 1. If it has, we must be able to surround 1 by an

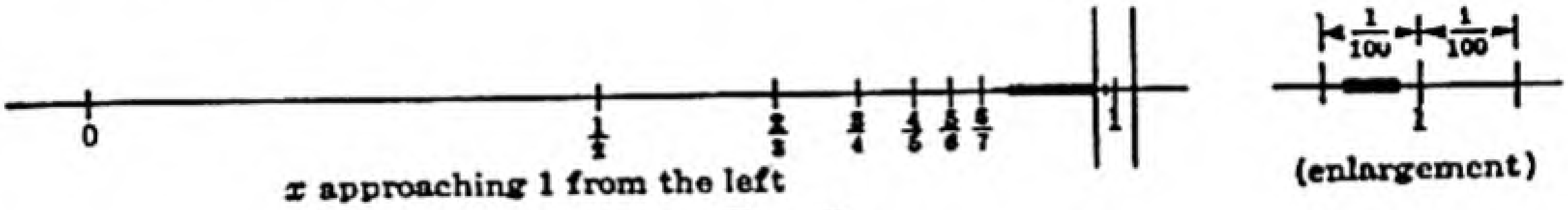


FIG. 7.

interval, as small as we please, so that eventually (that is, for sufficiently remote numbers in the sequence $x_1, x_2, x_3, \ldots$) the remaining plotted points will all fall inside the interval. Let us try an interval of width $\frac{2}{100}$, of which 1 is the middle point. Now we expect that the differences

$$1 - \frac{1}{2}, \quad 1 - \frac{2}{3}, \quad 1 - \frac{3}{4}, \ldots, 1 - \frac{n}{n+1}, \ldots$$

will become and remain less than $\frac{1}{100}$. To see whether they will, consider

$$1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}$$

Evidently $\dfrac{1}{n+1} < \dfrac{1}{100}$ if $n + 1 > 100$, that is, if $n > 99$. Therefore, we may say with certainty that $1 - x_n$ will be numerically less than $\frac{1}{100}$ provided only that n is greater than 99. Hence the differences in question eventually become and remain numerically less than $\frac{1}{100}$. This can be conveniently stated in symbols. We write

$$|1 - x_n| < \tfrac{1}{100} \qquad \text{for all } n > 99$$

However, we are still not ready to say that the limit of $\{x_n\}$ is 1. All we have said is that x eventually differs from 1 by less than $\frac{1}{100}$; x might eventually become and remain equal to $\frac{999}{1000}$ and still satisfy the condition. We must be assured that the difference between 1 and x will become and remain less than *any* arbitrarily small positive number. This is not difficult to show. Suppose $\epsilon = 1/G$ is some small number. Now

$$|1 - x_n| = 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

will be less than $\epsilon = 1/G$ if $n + 1 > G = 1/\epsilon$. Therefore, we may write, using symbols for brevity,

$$|1 - x_n| < \epsilon = \frac{1}{G} \qquad \text{for all } n > \frac{1}{\epsilon} - 1 = G - 1$$

We are now *sure* than $\{x_n\}$ has limit 1 because, no matter how small an interval (of width 2ϵ) is chosen to surround 1, *all* the values, after a certain one, assumed by x_n

will fall inside this interval. Evidently, this certain value of x_n depends upon the size of the interval chosen. An inspection of Fig. 7 leads us to say that, when x assumes the values in this sequence, it approaches 1 *from the left.*

Example 2. Let x assume the sequence of values

$$x_1 = \frac{2}{1}, \quad x_2 = \frac{3}{2}, \quad x_3 = \frac{4}{3}, \quad \dots, \quad x_n = \frac{n+1}{n}, \quad \dots$$

Again $\{x_n\}$ appears to have limit 1. Proceeding as in Example 1, we find that for

$$|1 - x_n| = x_n - 1 = \frac{n+1}{n} - 1 = \frac{1}{n}$$

to be less than ϵ we must have $1/n < \epsilon$, or $n > 1/\epsilon$. For instance, if $\epsilon = \frac{1}{100}$, all differences $|1 - x_n|$ with $n > 100$ will be less than $\frac{1}{100}$; hence, all points whose abscissas are $x_{101}, x_{102}, x_{103}, \dots$ will fall inside the interval of width $\frac{2}{100}$ whose middle point has abscissa 1. The reader should make a figure similar to Fig. 7 and observe that, when x assumes values in this sequence, it approaches 1 *from the right.*

Example 3. Let x assume the sequence of values

$$x_1 = 2, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{4}{3}, \quad x_4 = \frac{3}{4}, \quad x_5 = \frac{6}{5}, \quad x_6 = \frac{5}{6}, \quad \dots,$$
$$x_n = \frac{n+1}{n} \quad \text{(for } n \text{ odd)}, \qquad x_n = \frac{n-1}{n} \quad \text{(for } n \text{ even)}, \quad \dots$$

Note that, in any case, $x_n = 1 + \frac{(-1)^{n+1}}{n}$. Again x_n appears to have limit 1. Proceeding as in Example 1, we find that for n odd

$$|1 - x_n| = x_n - 1 = \frac{1}{n}$$

and for n even

$$|1 - x_n| = 1 - x_n = \frac{1}{n}$$

Hence $|1 - x_n|$ will be less than any arbitrarily assigned small number ϵ provided only that $1/n < \epsilon$, that is $n > 1/\epsilon$. The reader should make a figure similar to Fig. 7 and describe the behavior of x as it assumes values in this sequence.

In these examples, we have seen three different sequences, each one of which has limit 1. Many other sequences with limit 1 could be devised; since we have been considering x an *independent* variable, we might choose any of these sequences for the purpose of making x approach 1.

Now suppose that y is a function of an independent variable x. As x is given values in some sequence whose limit is a, y will assume a corresponding sequence of values. Does the sequence of y values have a limit b? If x is given any other sequence of values with limit a, does the corresponding y sequence still have the same limit b? If the answer to both questions is yes, then we say that the limit of y is b as x approaches a.

Let us suppose, for example, that $y = 8/x$, and inquire what happens to y as we let x approach 2. We choose some sequence of x values having limit 2, for instance the sequence 2.1, 2.01, 2.001, . . . The corresponding values of y appear in the table

x	2.1	2.01	2.001	2.0001	2.00001
y	3.8	3.98	3.998	3.9998	3.99998

The sequence of y's appears to have limit 4. If this is the case, then the difference between y and 4 should become and remain numerically less than any assigned small positive number, provided x is taken close enough, but not equal, to 2. To test this, we observe that

$$|y - 4| = \left|\frac{8}{x} - 4\right| = \left|\frac{4}{x}(2 - x)\right|$$

This will become and remain less than any arbitrarily small positive number ϵ if x becomes and remains close enough to 2.* But, since x approaches 2 (that is, assumes values in a sequence whose limit is 2), this is exactly what takes place, and we are justified in saying that this sequence of y's has limit 4. We also note that the result is in no way affected by what sequence of values of x is chosen so long as $x \neq 2$ and the limit of the sequence is 2. This is all just a precise way of saying that as x gets closer and closer to 2, y gets closer and closer to 4. It is very important for us to keep clearly in mind the fact that nothing whatever has been said about the value of y when x *equals* 2; in fact, $x = 2$ has been carefully excluded from the entire discussion. We are not interested in what happens to y for x equals 2, but in what y is doing as x gets closer and closer

* In fact, for $0 < x < 2$, $|y - 4|$ will be less than ϵ for all x such that

$$|y - 4| = \frac{4}{x}(2 - x) < \epsilon$$

that is, such that

$$8 - 4x < \epsilon x \qquad \text{or} \qquad 8 < x(\epsilon + 4) \qquad \text{or} \qquad \frac{8}{\epsilon + 4} < x < 2$$

For $x > 2$, $|y - 4|$ will be less than ϵ for all x such that

$$|y - 4| = \frac{4}{x}(x - 2) < \epsilon$$

that is, such that

$$4x - 8 < \epsilon x \qquad \text{or} \qquad x(4 - \epsilon) < 8 \qquad \text{or} \qquad 2 < x < \frac{8}{4 - \epsilon}$$

(We shall suppose ϵ to have been chosen less than 4.) Hence $|y - 4|$ will be less than ϵ, provided only that

$$\frac{8}{4 + \epsilon} < x < \frac{8}{4 - \epsilon}$$

to 2. To put it another way, it is the *neighborhood* of 2 that interests us, not 2 itself. The significance of this fact will appear presently.

What we have just been saying can be interpreted graphically as follows. First draw the graph of $y = 8/x$ as in Fig. 8. Next decide how close we wish y to be to 4, say closer than ϵ units. With the point (0,4) as mid-point and with 2ϵ as width, lay off an interval AB on the y axis. Through A and B, draw horizontal lines. They form a strip of width 2ϵ. Obviously, all points inside this strip have ordinates that differ from 4 by less than ϵ. For example, if ϵ is chosen equal to $\frac{1}{100}$, every point in this strip would have its ordinate greater than 3.99 but less than 4.01. Where these horizontal lines cut the graph (at C and D), draw vertical lines. They cut the x axis at E and F on either side of 2. Now, if we take any point on the x axis between E and 2 or between 2 and F and substitute its abscissa for x in the equation $y = 8/x$, the result will be the ordinate of a point inside the horizontal strip. In other words, y will be closer to 4 than ϵ, if x is close enough to 2, and we say that *the limit of y is 4 as x approaches* 2. Note particularly that the point whose abscissa is 2 has been very carefully excluded from the entire discussion.

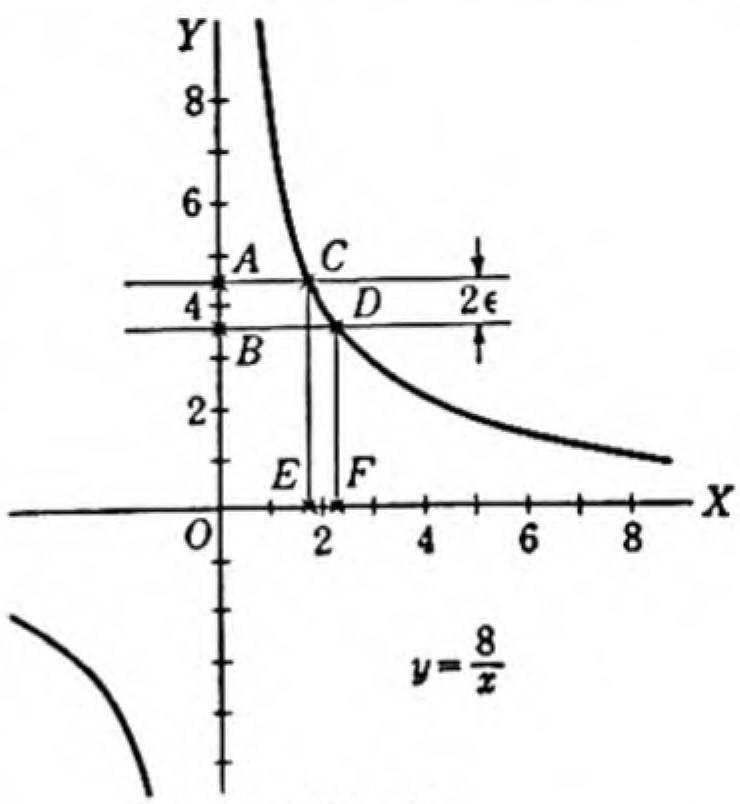


FIG. 8.

We can now state our **definition of the limit of a function of x**: *If $y = f(x)$ is a function of x, then y is said to have a limit b as x approaches a, provided that the numerical value of the difference between y and b becomes and remains less than any arbitrarily assigned small positive number for all values of x close enough, but not equal, to a.** We write

$$\lim_{x \to a} y = b \qquad \text{or} \qquad \lim_{x \to a} f(x) = b$$

It cannot be too strongly emphasized that this definition and the idea of the limit of y in general have nothing whatever to do with the *value* of the function y for $x = a$. It must also be pointed out that x may approach *a in any manner*, that is, through any desired sequence of values (all different from a) that has a as a limit.

Example 4. Suppose $y = \dfrac{2x^2 - 2x}{x - 1}$, and let us see if y has a limit as $x \to 1$. Note that if $x = 1$, this function is undefined, for the expression on the right is the meaning-

* This definition could be expressed in symbols as follows: The limit of $f(x)$ is said to be b as x approaches a if, for any preassigned number $\epsilon > 0$, there exists a number $\delta > 0$, such that for $0 < |x - a| < \delta$ we have $|f(x) - b| < \epsilon$.

less fraction 0/0. But, we note that for every $x \neq 1$,

$$y = \frac{2x(x-1)}{x-1} = 2x$$

As $x \to 1$, y seems to be approaching 2 as a limit. Since, in testing for a limit as $x \to 1$, we *exclude* $x = 1$, we need consider only

$$|y - 2| = |2x - 2| = 2\,|x - 1|$$

This can evidently be made less than any arbitrarily small positive number ϵ simply by taking x near enough (but not equal) to 1. Hence the limit of y is 2.

The graph of this function consists of the line $y = 2x$ with one point (with abscissa 1) omitted. The reader should draw the graph and carry through a discussion similar to that illustrated by Fig. 8. This example illustrates the striking fact that the function y may have a *limit* as x approaches some particular value, although it may be entirely undefined for the value of x in question.

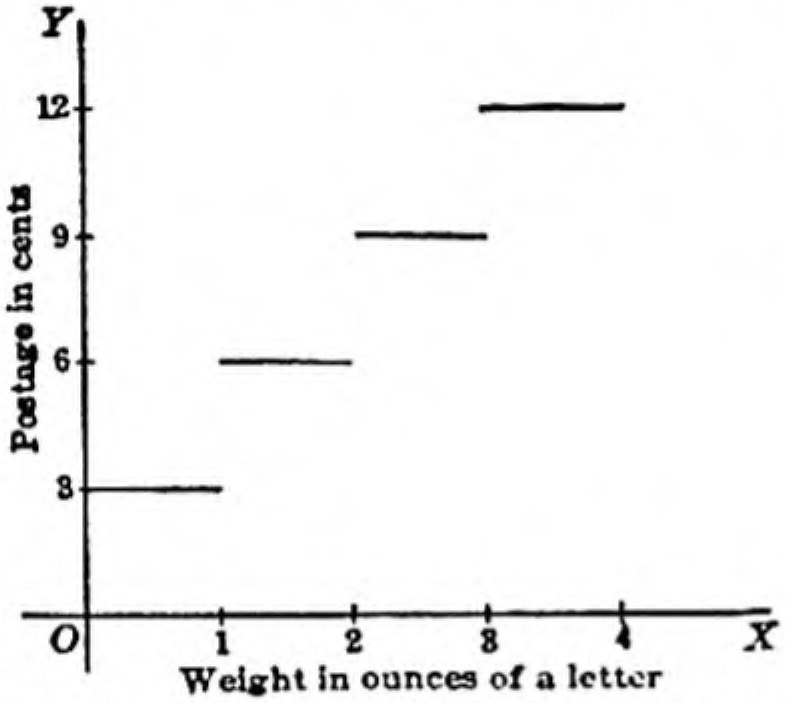


FIG. 9. Postage on a letter as a function of its weight.

Example 5. Consider the function described in Fig. 9, namely, the postage as a function of the weight of a letter. We have

$$\begin{aligned} y = f(x) &= 3 \quad \text{for } 0 < x \leqq 1 \\ y &= 6 \quad \text{for } 1 < x \leqq 2 \\ y &= 9 \quad \text{for } 2 < x \leqq 3 \\ &\ldots \quad \ldots\ldots\ldots \end{aligned}$$

Does y have a limit as x approaches 1? Evidently $|y - 3|$ becomes and remains less than any arbitrarily small positive number (in fact, it is equal to 0) for x approaching 1 through values less than 1 ("from the left"). But $|y - 3| = 3$ for x approaching 1 through values greater than 1 ("from the right"), and therefore cannot become and remain less than any arbitrarily small positive number. Since, if y is to have 3 as a limit, $|y - 3|$ must become less than any arbitrarily small positive number *no matter how* x approaches 1, the limit of y is not 3. For similar reasons, it cannot be 6, or indeed any other number; hence, y does not have a limit as x approaches 1.

In this case, we might speak of the *left-hand limit* of y, and write $\lim\limits_{x \to 1^-} y = 3$. Here we indicate by the symbol $x \to 1^-$ that x approaches 1 through values *less* than 1. Similarly, we might speak of the *right-hand limit* of y, and write $\lim\limits_{x \to 1^+} y = 6$, indicating by the symbol $x \to 1^+$ that x approaches 1 through values *greater* than 1. A function has a limit for $x \to a$ only if the left- and right-hand limits exist and are equal.

8. Theorems Concerning Limits. There are three general theorems that will be found useful in evaluating limits.

THEOREM 1: *If each one of a finite number of functions of x has a limit as x approaches a, then the limit of the sum of the functions is equal to the sum of their limits.*

Thus, in the case of two functions,

$$\lim_{x \to a} [f_1(x) + f_2(x)] = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x)$$

For example, consider the function $y = f(x) = x^3 + x^2$, and let $x \to 3$. Then $\lim\limits_{x \to 3} x^3 = 27$, $\lim\limits_{x \to 3} x^2 = 9$, and

$$\lim_{x \to 3} (x^3 + x^2) = \lim_{x \to 3} x^3 + \lim_{x \to 3} x^2 = 27 + 9 = 36$$

THEOREM 2: *If each one of a finite number of functions of x has a limit as x approaches a, then the limit of the product of the functions is equal to the product of their limits.*

Thus, in the case of two functions,

$$\lim_{x \to a} [f_1(x) \cdot f_2(x)] = \lim_{x \to a} f_1(x) \cdot \lim_{x \to a} f_2(x)$$

For example, consider the function $y = f(x) = x^3 + x^2 = x^2(x + 1)$ and let $x \to 3$. Then $\lim\limits_{x \to 3} x^2 = 9$, $\lim\limits_{x \to 3} (x + 1) = 4$, and

$$\lim_{x \to 3} x^2(x + 1) = \lim_{x \to 3} x^2 \cdot \lim_{x \to 3} (x + 1) = (9)(4) = 36$$

THEOREM 3: *If each of two functions of x has a limit as x approaches a, then the limit of the quotient of the functions is equal to the quotient of their limits, provided that the limit of the denominator is not zero.*

Thus $$\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim\limits_{x \to a} f_1(x)}{\lim\limits_{x \to a} f_2(x)} \qquad \text{provided } \lim_{x \to a} f_2(x) \neq 0$$

For example, $$\lim_{x \to 3} \frac{x^2 + 2}{x^2 - 1} = \frac{\lim\limits_{x \to 3} (x^2 + 2)}{\lim\limits_{x \to 3} (x^2 - 1)} = \frac{11}{8}$$

Now consider the quotient $y = \dfrac{2x^2 - 2x}{x - 1} = \dfrac{g_1(x)}{g_2(x)}$. We have seen in Example 4 of Art. 7 that the limit of this quotient is 2 as $x \to 1$. But, since $\lim\limits_{x \to 1} (x - 1) = \lim\limits_{x \to 1} g_2(x) = 0$, the quotient $\dfrac{\lim\limits_{x \to 1} g_1(x)}{\lim\limits_{x \to 1} g_2(x)}$ has no meaning. In other words, the limit of this quotient cannot be found by taking the quotient of the limits. Incidentally, note that

$$\lim_{x \to 1} (2x^2 - 2x) = \lim_{x \to 1} g_1(x) = 0$$

Again, consider the quotient

$$\frac{x^2 + 2}{x^2 - 1} = \frac{f_1(x)}{f_2(x)}$$

and let $x \to 1$. Then we see that the fraction $\left|\dfrac{x^2 + 2}{x^2 - 1}\right|$ increases with-

out limit, for $f_1(x) \to 3$ while $f_2(x) \to 0$. Also, the quotient $\dfrac{\lim\limits_{x\to 1} f_1(x)}{\lim\limits_{x\to 1} f_2(x)}$ has no meaning since its denominator is 0. In general, if $f_1(x)$ has a limit $\neq 0$ while $f_2(x)$ has limit 0, the quotient $\dfrac{f_1(x)}{f_2(x)}$ has no limit.

These examples indicate that, if the limit of the denominator is zero, a quotient may or may not have a limit, but if it has a limit, this limit cannot be found by forming the quotient of the limits of numerator and denominator. If the limit of the quotient exists, some other device must be employed to find its value. These facts are included in a corollary to Theorem 3:

COROLLARY: *If the limit of the denominator of a quotient is zero and* (a) *the limit of the numerator is also zero, the quotient may or may not have a limit;* (b) *the limit of the numerator is not zero, the quotient has no limit.*

The proof of only theorem 1 will be given. Proofs of the other two theorems are similar in nature. As a first step, the student should satisfy himself that, if A and B are any two numbers, then $|A + B| \leqq |A| + |B|$. For example, if $A = -5$ and $B = 3$, then $|A + B| = |-2| = 2$. Also, $|A| = 5$ and $|B| = 3$; hence $|A| + |B| = 8$. Since $2 < 8$, our general statement is verified in this case.

Proof of Theorem 1: Consider two functions, $f_1(x)$ and $f_2(x)$, and let it be given that $\lim\limits_{x\to a} f_1(x) = b_1$ and $\lim\limits_{x\to a} f_2(x) = b_2$. In order to prove that

$$\lim_{x\to a} [f_1(x) + f_2(x)] = b_1 + b_2$$

we must show that $|f_1(x) + f_2(x) - (b_1 + b_2)|$ can be made less than any arbitrarily small positive number simply by taking x close enough to a. Now, we have

$$\begin{aligned} |f_1(x) + f_2(x) - (b_1 + b_2)| &= |f_1(x) - b_1 + f_2(x) - b_2| \\ &\leqq |f_1(x) - b_1| + |f_2(x) - b_2| \end{aligned}$$

Let ϵ be an arbitrarily small positive number. Since $\lim\limits_{x\to a} f_1(x) = b_1$, there exists some number δ_1 such that

$$|f_1(x) - b_1| < \frac{\epsilon}{2} \qquad \text{for all } x \text{ such that } 0 < |x - a| < \delta_1$$

Similarly, there exists some number δ_2 such that

$$|f_2(x) - b_2| < \frac{\epsilon}{2} \qquad \text{for all } x \text{ such that } 0 < |x - a| < \delta_2$$

Hence for all x sufficiently close to a, that is, for all $0 < |x - a| < \delta$ where δ is the smaller of δ_1, δ_2, we shall have

$$|f_1(x) + f_2(x) - (b_1 + b_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{Q.E.D.}$$

Evidently the proof can be extended to the sum of any finite number of functions.

Example 1. Limit of a Constant Times a Function. Suppose we wish to find $\lim_{x \to 2} 5x^2$. This may be regarded as a special case of theorem 2 where $f_1(x) = 5$ and $f_2(x) = x^2$. Hence

$$\lim_{x \to 2} 5x^2 = \lim_{x \to 2} 5 \cdot \lim_{x \to 2} x^2 = 5 \cdot \lim_{x \to 2} x^2 = 5 \cdot 4 = 20$$

In general, if k is any constant, then

$$\lim_{x \to a} k \cdot f(x) = \lim_{x \to a} k \cdot \lim_{x \to a} f(x) = k \cdot \lim_{x \to a} f(x)$$

In other words, *the limit of a constant times a function is that constant times the limit of the function.*

Example 2. Find

$$\lim_{x \to 3} \frac{x^3 - x^2 - 7x + 3}{x^2 - 9}$$

Since this function is the quotient of two functions, we first try to apply theorem 3 about the quotient of the limits. The numerator is the sum of four functions, namely, x^3, $-x^2$, $-7x$, 3. Hence, by theorem 1,

$$\lim_{x \to 3} (x^3 - x^2 - 7x + 3) = \lim_{x \to 3} x^3 + \lim_{x \to 3} (-x^2) + \lim_{x \to 3} (-7x) + \lim_{x \to 3} 3 = 0$$

Similarly $$\lim_{x \to 3} (x^2 - 9) = 9 - 9 = 0$$

But, since the limit of the denominator is 0, we cannot find the limit of the quotient by taking the quotient of the limits. Since the limit of the numerator is 0, it is possible that the fraction has a limit. If it has, some other method must be used to find the limit.

We observe that the fraction could be written

$$\frac{x^3 - x^2 - 7x + 3}{x^2 - 9} = \frac{(x - 3)(x^2 + 2x - 1)}{(x - 3)(x + 3)} = \frac{x^2 + 2x - 1}{x + 3} \qquad \text{for all } x \neq 3$$

It is now possible to apply the theorem about the quotient of the limits,

$$\lim_{x \to 3} \frac{x^2 + 2x - 1}{x + 3} = \frac{14}{6} = \frac{7}{3}$$

and this is the limit of the fraction originally given since, in the limiting process, x is never *equal* to 3. Note especially that we reduce the fraction *before* proceeding to the *limit* and that the reduction is valid for every value of x except $x = 3$. This is essentially different from reducing the fraction and then *substituting* $x = 3$.

Example 3. Find the limit

$$\lim_{x \to 3} \frac{x^3 - x^2 - 7x + 3}{x^2 - 6x + 9}$$

Again we cannot obtain any result by trying to divide the limit of the numerator by the limit of the denominator, since the latter is 0. But we do have

$$\frac{x^3 - x^2 - 7x + 3}{x^2 - 6x + 9} = \frac{(x - 3)(x^2 + 2x - 1)}{(x - 3)(x - 3)} = \frac{x^2 + 2x - 1}{x - 3}$$

for every value of x except $x = 3$. In this new fraction the limit of the denominator is 0 and the limit of the numerator is 14. Hence the fraction has no limit, and the same holds for the original fraction.

Example 4. Find the limit

$$\lim_{x \to 0} \frac{\sin x}{\tan x}$$

Here we may write, using the identities of trigonometry,

$$\frac{\sin x}{\tan x} = \frac{\sin x \cos x}{\sin x} = \cos x$$

for all values of x between $-\pi/2$ and $\pi/2$ except $x = 0$. Hence

$$\lim_{x \to 0} \frac{\sin x}{\tan x} = \lim_{x \to 0} \cos x = 1$$

EXERCISES

1. Show that each of the following sequences has limit zero. Represent values of x by points on the x axis.

(a) $x_1 = 1, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{1}{3}, \quad \ldots, \quad x_n = \frac{1}{n}, \quad \ldots$

(b) $x_1 = -1, \quad x_2 = -\frac{1}{2}, \quad x_3 = -\frac{1}{3}, \quad \ldots, \quad x_n = -\frac{1}{n}, \quad \ldots$

(c) $x_1 = 1, \quad x_2 = -\frac{1}{2}, \quad x_3 = \frac{1}{3}, \quad x_4 = -\frac{1}{4}, \quad \ldots, \quad x_n = (-1)^{n-1}\frac{1}{n}, \quad \ldots$

(d) $x_1 = 1, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{1}{9}, \quad \ldots, \quad x_n = \frac{1}{n^2}, \quad \ldots$

(e) $x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{1}{8}, \quad x_4 = \frac{1}{16}, \quad \ldots, \quad x_n = \frac{1}{2^n}, \quad \ldots$

2. Compute six terms of each of the following sequences and guess from these terms what the limit of the sequence is; prove your guess. (*Note:* It may be necessary to start some of the sequences with $n = 2$ instead of $n = 1$.)

(a) $x_n = \dfrac{n}{2n - 1}$

(b) $x_n = \dfrac{n + 1}{n - 1}$

(c) $x_n = \dfrac{2n - 1}{2n + 1}$

(d) $x_n = \dfrac{(-1)^n}{n^2}$

(e) $x_n = \dfrac{2n + 1}{n}$

(f) $x_n = \dfrac{1}{\sqrt{n}}$

(g) $x_n = \dfrac{(2n - 1)^2}{4n^2}$

(h) $x_n = \dfrac{(-1)^n}{n^2 - 1}$

3. Discover x_n in each of the following sequences; then proceed as in Exercise 2.

(a) $\frac{2}{2}, \frac{4}{5}, \frac{6}{8}, \frac{8}{11}, \frac{10}{14}, \ldots$

(b) $\frac{1}{3}, \frac{4}{6}, \frac{9}{11}, \frac{16}{18}, \frac{25}{27}, \ldots$

(c) $\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{8}}, \dfrac{1}{\sqrt{15}}, -\dfrac{1}{\sqrt{24}}, \dfrac{1}{\sqrt{35}}, \ldots$

(d) $\frac{1}{2}, \frac{7}{5}, \frac{17}{10}, \frac{31}{17}, \frac{49}{26}, \ldots$

4. Discover x_n in each of the following sequences; then if the sequence has a limit, find it.

(a) $\frac{1}{2}, -\frac{2}{5}, \frac{3}{10}, -\frac{4}{17}, \frac{5}{26}, \ldots$

(b) $1, -2, 3, -4, 5, \ldots$

(c) $1, -1, 1, -1, 1, \ldots$

(d) $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots$

5. (a) Given $y = \dfrac{x^2 + 3x - 10}{x - 2}$. Let x take the sequence of values 2.1, 2.01, 2.001, 2.0001, ... Calculate the corresponding values of y. Guess the limit of y and show that your guess is correct. Draw the graph of this function.

(b) Given $y = \dfrac{x^2 - x - 2}{x + 1}$. Let x take the sequence of values -1.1, -1.01, -1.001, -1.0001, ... Treat this function as you have treated the function of (a).

6. By letting x take a sequence of values with limit 3, illustrate the fact that $y = \dfrac{2x + 3}{x^2 - 9}$ does not approach a limit as x approaches 3.

Use the theorems of Art. 8 to evaluate the following limits (Ex. 7 to 17):

7. $\lim\limits_{x \to 2} (x^4 - 3x^3 + x^2 + 7x - 11)$

8. $\lim\limits_{t \to -1} (2t^5 + 3t^4 - 4t^3 + 5t^2 - t - 11)$

9. $\lim\limits_{z \to 2} \dfrac{(z^2 + 9)(z^2 - 4)}{4}$

10. $\lim\limits_{\varphi \to \pi/4} \sin 2\varphi \tan \varphi$

11. $\lim\limits_{x \to 3\pi/4} x \cos x \tan x$

12. $\lim\limits_{x \to 3} \dfrac{2x^2 - x + 3}{x^2 + 9}$

13. $\lim\limits_{x \to 4} \dfrac{x^3 - 3x^2 - 3x - 4}{x^2 - 4x}$

14. $\lim\limits_{x \to -3} \dfrac{2x^3 + 6x^2 - x - 3}{x^3 + 4x^2 + 2x - 3}$

15. $\lim\limits_{x \to -2} \dfrac{x^4 + 5x^3 + 5x^2 - x + 2}{x^4 + 4x^3 + 8x^2 + 16x + 16}$

16. $\lim\limits_{x \to 1} \dfrac{2x^3 - x^2 - 1}{x^3 - x^2 - x + 1}$

17. $\lim\limits_{x \to 2} \dfrac{x^3 + x^2 - 16x + 20}{x^3 + x^2 - 12}$

18. (a) If n is a positive integer, show that $\lim\limits_{x \to a} x^n = a^n$.

(b) If n is a negative integer and $a \neq 0$, show that $\lim\limits_{x \to a} x^n = a^n$.

19. (a) If $P(x)$ is any polynomial, show that $\lim\limits_{x \to a} P(x) = P(a)$.

(b) If $P(x)$ and $Q(x)$ are any two polynomials and if a is not a root of $Q(x)$, show that $\lim\limits_{x \to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$.

20. Construct examples illustrating the fact that $P(x)/Q(x)$ may or may not have a limit as x approaches a when a is a root of both $P(x)$ and $Q(x)$.

Evaluate the following limits by use of the theorems of Art. 8 (Ex. 21 to 29):

21. $\lim\limits_{x \to 3^+} \dfrac{\sqrt{x^2 - 9}}{x - 3}$

22. $\lim\limits_{x \to 3^+} \dfrac{x - 3}{\sqrt{x^2 - 9}}$

23. $\lim\limits_{x \to -1} \dfrac{(x^3 + 1)^{1/3}}{(x + 1)^{1/3}}$

24. $\lim\limits_{\theta \to \pi} \dfrac{\tan 2\theta}{\tan \theta}$

25. $\lim\limits_{\theta \to \pi/2} \dfrac{1 - \sin \theta}{\cos^2 \theta}$

26. $\lim\limits_{\theta \to \pi/4} \dfrac{\cos 2\theta}{\cot 2\theta}$

27. $\lim\limits_{\theta \to \pi/2} \dfrac{\sin 3\theta}{\sin \theta}$

28. $\lim\limits_{\varphi \to \pi/4} \dfrac{\sin \varphi}{\tan\left(\dfrac{\pi}{4} - \varphi\right)}$

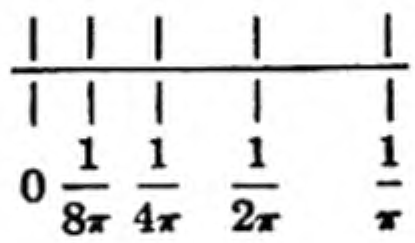
29. $\lim\limits_{\theta \to 0} \dfrac{1 - \sec^2 \theta}{\sin^2 \theta}$

30. Given $y = \sin (1/x)$. Consider the successive values of x,

$$\frac{2}{\pi}, \frac{1}{\pi}, \frac{2}{3\pi}, \frac{1}{2\pi}, \frac{2}{5\pi}, \frac{1}{3\pi}, \frac{2}{7\pi}, \ldots$$

and write down the corresponding values of y. Draw the graph of this function. Does y have a limit as $x \to 0$? Why?

Hint: Mark the x axis as follows:



9. Variables Becoming Infinite. Consider the function $y = 8/x$, and let $x \to 0$. Evidently we can make y numerically as large as we like by making x close enough to 0. When this happens, it is very convenient to make use of the expression "y becomes *infinite* as x approaches 0," which we write in symbols $y \to \infty$ as $x \to 0$.* It is of the utmost importance to remember that this is merely a short way of saying "y increases numerically without limit." To be still more precise, it is another way of saying "If G is any positive number, no matter how large, then $|y| > G$ for all x close enough to 0." We must be very careful not to fall into the error of thinking that the symbol ∞ represents some very large, although unknown, number.

It is helpful to emphasize two special cases. If $y = 8/x$ and we make x approach 0 through values greater than 0, then y remains positive and increases without limit. We say that y becomes *positively infinite* and write $y \to +\infty$ as $x \to 0^+$. On the other hand, if x is made to approach 0 through values less than 0, then y increases numerically but remains negative. We may say that y becomes *negatively infinite* and write $y \to -\infty$ as $x \to 0^-$. It is instructive to look again at the graph of this function (Fig. 8).†

* The symbol $y \to \infty$ should be read "y becomes infinite," *not* "y approaches infinity."

† A variable may become infinite without becoming either positively or negatively infinite. For example, y might take values $1, -2, 3, -4, 5, -6, \ldots, (-1)^{n+1} n, \ldots$. Here y becomes and remains numerically greater than any assigned number, provided that n is large enough, but it remains neither positive nor negative. In such cases, y is said to *oscillate* infinitely. Observe that the notation $y \to \infty$ still includes this case.

In general, a variable is said to become *infinite* if, under the law that governs its variation, it becomes and remains numerically greater than any assigned constant however large.*

10. Functions Whose Arguments Become Infinite. We may well ask, "What happens to the function $y = 8/x$ if the numerical value of x itself is made to increase without limit?" To answer this question, first let x take the sequence of values 1, 10, 100, 1000, 10,000, ... Corresponding values of y are 8, 0.8, 0.08, 0.008, 0.0008, ... Apparently y is decreasing toward 0. In fact, we can make $8/x$ as close to 0 as we please by taking x sufficiently great in numerical value. If we wish to have $\left|\frac{8}{x}\right| < \epsilon$, it is sufficient to have x such that $|x| > \frac{8}{\epsilon}$. This is exactly what is meant by the statement "The limit of $8/x$ is zero." However, x is not approaching a definite constant but is increasing indefinitely. We therefore shall express this fact by saying "The limit of y is 0 as x becomes infinite" and write $\lim\limits_{x\to\infty} y = 0$.

If we let x increase numerically and remain positive, we can make $\left|\frac{8}{x}\right|$ as small as we please. We then write $\lim\limits_{x\to+\infty} \frac{8}{x} = 0$. Similarly, if we let x increase numerically but remain negative, we can still make $\left|\frac{8}{x}\right|$ as small as we please. We then write $\lim\limits_{x\to-\infty} \frac{8}{x} = 0$.

The graphical interpretation of these remarks is not difficult. The fact that $y \to +\infty$ as $x \to 0^+$ and that $y \to -\infty$ as $x \to 0^-$ means that the line $x = 0$ is a vertical asymptote to the curve, whereas $\lim\limits_{x\to+\infty} y = 0$ and $\lim\limits_{x\to-\infty} y = 0$ mean that the line $y = 0$ is a horizontal asymptote (Fig. 8).

Example 1. Evaluate the limit $\lim\limits_{x\to\infty} \frac{x^2+4}{x^2-9}$. Evidently we cannot take the quotient of the limits, for neither numerator nor denominator has a limit. But this difficulty can be overcome by first dividing numerator and denominator by x^2, thus

* Various notations are in vogue in which the symbol ∞ appears. For instance, if $y \to \infty$ as $x \to 0$, it is customary to write $\lim\limits_{x\to 0} y = \infty$. Although this notation may seem paradoxical, since $y \to \infty$ means that y does not have a limit for the value of x in question, it is, nevertheless, convenient and can be used if properly understood. However, it is unfortunate that $\frac{8}{0} = \infty$ is written. The fraction $\frac{8}{0}$ simply has no meaning at all. It would be correct to say that $8/x \to \infty$ as $x \to 0$, for $8/x$ does increase numerically without limit as x approaches 0. "Infinity" is concerned with variables and may properly be associated with fractions whose denominators *approach* zero, but not with fractions whose denominators are *equal to* zero.

$$\lim_{x \to \infty} \frac{x^2 + 4}{x^2 - 9} = \lim_{x \to \infty} \frac{1 + \frac{4}{x^2}}{1 - \frac{9}{x^2}} = 1$$

The graph (Fig. 10) of the function $y = \dfrac{x^2 + 4}{x^2 - 9}$ illustrates the fact that $y = 1$ is a horizontal asymptote. Also, since $y \to +\infty$ as $x \to 3^+$ and $y \to -\infty$ as $x \to 3^-$, the line $x = 3$ is a vertical asymptote. Similar remarks apply to the line $x = -3$.

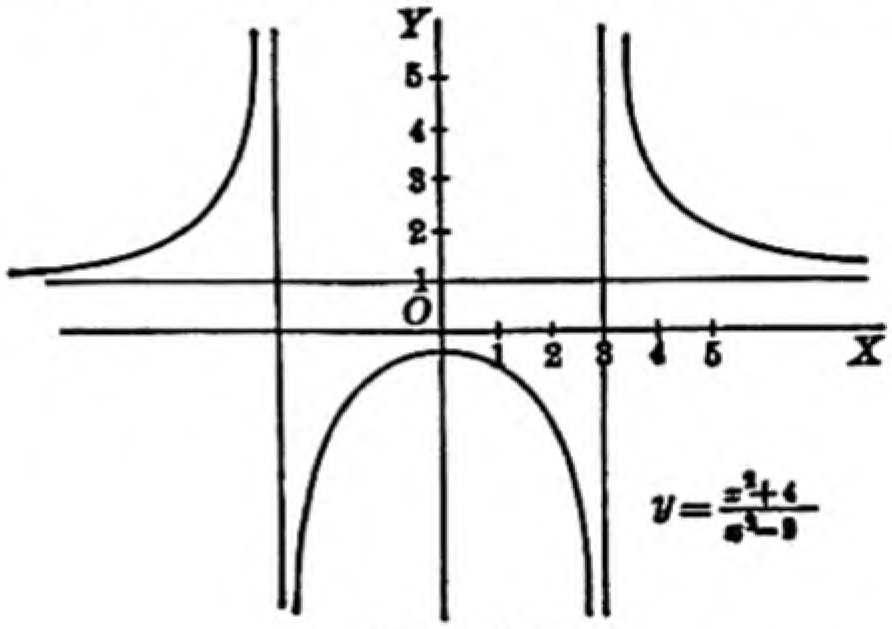


FIG. 10.

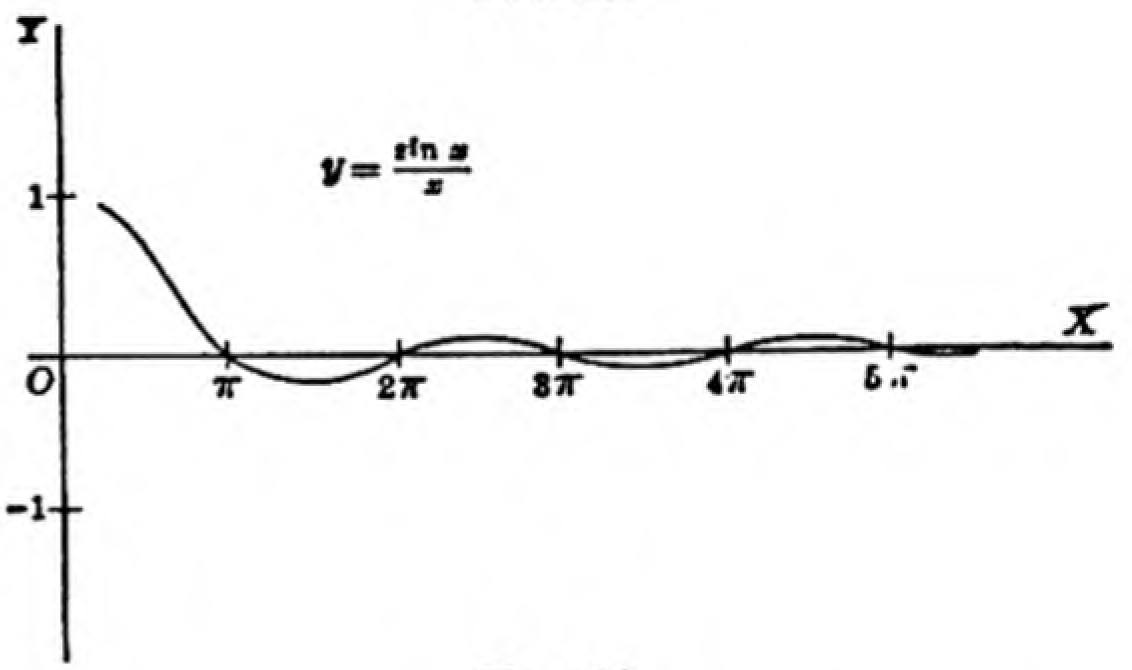


FIG. 11.

Example 2. Evaluate the limit $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$. Again we cannot take the quotient of the limits since the numerator *oscillates* between -1 and 1, while the denominator increases without limit. However, since x is eventually positive, we have

$$-1 \leqq \sin x \leqq 1 \qquad \text{and} \qquad -\frac{1}{x} \leqq \frac{\sin x}{x} \leqq \frac{1}{x}$$

Observe that $\dfrac{\sin x}{x}$ is contained between $-1/x$ and $1/x$. But as x increases indefinitely, both $-1/x$ and $1/x$ approach 0. Therefore, $\dfrac{\sin x}{x}$ must also approach 0. In symbols,

$$0 = \lim_{x \to \infty} \left(-\frac{1}{x}\right) \leqq \lim_{x \to \infty} \frac{\sin x}{x} \leqq \lim_{x \to \infty} \frac{1}{x} = 0$$

Hence, $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$. The graph (Fig. 11) illustrates the situation.

EXERCISES

Evaluate the following limits (Ex. 1 to 19):

1. $\lim\limits_{x\to\infty} \dfrac{3}{x^2+9}$

2. $\lim\limits_{x\to\infty} \dfrac{17x+9}{x^2+8x-2}$

3. $\lim\limits_{x\to\infty} \dfrac{7x^2-5}{x^2+25}$

4. $\lim\limits_{x\to\infty} \dfrac{3x^4-2x^3+x^2-5x+1}{2x^4+x^3-8x+10}$

5. $\lim\limits_{x\to\infty} \dfrac{15x^3+x^2-12x+4}{27x^2-15x+7}$

6. $\lim\limits_{x\to\infty} \dfrac{5x^2+7x+9}{\sqrt{x^4-81}}$

7. $\lim\limits_{x\to\infty} \dfrac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$ where $a_0 \neq 0$, $b_0 \neq 0$, and n and m are positive integers. Consider three cases: (a) $n < m$, (b) $n = m$, (c) $n > m$.

8. $\lim\limits_{x\to\infty} \dfrac{(4-x)^2}{(1+3x)^2}$

9. $\lim\limits_{x\to-\infty} 10^x$, also $\lim\limits_{x\to+\infty} 10^x$

10. $\lim\limits_{x\to+\infty} \dfrac{2^x}{3^x}$

11. $\lim\limits_{x\to+\infty} \dfrac{3^x}{2^x}$

12. $\lim\limits_{x\to+\infty} 10^{16-x}$

13. $\lim\limits_{x\to\infty} 10^{\frac{1}{x}}$

14. $\lim\limits_{x\to0} 10^{\frac{1}{x^2}}$

15. $\lim\limits_{x\to\infty} \dfrac{\cos x}{x}$

16. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{x}$

17. $\lim\limits_{x\to\infty} \sin x$; interpret graphically.

18. $\lim\limits_{x\to\infty} \tan x$; interpret graphicallly.

19. $\lim\limits_{x\to\infty} \dfrac{\tan x}{x}$

20. Discuss the behavior of $x \sin(\pi/x)$ as $x \to 0$; interpret graphically.

21. Find horizontal and vertical asymptotes for the curves

(a) $y = \dfrac{x^2-9}{x^2-3x-4}$ (b) $y = \dfrac{5x^4+1}{x^4-x^3-7x^2+x+6}$

11. Continuity. Because we are to be concerned in this book with functions that are *continuous*, except perhaps for certain values of the independent variable, it is important for us to have a clear understanding of the meaning of this term. We try to answer the question "When is a function continuous in the interval from $x = a$ to $x = b$?" Perhaps the most natural answer would be "When its graph for values of x in this interval can be drawn without lifting the pencil from the paper." This can be done, for instance, for the function $y = 8/x$ in the interval from $x = 1$ to $x = 3$ (see Fig. 8). But it cannot be done in the interval from $x = -1$ to $x = 1$, for something goes wrong at the point where $x = 0$. We might say that "the function is discontinuous at $x = 0$." Again, for the function giving the postage in terms of the weight of a letter (see

Fig. 9), we must make sudden jumps at the points where $x = 1, 2, 3, \ldots$, and these we are naturally led to call *points of discontinuity.*

What, then, is the difficulty at such points? In the case $y = 8/x$, we note that the function is *not defined* at all for $x = 0$. In the case of the postage function, if we fix our attention upon the point where $x = 1$, we find that we can draw the graph from the left up to and including the point (1,3), but then we must make a sudden jump if we wish to go on for values of x greater than 1. If we draw from the right, we approach a point (1,6) instead of (1,3). The difficulty evidently lies in the fact that the function does not have a limit for x approaching 1. In general, if $y = f(x)$, the point for which $x = a$ will be a point of discontinuity if

1. The function is not defined for $x = a$.
2. The function is defined for $x = a$ but has no limit as $x \to a$.
3. The function is defined for $x = a$ and has a limit for $x \to a$, but the limit is not equal to the value of the function.

Now consider again the function $y = 8/x$. We said that we could draw its graph without lifting the pencil from the paper in the interval from $x = 1$ to $x = 3$. This leads us to regard this function as continuous at the point where $x = 2$. In fact, as we draw the curve we find that it goes through the point (2,4) without making any jumps. Observe that $\lim\limits_{x \to 2} \dfrac{8}{x} = 4$ and that, therefore, the limit of the function as $x \to 2$ equals the value of the function for $x = 2$. We are now ready to give a precise and easily applied general definition of continuity. We first **define continuity at a point**: *A function $f(x)$ is said to be continuous at a point $x = a$ if and only if the function is defined at the point and the limit of the function as $x \to a$ is equal to the value of the function for $x = a$.* This may be expressed in symbols:

$$f(x) \textit{ is continuous at } x = a \textit{ if and only if } \lim_{x \to a} f(x) = f(a)$$

Having defined continuity at a point, we are ready for a **definition of continuity in an interval**: *A function $f(x)$ is said to be continuous in an interval $a \leqq x \leqq b$ if it is continuous at every point of the interval.*

Example 1. Show that $y = f(x) = \dfrac{x^2 + 4}{x^2 - 9}$ is continuous at $x = 2$, and find any points of discontinuity. Here $\lim\limits_{x \to 2} f(x) = -\frac{8}{5}$. Also, $f(2) = -\frac{8}{5}$. Hence $f(x)$ is continuous at $x = 2$. Since $x^2 - 9 = 0$ if $x = \pm 3$, the function is not defined for $x = \pm 3$. Therefore, $x = \pm 3$ are points of discontinuity. Furthermore, as x approaches either 3 or -3, $y = f(x)$ becomes infinite. For this reason, these points are called *points of infinite discontinuity* (see Fig. 10).

Example 2. Suppose $y = f(x) = \dfrac{x^2 - 2x}{x - 2}$. This function is discontinuous for $x = 2$ since it is undefined for this value of x. We see that $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x - 2} = 2$; so, if we

define $f(2) = 2$, then the function becomes continuous at $x = 2$. When a function is undefined at a given point, it is usually helpful, if circumstances permit, to frame its definition so that it becomes continuous at the point.

Functions that are continuous throughout an interval $a \leqq x \leqq b$ may be shown to have many interesting properties. Two such properties which are of considerable importance are

1. As x varies from a to b, $f(x)$ must assume at least once each value between $f(a)$ and $f(b)$.

2. For x in the interval $a \leqq x \leqq b$, $f(x)$ must assume at least once a largest and a smallest value.*

For example, the function giving postage y as a function of the weight x of a letter does not possess property (1), for it assumes values 3 and 6 for $x = 1$ and $x = 2$, respectively, but it does not assume the value 5 for any x in the interval. Neither of these properties will be established here, for their proofs are best deferred to a more advanced course.

EXERCISES

1. Show that $y = \dfrac{x-3}{x^2-16}$ is continuous at $x = 3$ and at $x = 5$. Find any points of discontinuity, and sketch the graph.

2. Show that $y = x^2 - 4x + 5$ is continuous at the points where $x = 0, -1, 3, 6$. Are there any points of discontinuity?

3. Find any points of discontinuity in $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \csc x$. Illustrate by drawing graphs.

Locate any points of discontinuity in the following functions (Ex. 4 to 8):

4. $y = \dfrac{x+2}{x^2-13x+42}$

5. $y = \dfrac{x-4}{x^2+6x+9}$

6. $y = \dfrac{x^3-5x^2+10x-8}{x^2+x-6}$

7. $y = \dfrac{3x^2-x+1}{x^2+10x+27}$

8. $y = \dfrac{2x^2+3x-7}{5x^2+9x-2}$

9. Show that a polynomial $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is continuous for all values of x.

10. Show that the rational fraction $y = \dfrac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$ is continuous for all values of x except those for which the denominator is zero.

11. Consider the behavior of the function $y = 10^{\frac{1}{x}}$ as $x \to 0$. Illustrate graphically. Is the function continuous at $x = 0$?

12. Same as Exercise 11 for $y = 10^{\frac{1}{x^2}}$

13. Same as Exercise 11 for $y = \dfrac{1}{1 + 10^{\frac{1}{x}}}$

14. Is the function $y = \sin(\pi/x)$ continuous for all values of x?

* The largest and smallest values might be equal as, for instance, in case $f(x) =$ a constant.

15. Show that, if $f(x)$ and $g(x)$ are continuous at a point $x = a$, then their sum, product, and quotient are continuous at $x = a$. Is any modification needed in this statement regarding the quotient?

16. Given $y = f(x)$ and $\lim_{x \to a} y = \lim_{x \to a} f(x) = b$. Also, given that $F(y)$ is continuous at $y = b$, that is, $\lim_{y \to b} F(y) = F(b)$. Prove that

$$\lim_{x \to a} F[f(x)] = F[\lim_{x \to a} f(x)]$$

Briefly, this says that the *limit of a continuous function of a variable equals that function of the limit of the variable.* Illustrate graphically.

MISCELLANEOUS EXERCISES

1. The strength of a beam of rectangular cross section is proportional to the width and to the square of the depth. Express the strength as a function of the width if the beam is cut from a circular log of diameter 3 ft.

2. The material in the top and sides of a rectangular box costs 30 cents per square foot, whereas that in the bottom costs 20 cents per square foot. The width and depth are the same, but the length is three times the width. Express the cost of the box as a function of its width.

3. Find $f(0), f\left(\frac{\pi}{2}\right), f\left(\frac{\pi}{4}\right), f(\pi - x)$ if $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$.

4. If $g(x) = a^x$, show that $g(x + y) = g(x)g(y)$.

5. Find $F(\sin x)$ if $F(y) = \arcsin y$.

6. Given two functions $f(x)$ and $g(x)$. Consider each of the following cases: (*a*) both even functions, (*b*) both odd functions, (*c*) one even and one odd function. Discuss fully whether the sum, difference, product, and quotient of the two functions are even, odd, or neither.

7. Show that, if $y = f(x) = \dfrac{2x + 3}{5x - 2}$, then $x = f(y)$. Generalize this statement. (*Hint:* Use letters instead of 2, 3, 5.)

8. Express y as an explicit function of x in each of the following cases:

(*a*) $x^3 + 3xy^2 - 1 = 0$ (*b*) $3x^2 - 4xy + y^2 + 2x - 8y + 1 = 0$

Evaluate the limits for the following (Ex. 9 to 28):

9. $\displaystyle\lim_{x \to 1} \frac{x^3 - x^2 + 4x - 4}{x^2 - 1}$

10. $\displaystyle\lim_{x \to -3} \frac{x^2 + 9}{x^2 - 9}$

11. $\displaystyle\lim_{x \to 2} \frac{x^3 - 8}{x^3 - 5x^2 + 8x - 4}$

12. $\displaystyle\lim_{x \to 1} \frac{x^4 - 2x^3 + 5x^2 - 8x + 4}{x^4 - x^3 + 8x - 8}$

13. $\displaystyle\lim_{\theta \to \pi} \frac{\tan 2\theta}{\sin \theta}$

14. $\displaystyle\lim_{\theta \to 0} \frac{\sin \theta - \tan \theta}{\cos \theta - \sec \theta}$

15. $\displaystyle\lim_{x \to \infty} \frac{2x^2 + 3\sqrt{x} + x^{1/2}}{\sqrt{x^4 - x^2 + 1}}$

16. $\displaystyle\lim_{\alpha \to \pi/2} \frac{\tan \alpha}{\sec \alpha}$

17. $\displaystyle\lim_{x \to \infty} \cos x$

18. $\displaystyle\lim_{x \to \infty} \frac{\sin x - \cos x}{x^2}$

19. $\displaystyle\lim_{x \to \infty} \frac{\tan x - \sin x}{x}$

20. $\displaystyle\lim_{x \to \infty} \frac{5x^5 - 4x^4 - 3x^3 - 2x^2 - x - 1}{1000x^4 + 7x^3 + 6x^2 + x + 3}$

21. $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^6 - x^4 + x^2 - 1}}{x^4 + 1}$

22. $\lim\limits_{x \to 0} \dfrac{x}{1 - \sqrt{1 - x}}$ (*Hint:* Rationalize the denominator.)

23. $\lim\limits_{x \to 2} \left(\dfrac{x^2 + 1}{x - 2} - \dfrac{2x^3 + 2x}{x^2 - 4} \right)$

24. $\lim\limits_{x \to 3} \left(\dfrac{x - 2}{x^2 - 9} - \dfrac{2x + 1}{x - 3} \right)$

25. $\lim\limits_{x \to 1} \left(\dfrac{5x^2 - x}{x^2 - 1} - \dfrac{20x^2 + x - 1}{x^3 + 4x^2 - x - 4} \right)$

26. $\lim\limits_{z \to 2} \dfrac{1 - \sqrt{3 - z}}{z - 2}$

27. $\lim\limits_{x \to 4} \dfrac{3 - \sqrt{5 + x}}{x - 4}$

28. $\lim\limits_{x \to 2} \dfrac{4 + \sqrt{20 - x^2}}{4 - \sqrt{20 - x^2}}$

29. Find any horizontal and vertical asymptotes for the curves

(*a*) $y = \dfrac{x^2 + 5x + 6}{x^3 - 6x^2 + 11x - 6}$

(*b*) $y = \dfrac{2x}{\sqrt{x^2 - 9}}$

(*c*) $y = x\sqrt{\dfrac{x - 9}{x + 3}}$

(*d*) $y = \dfrac{x^2 - 4}{x^4 + 16}$

30. Locate any points of discontinuity in the function $y = \dfrac{2x^2 - x - 6}{4x^2 - 7x - 2}$. What is the nature of each discontinuity? Can the definition of the function be changed to remove either discontinuity?

31. Same as Exercise 30 for the function $y = \dfrac{x^2 + x + 1}{4x^2 - 7x - 2}$

32. Find any points of discontinuity in the following: $y = \sin \dfrac{\pi}{x - 2}$.

33. Is the function $y = x \sin (\pi/x)$ continuous at the origin? If not, can its definition be extended to make it continuous? (See Exercise 20, page 25.)

CHAPTER 3

THE DERIVATIVE

We shall confine our study to functions that are single-valued and, in general, continuous. If a function is many-valued, we shall study individually one or more of the single-valued branches of which it is composed. If points of discontinuity occur, their effect upon the function under consideration will be treated separately.

12. Tangent Line to a Curve. A natural way to investigate the properties of any given function is to consider its graph. As we look at the graph of the function, for example that of $y = \frac{1}{2}x^2$ (Fig. 12), we may observe that, at some particular point, it appears to have a definite direction. It is natural to think of this direction as being the direction of the tangent line to the curve at the point in question. We are, therefore, concerned with the question, under what circumstances does a curve have a tangent line at a given point, and what is the direction of this line?

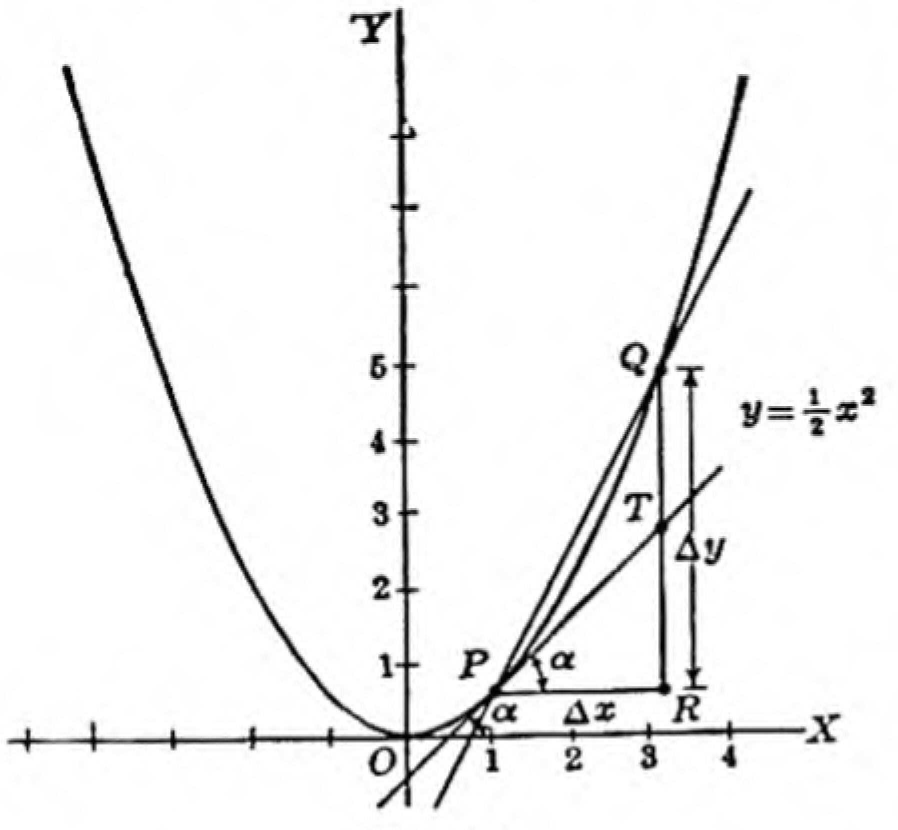


FIG. 12.

In other words, we seek to solve the *problem of tangents* mentioned in the first chapter.

Let us start with the function $y = \frac{1}{2}x^2$ and consider the point $P(1,\frac{1}{2})$, the graph being drawn in a rectangular coordinate system with equal horizontal and vertical scales. We shall define the *tangent line to the curve at point* P as follows: Let P be a fixed point on the curve and Q any other point on the curve. The tangent at P is the limiting position PT (Fig. 12) of the secant PQ as Q approaches P along the curve from either side of P. Now draw PR parallel to the x axis and RQ parallel to the y axis. Our definition is equivalent to saying that the limit of the variable angle RPQ is a fixed angle α as Q approaches P along the curve from either side; and if $\alpha \neq 90$ deg., this is equivalent to saying that the limit of $RQ/PR = \tan RPQ$ is $\tan \alpha$ as Q approaches P along the

curve. Returning now to the point $P(1,\frac{1}{2})$ on the curve $y = \frac{1}{2}x^2$, let us designate the distance PR by Δx (read "delta x")* and the distance RQ by Δy (Fig. 12). We note that Q has coordinates $1 + \Delta x$, $\frac{1}{2} + \Delta y$. Then $\tan RPQ = \dfrac{RQ}{PR} = \dfrac{\Delta y}{\Delta x}$. Note that, if Q is to the right and above P, then RQ and PR (being directed line segments) are both positive, whereas if Q is to the left and below P, then RQ and PR are both negative, and in both cases $\dfrac{RQ}{PR} = \dfrac{\Delta y}{\Delta x}$ is positive. If Q approaches P along the curve from either side of P, then Δx approaches 0, and conversely. Hence, if $\dfrac{\Delta y}{\Delta x}$ has a limit as Δx approaches 0, this limit must be $\tan \alpha$. We calculate the value of $\dfrac{\Delta y}{\Delta x}$ as follows:

$$\begin{aligned}\text{Ordinate of } Q &= \tfrac{1}{2}(1 + \Delta x)^2 = \tfrac{1}{2}(1 + 2\Delta x + \Delta x^2)\\ &= \tfrac{1}{2} + \Delta x + \tfrac{1}{2}\Delta x^2\end{aligned}$$

To find Δy, we subtract from this ordinate of Q the ordinate of P (namely, $\frac{1}{2}$), obtaining

$$\Delta y = \Delta x + \tfrac{1}{2}\Delta x^2$$

Hence
$$\frac{\Delta y}{\Delta x} = 1 + \frac{1}{2}\Delta x$$

and therefore
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(1 + \frac{1}{2}\Delta x\right) = 1$$

Thus we are sure that the curve has a tangent line at $P(1,\frac{1}{2})$. In addition, we know the direction of this tangent line; for since its slope is 1, it makes an angle α of 45 deg. with the horizontal. If the scale used for ordinates is not the same as that used for abscissas, this number 1 cannot be called the *slope* of the tangent line; it should be called, say, the rise of the tangent per horizontal unit. Unless expressly stated otherwise, the same scale will always be used for ordinates and abscissas.

We may investigate points other than $(1,\frac{1}{2})$ on the curve $y = \frac{1}{2}x^2$. Suppose P is the point on the curve whose coordinates are (x,y). If Q is some other point on the curve (Fig. 12), its coordinates are $x + \Delta x$, $y + \Delta y$. As before, to find the ordinate of Q we replace x by $x + \Delta x$ in the equation $y = \frac{1}{2}x^2$, obtaining

$$y + \Delta y = \tfrac{1}{2}(x + \Delta x)^2 = \tfrac{1}{2}x^2 + x\,\Delta x + \tfrac{1}{2}\Delta x^2$$

To find Δy, we must subtract the ordinate of P (namely, $\frac{1}{2}x^2$) from the ordinate of Q. This gives

$$\Delta y = x\,\Delta x + \tfrac{1}{2}\,\Delta x^2$$

* Δx is to be regarded as a single symbol; thus Δx^2 means $(\Delta x)^2$, etc.

Hence
$$\frac{\Delta y}{\Delta x} = x + \frac{1}{2}\Delta x$$

and therefore
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(x + \frac{1}{2}\Delta x\right) = x$$

In other words, at any point on the curve, $\tan \alpha$ is equal to the abscissa of that point. For instance, at the point $(1,\frac{1}{2})$, $\tan \alpha = 1$ which, of course, agrees with our previous result. We therefore see that the graph of $y = \frac{1}{2}x^2$ has a tangent line at every point of the curve.

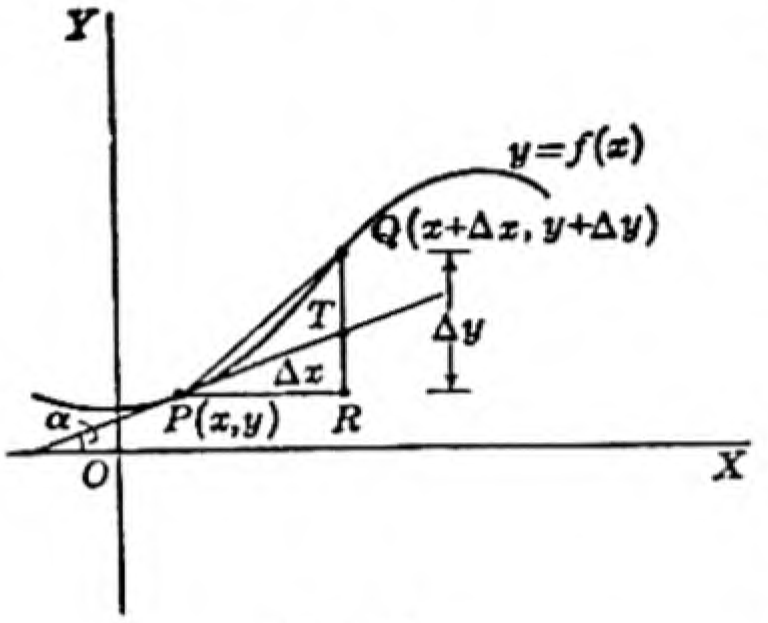


FIG. 13.

We may generalize these remarks. Suppose we have a function $y = f(x)$ that is continuous and single-valued throughout an interval, say $a \leqq x \leqq b$ (Fig. 13). Let P be the point (x,y), and let Q be any other point on the curve. The coordinates of Q are then $(x + \Delta x, y + \Delta y)$. To calculate the ordinate of Q, we replace x by $x + \Delta x$, obtaining $y + \Delta y = f(x + \Delta x)$. To find Δy, we subtract the ordinate of P from this ordinate, obtaining

$$\Delta y = f(x + \Delta x) - f(x)$$

Hence
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and therefore
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If this limit exists, the secant PQ has a limiting position as Q approaches P along the curve, and the value of the limit is the slope of the tangent line, or, if different vertical and horizontal scales are used, the rise of the tangent per horizontal unit. The point P is called the *point of contact* of the tangent. The *slope of a curve* at any point on the curve is defined to be the slope of the tangent line to the curve at that point.

EXERCISES

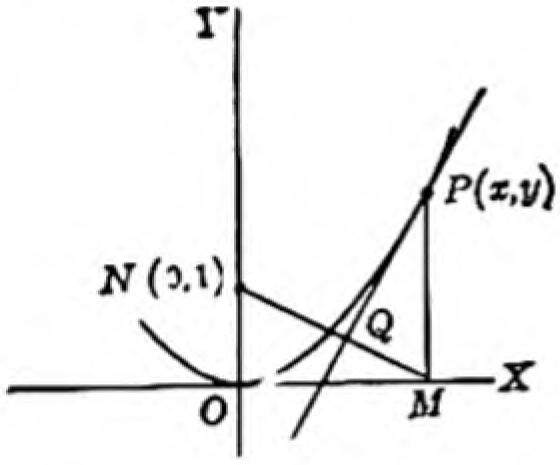
1. Draw the graph of the function $y = \frac{1}{16}x^4$. Let P be the point (2,1). Let Q be the point $(2 + \Delta x, 1 + \Delta y)$. Calculate the slope of the tangent line at P, first making a table showing values of $\frac{\Delta y}{\Delta x}$ for $\Delta x = 1$, 0.1, 0.01, . . . and then actually evaluating $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Draw the tangent line, and find its equation.

2. Same as Exercise 1 for $y = 2x^2$ and point $(-2,8)$

3. Same as Exercise 1 for $y = \frac{1}{9}x^3$ and point $(-3, -3)$

4. Same as Exercise 1 for $y = -\frac{1}{2}x^2$ and point $(2, -2)$

5. In the figure, the curve is the graph of $y = \frac{1}{2}x^2$. P is any point (x,y) on the curve. PM is perpendicular to OX; N is the point $(0,1)$. If PQ is drawn perpendicular to NM, show that it is the tangent at P.



13. Rate of Change of a Function. Suppose we regard the function $y = \frac{1}{2}x^2$ of the last section from a somewhat different point of view. How fast does y increase with increasing x? In other words, by how many units does y change with one unit change in x? Even the most perfunctory examination of the graph (Fig. 12) tells us that near $x = 0$ the curve rises slowly and hence y changes slowly with increasing x, whereas for large values of x the curve rises rapidly and y changes rapidly with increasing x. Evidently the *rate of change of the function* y is not a constant but varies with x. Let us find the average rate of change of y between $x = 1$ and $x = 2$. To do this, we evidently must compute the number of units change in y and divide by the number of units change in x. When $x = 1$, $y = \frac{1}{2}$; and when $x = 2$, $y = 2$. The change in y is $\Delta y = 2 - \frac{1}{2} = \frac{3}{2}$. The change in x is $\Delta x = 1$. The average rate of change of y between the two values of x is, therefore, $\dfrac{\Delta y}{\Delta x} = \dfrac{3}{2} \div 1 = \dfrac{3}{2}$.
Next let us compute the average rate of change of y between $x = 1$ and $x = 1 + \Delta x$. The computations are identical with those for the slope of the secant line to the curve $y = \frac{1}{2}x^2$. Thus, the average rate of change of y approaches 1 as a limit as Δx is made to approach 0. This limit we call the *instantaneous rate of change of* y *for* $x = 1$, or simply the *rate of change of* y *for* $x = 1$.

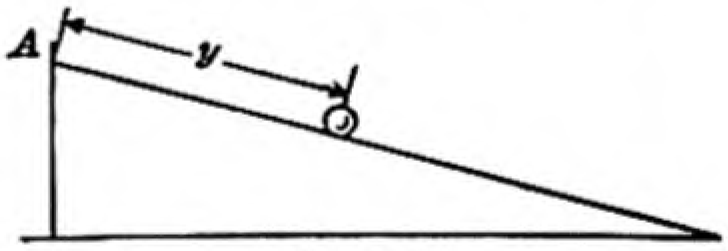


FIG. 14.

Evidently, in general, the rate of change of $y = f(x)$ for any given value of x is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, and it is computed in the same way as the slope of the tangent line to the graph of $y = f(x)$.

The determination of velocity is perhaps the most familiar example of finding the rate of change of a function. Suppose a ball is allowed to roll down an inclined plane, starting from rest at point A (Fig. 14), and suppose that x sec. later its distance y ft. from A is given by the equation $y = \frac{1}{2}x^2$. The graph of this equation (Fig. 12) now shows the dis-

tance y from A as a function of the time x elapsed since the ball was at A. Clearly, the fact that $(1,\frac{1}{2})$ and $(2,2)$ are points on this graph (Fig. 12) means that after 1 sec. the ball has rolled $\frac{1}{2}$ ft. down the incline from A and after 2 sec. it has reached a point 2 ft. from A. Its *average* velocity during the time elapsed from $x = 1$ to $x = 2$ is the distance traversed divided by the time elapsed, namely, $\frac{\Delta y}{\Delta x} = \frac{3}{2} \div 1 = \frac{3}{2}$ ft./sec. Evidently this is simply the average rate of change of the function y between $x = 1$ and $x = 2$. The shorter the interval of time over which we compute the average velocity, the closer the result is to the velocity at the instant beginning the interval. Hence the velocity at the instant when $x = 1$ is just the rate of change of y at that instant. This we have already seen to be $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1$. Therefore the velocity of the ball at the instant $x = 1$ is 1 ft. per second. If the ball could be fitted with an accurate speedometer, this is the velocity which would show on the speedometer at an instant exactly 1 sec. after starting down the incline.

EXERCISES

1. Find the average rate of change of the function $y = x^2$ between $x = 2$ and $x = 5$. Find the instantaneous rate of change at $x = 2$.
2. Find the average rate of change of the function $y = x^3$ between $x = 1$ and $x = 3$. Find the instantaneous rate of change at $x = 1$.
3. Find the average rate of change of the function $y = 16 - x^2$ between $x = 3$ and $x = 4$; also find the rate of change at $x = 4$. The result is negative; can you interpret the meaning of this fact?
4. A spherical toy balloon is inflated so that its volume (cubic inches) is a function of time (seconds) given by the formula $V = (\pi/6)t^3$. Find the rate of change of V at the time $t = 4$.
5. The surface area (square inches) of the balloon of Exercise 4 is $S = \pi t^2$. How fast is S changing at the time $t = 4$?
6. A ball is dropped from the roof of a building 144 ft. above the ground. Its distance (s ft.) from the starting point is a function of the time (t sec.) elapsed since its start: $s = 16t^2$. When will it strike the ground? Find its average speed during the time of the fall. Find its speed at the instant it strikes the ground.

14. Derivative of a Function. We have seen that the calculations involved in computing the slope of the tangent line to a curve and in calculating the rate of change of a function are exactly the same. The only difference is in the terminology and point of view. We can formulate a general process and frame a general definition entirely independently of these two ideas, but this general process will have the calculation of the rate of change of a function and the slope of its graph as two special applications. To this end, let $y = f(x)$ be a function that is single-valued and continuous in the interval $a \leqq x \leqq b$. Let (x,y) be any particular

pair of numbers satisfying the equation $y = f(x)$ and having x in the specified interval. Then

1. Let x receive an *increment* Δx. Replace x by $x + \Delta x$, obtaining a new value of y (denoted by $y + \Delta y$)

$$y + \Delta y = f(x + \Delta x)$$

2. Subtract the original value of y from the new value of y to obtain the increment in y, Δy

$$\Delta y = f(x + \Delta x) - f(x)$$

3. Form the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

4. Take the limit of this quotient as Δx approaches 0. The result is denoted by $\frac{dy}{dx}$ and is called the **derivative of y with respect to x at the point (x,y)**,

$$\bigstar \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided this limit exists. The process of finding the derivative is called *differentiation.* The symbol $\frac{dy}{dx}$ is to be regarded as a *single symbol* and is not to be thought of as a quotient of two numbers dy and dx. In particular, dy and dx are not to be thought of as the limits of Δy and Δx since each of these limits is 0.* It is very important to note that $\frac{dy}{dx}$ means the derivative at *the particular point whose coordinates are x, y.*

It is important to note that the definition of the derivative just set up is entirely independent of geometrical considerations. From the examples already given, it is clear that the derivative is equal to the slope of the graph of the function $y = f(x)$ at the point (x,y), or to the rate of change of the function y for a given value of x, but the derivative is defined and can be calculated quite without reference to these two ideas. They were presented first merely to indicate two reasons for taking the trouble of defining the derivative.

That $f(x)$ must be continuous if it is to possess a derivative is clear from the fact that

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

* Later we shall provide a definition of the symbols dy and dx so that they will have independent meanings. But this definition will be framed especially so that their quotient will equal the derivative of y with respect to x. It cannot be too strongly emphasized that the derivative is essentially not a quotient, but the *limit* of a quotient whose numerator and denominator both approach 0.

cannot have a limit unless $\lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = 0$, since the denominator approaches 0. But this is equivalent to

$$\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$$

which is true if and only if $f(x)$ is continuous for the value of x in question.* Hence, if a function has a derivative for a certain value of the argument, it is continuous for that value of the argument. However, the converse is not necessarily true, as can be shown by any number of examples. For instance, suppose that

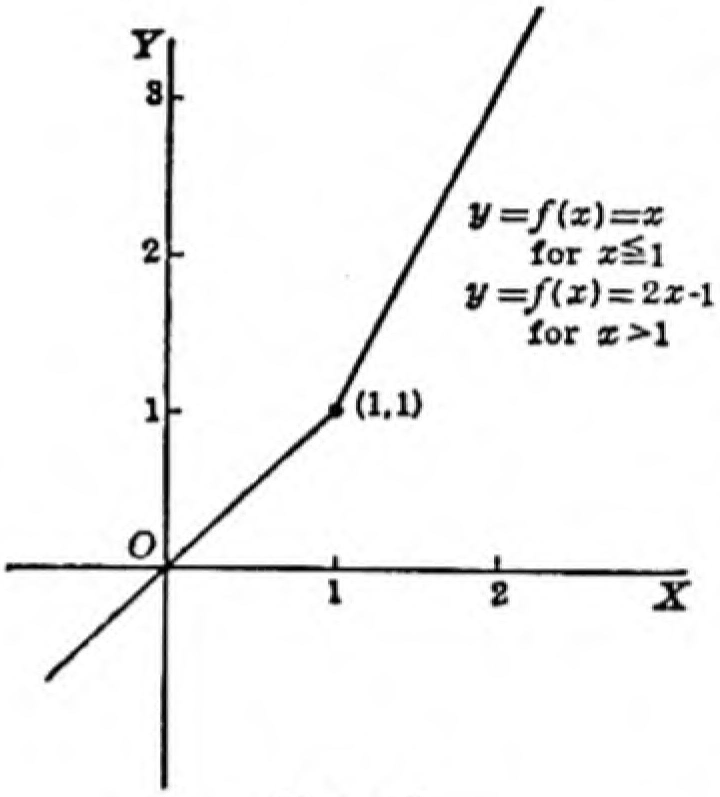


FIG. 15.

$$y = f(x) = x \qquad \text{for } x \leqq 1$$
$$y = f(x) = 2x - 1 \qquad \text{for } x > 1 \text{ (Fig. 15)}$$

This function is continuous for $x = 1$, since

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 1) = 1$$

Hence

$$\lim_{x \to 1} f(x) = 1$$

Since $f(1) = 1$, we have $f(x)$ continuous for $x = 1$. However, if Δx approaches 0 through negative values,

$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{1 + \Delta x - 1}{\Delta x} \qquad \text{has limit 1}$$

But if Δx approaches 0 through positive values,

$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{2(1 + \Delta x) - 1 - 1}{\Delta x} \qquad \text{has limit 2}$$

Therefore $\dfrac{f(1 + \Delta x) - f(1)}{\Delta x}$ does not have a limit as Δx approaches 0; and the function $f(x)$ does not have a derivative at $x = 1$.†

Functions that possess a derivative for all values of the independent variable within a certain interval are said to be *differentiable* in that interval. We shall hereafter be dealing largely with such functions.

* Suppose x fixed, then $x + \Delta x = \xi$ is a variable. If $\Delta x \to 0$, then $\xi \to x$. Now if, and only if, $\lim_{\xi \to x} f(\xi) = f(x)$, then $f(\xi)$ is continuous for $\xi = x$.

† Under such circumstances, $f(x)$ is often said, for obvious reasons, to have a *left-hand derivative* equal to 1 and a *right-hand derivative* equal to 2 at the point where $x = 1$. It can have a derivative only if the left- and right-hand derivatives exist and are equal. In 1861, Weierstrass gave an example of a function which is continuous at every point of an interval but which has a derivative at *no* point of the interval.

15. Notation for Derivatives. The most widely used notation for the derivative with respect to x of the function $y = f(x)$ is $\frac{dy}{dx}$. It is clear from the examples of Arts. 12 and 13 that the derivative is itself a function of x, obtained by carrying out certain operations upon $f(x)$. It is, in fact, "derived" from $f(x)$. For this reason the symbol $f'(x)$ is a suggestive and useful designation for the derivative of y with respect to x. A notation employed for similar reasons is y', but this has the disadvantage of not indicating the variable with respect to which the differentiation is carried out. It is convenient to use the symbol $\frac{d}{dx}$ to mean "the operation of finding the derivative with respect to x" of whatever follows it. Such a symbol is called an *operator*. For example, $\frac{d}{dx}(x^2 - 3x + 2)$ means "Find the derivative of $x^2 - 3x + 2$ with respect to x." The symbol D_x (or simply D) is often used instead of $\frac{d}{dx}$. Hence, all the following are notations for the derivative with respect to x of $y = f(x)$:

$$\frac{dy}{dx}, \quad f'(x), \quad y', \quad \frac{d}{dx}(y), \quad \frac{d}{dx}f(x), \quad D_xy, \quad D_xf(x), \quad D(y)$$

16. Calculation of the Derivative. Several examples of the process outlined in Art. 14 will now be given.

Example 1. *y* **a Positive Integral Power of** *x*. If $y = x^4$, find $\frac{dy}{dx}$. Also find the value of the derivative at the point $(2,16)$. Let x receive an increment Δx. We then have the new value of y,

$$y + \Delta y = (x + \Delta x)^4$$

Subtracting the original value of y $(= x^4)$ from this new value, we obtain Δy,

$$\Delta y = (x + \Delta x)^4 - x^4 = 4x^3\,\Delta x + 6x^2\,\Delta x^2 + 4x\,\Delta x^3 + \Delta x^4$$

We form the quotient

$$\frac{\Delta y}{\Delta x} = 4x^3 + 6x^2\,\Delta x + 4x\,\Delta x^2 + \Delta x^3$$

and take the limit as Δx approaches 0,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4x^3$$

The value of $\frac{dy}{dx}$ at the point $(2,16)$ is found by setting $x = 2$. Thus $\left.\frac{dy}{dx}\right]_{x=2} = 32$. Note the symbol $\left.\frac{dy}{dx}\right]_{x=2}$ which is a convenient way of designating the value of $\frac{dy}{dx}$ at $x = 2$.

Example 2. *y* **a Polynomial in** *x*. If $y = x^2 - 5x + 4$, find $\frac{dy}{dx}$. Let x receive an increment Δx. Then

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 - 5(x + \Delta x) + 4 \\ \Delta y &= (x + \Delta x)^2 - 5(x + \Delta x) + 4 - (x^2 - 5x + 4) \\ &= 2x\,\Delta x + \Delta x^2 - 5\Delta x \end{aligned}$$

Therefore $\quad \frac{\Delta y}{\Delta x} = 2x + \Delta x - 5 \quad$ and $\quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x - 5$

Example 3. *y* **a Negative Integral Power of** *x*. If $y = 1/x^2 = x^{-2}$, find $\frac{dy}{dx}$. Let x receive an increment Δx. Then

$$y + \Delta y = \frac{1}{(x + \Delta x)^2} \qquad \Delta y = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2} \qquad \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\left(\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}\right)$$

It is evident that we cannot evaluate this limit by taking the quotient of the limits, since the denominator and numerator both approach 0. We therefore change the form of the right-hand member of the equation as follows:

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\left(\frac{x^2 - (x + \Delta x)^2}{x^2(x + \Delta x)^2}\right) = \frac{1}{\Delta x}\left(\frac{-2x\,\Delta x - \Delta x^2}{x^2(x + \Delta x)^2}\right) = -\frac{2x + \Delta x}{x^2(x + \Delta x)^2}$$

It is now possible to find the limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{2x}{x^4} = -\frac{2}{x^3} = -2x^{-3}$$

Example 4. *y* **a Rational Function of** *x*. If $y = \frac{2x^2 + 1}{3x - 4}$, find $\frac{dy}{dx}$. Let x receive an increment Δx. We then have

$$\begin{aligned} y + \Delta y &= \frac{2(x + \Delta x)^2 + 1}{3(x + \Delta x) - 4} \\ \Delta y &= \frac{2(x + \Delta x)^2 + 1}{3(x + \Delta x) - 4} - \frac{2x^2 + 1}{3x - 4} \\ &= \frac{(6x^2 - 16x - 3)\,\Delta x + (6x - 8)\,\Delta x^2}{[3(x + \Delta x) - 4](3x - 4)} \end{aligned}$$

Hence
$$\frac{\Delta y}{\Delta x} = \frac{6x^2 - 16x - 3 + (6x - 8)\,\Delta x}{[3(x + \Delta x) - 4](3x - 4)}$$

and
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{6x^2 - 16x - 3}{(3x - 4)^2}$$

Example 5. *y* **the Square Root of a Function of** *x*. If $y = \sqrt{x - 4}$, find $\frac{dy}{dx}$. Let x receive an increment Δx. We then have

$$\begin{aligned} y + \Delta y &= \sqrt{(x + \Delta x) - 4} \\ \Delta y &= \sqrt{(x + \Delta x) - 4} - \sqrt{x - 4} \end{aligned}$$

Again, as in Example 3, if we divide by Δx and try to evaluate the limit, both numerator and denominator will have the limit 0. So we shall employ an artifice, namely,

we shall mutliply numerator and denominator by

$$\sqrt{(x + \Delta x) - 4} + \sqrt{x - 4}$$

We then obtain

$$\Delta y = \frac{(x + \Delta x) - 4 - (x - 4)}{\sqrt{(x + \Delta x) - 4} + \sqrt{x - 4}} = \frac{\Delta x}{\sqrt{(x + \Delta x) - 4} + \sqrt{x - 4}}$$

Hence
$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{(x + \Delta x) - 4} + \sqrt{x - 4}}$$

and
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x - 4}}$$

Example 6. A circular metal plate of radius r in. and area A sq. in. expands as a result of heating. Find the rate at which the area changes with the radius. We have $A = \pi r^2$. For any given value of r the required rate of change of A is simply the derivative of A with respect to r. (The reader will recall that the derivative can be interpreted as the instantaneous rate of change of a function.) Let r receive an increment Δr; then

$$\begin{aligned} A + \Delta A &= \pi(r + \Delta r)^2 \\ \Delta A &= \pi(r + \Delta r)^2 - \pi r^2 \\ &= 2\pi r\,\Delta r + \pi\,\Delta r^2 \end{aligned}$$

and
$$\frac{\Delta A}{\Delta r} = 2\pi r + \pi\,\Delta r$$

Hence
$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = 2\pi r$$

Thus, the rate at which the area of the plate changes is $2\pi r$ sq. in. (of area) per inch (of radius), where r is the radius of the plate at the instant in question.

17. Sign of the Derivative. A function of x, $y = f(x)$, is called an *increasing function* throughout any interval in which an increase in x causes an increase in the function. A function of x is called a *decreasing function* in any interval in which an increase in x causes a decrease in the function. The graph of an increasing function rises as x increases; the graph of a decreasing function falls as x increases.

Now it is readily seen from the examples given in the last section that the derivative of a given function of x may be positive, negative, or zero for different values of x. The sign of the derivative enables us to say whether a function is increasing or decreasing in the neighborhood of a given value of x. Let us first suppose that the derivative of $y = f(x)$ is *positive* for a certain value of x. Then, since $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, it is clear that, for sufficiently small values of Δx, Δy and Δx must have the same sign. The effect is illustrated in Fig. 16, in which $x_1 < x < x_2$; if $\Delta x = x_2 - x$ is positive, then $\Delta y = y_2 - y$ must be positive (for all values of x_2 close enough to x) and therefore $y_2 > y$. On the other hand, if $\Delta x = x_1 - x$ is negative, then $\Delta y = y_1 - y$ must be negative (for all values of x_1 close enough to x) and therefore $y_1 < y$. Hence $y_1 < y < y_2$ if $x_1 < x < x_2$,

and y is an *increasing* function of x. Of course, this result is intuitively evident since $\frac{dy}{dx} > 0$ expresses the fact that the tangent to the graph of $y = f(x)$ makes a positive acute angle with the x axis, and consequently the curve must be rising in the neighborhood of the point of contact of the tangent line.

In a similar manner, if the derivative of $y = f(x)$ is *negative* for a certain value of x, then, for sufficiently small values of Δx, Δy and Δx must have opposite signs. Thus, if Δx is positive, then Δy must be negative and y decreasing. Let the reader illustrate this case geometrically.

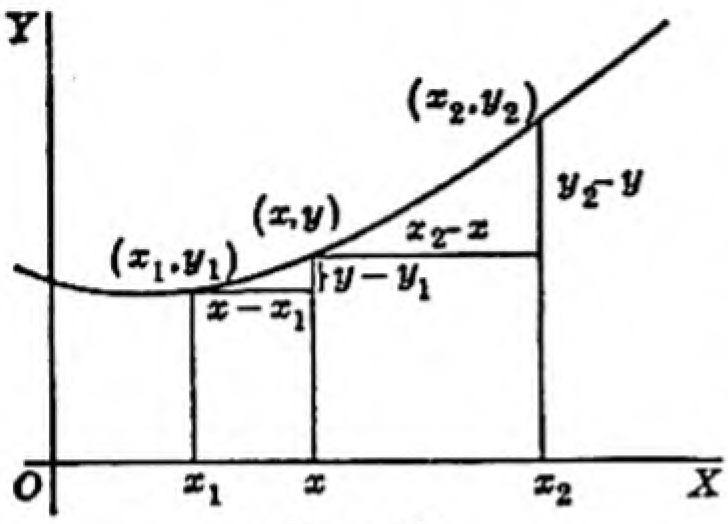


FIG. 16.

We have now shown that, *if the derivative of a function is positive at a given point, the function is increasing in the neighborhood of this point; if the derivative is negative, the function is decreasing in the neighborhood.* The situation arising when the derivative is zero will be considered in a later chapter.

Further, if $f'(x)$ is positive for all values of x in a certain interval, then $f(x)$ is increasing *throughout this interval.* A corresponding theorem is true for $f'(x)$ negative.

For example, if $y = \frac{1}{2}x^2$, then $\frac{dy}{dx} = x$; consequently y is increasing for $x > 0$ and decreasing for $x < 0$ (see Fig. 12).

It is quite evident from the examples given that it is desirable to develop methods which will enable us to calculate more conveniently the derivative of a function. The direct evaluation of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ in each special case would consume a great deal of time and would involve tedious computation which can be avoided. In the next few chapters, we shall, therefore, establish several standard formulas of a quite general character by means of which all the familiar elementary functions can be differentiated, and at the same time we shall illustrate some of the many ways in which the derivative can be used to discover the properties of various functions and to solve problems of practical importance.

EXERCISES

Find the derivatives of the following with respect to x, t, or u as appropriate by the "method of increments" illustrated in Art. 16 (Ex. 1 to 20):

1. $y = x^6$	2. $y = x^2 - 9$
3. $y = 8x^3$	4. $y = x^3 - 3x^2 + 8x - 11$
5. $s = 100 - 3t^2$	6. $y = 4u^2 - 5u + 6$
7. $y = 1/x$	8. $y = \frac{4}{x^2} - x^2$

9. $y = 1/x^4$

10. $y = \dfrac{4x + 3}{5x - 7}$

11. $y = \dfrac{2x^2}{x^2 - 4}$

12. $s = \dfrac{t^3 - 8}{t + 1}$

13. $y = \sqrt{x}$

14. $y = \sqrt{3x + 1}$

15. $y = \sqrt{x^2 - 25}$

16. $y = \sqrt{u^3}$

17. $s = t^{1/3}$

18. $y = (x + 1)^{1/3}$

19. $y = 1/\sqrt{x}$

20. $y = \dfrac{1}{\sqrt{3u + 5}}$

21. $y = \dfrac{1}{x^{1/3}}$

22. $y = \dfrac{3}{\sqrt{x^2 + 4}}$

23. $y = \dfrac{5}{\sqrt{x^3 - 27}}$

24. A steel tire of radius r ft. expands as a result of heating. Find the rate at which the circumference changes with the radius. Does this rate depend upon the radius? (Compare with Example 6, Art. 16.)

25. A spherical toy balloon has a radius r. Find the rate at which (*a*) the surface area and (*b*) the volume change with the radius. Find these rates at the instant when the radius is 3 in.; 6 in.

26. The gravitational attraction between two particles is a function of the distance x between them, namely, $F = k/x^2$. Find the rate of change of F with x. If x increases, is F increasing or decreasing?

27. Determine whether the function is increasing or decreasing at the points specified:

(*a*) Exercise 3, points $(2,64)$, $(-1,-8)$, any other point (except the origin).
(*b*) Exercise 4, points $(1,-5)$, $(-1,-23)$, any point.
(*c*) Exercise 7, points $(2,\frac{1}{2})$, $(-3,-\frac{1}{3})$, any point except $x = 0$.
(*d*) Exercise 9, x negative, x positive.

Find the slope of the curve at the point indicated (Ex. 28 to 33):

28. $y = x^2 - 3x + 2$ at $(3,2)$

29. $y = \dfrac{1}{3 + x}$ at $(-4,-1)$

30. $y = x^3 - x^2 + 3x - 1$ at $(1,2)$

31. $y = \dfrac{2x}{x^2 + 1}$ at $(0,0)$

32. $y^2 = x - 1$ at $(5,2)$

33. $y^2 = x - 1$ at $(5,-2)$

Find the rate of change of s at the indicated time if s is measured in feet and t in seconds (Ex. 34 to 40):

34. $s = 12t - 16t^2$ at $t = 2$

35. $s = \dfrac{t + 1}{t - 2}$ at $t = 4$

36. $s = \sqrt{t^2 - 9}$ at $t = 5$

37. $s = \dfrac{2}{\sqrt{t - 1}}$ at $t = 2$

38. $s = \dfrac{5}{\sqrt{t^2 + 9}}$ at $t = 4$

39. $s = \dfrac{t^2 - 1}{t + 2}$ at $t = 2$

40. $s = \dfrac{t}{\sqrt{t + 1}}$ at $t = 3$

CHAPTER 4

GENERAL RULES FOR DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

We are now ready to develop a number of *standard formulas for differentiation* by means of which the derivative of any elementary function can be found without the necessity of evaluating $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ in each separate case. In the development of most of the formulas of this chapter the functions involved are required to be not algebraic but merely differentiable. Consequently, these formulas apply to transcendental as well as to algebraic functions. We shall, however, defer the study of transcendental functions to Chaps. 6 and 7.

18. The Derivative of x^n. We have already seen (page 37) that if $y = x^4$, then $\frac{dy}{dx} = 4x^3$. We observe that $y = x^4$ is a special case of $y = x^n$ where n is a positive integer. It is not difficult to find a general formula that will give $\frac{dy}{dx}$ in terms of x and n. We shall then be in a position to write the derivative of any positive integral power of x at once simply by replacing n by the proper value. Assuming n to be a positive integer, we make use of the binomial theorem,

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^n \\ &= x^n + nx^{n-1}\,\Delta x + \frac{n(n-1)}{2!}x^{n-2}\,\Delta x^2 + \cdots + \Delta x^n \\ \Delta y &= nx^{n-1}\,\Delta x + \frac{n(n-1)}{2!}x^{n-2}\,\Delta x^2 + \cdots + \Delta x^n \\ \frac{\Delta y}{\Delta x} &= nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\,\Delta x + \cdots + \Delta x^{n-1} \end{aligned}$$

To evaluate the limit of the right-hand member, we recall that the limit of the sum of a finite number of functions is the sum of their separate limits. Since every term after nx^{n-1} contains some positive power of Δx as a factor, each of these has limit 0 when $\Delta x \to 0$. Hence

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

Example 1. If $y = x^7$, then $n = 7$, and $\frac{dy}{dx} = 7x^6$.

Example 2. If $y = x$, then $n = 1$, and $\frac{dy}{dx} = x^0 = 1$.

In case n is *not* a positive integer, or zero, the expansion of $(x + \Delta x)^n$ will not contain a *finite number* of terms. For example,

$$(x + \Delta x)^{-1} = x^{-1} - x^{-2}\,\Delta x + x^{-3}\,\Delta x^2 - x^{-4}\,\Delta x^3 + \cdots$$

Therefore, neither Δy nor $\frac{\Delta y}{\Delta x}$ will contain a finite number of terms but will consist of never-ending series. Consequently, we cannot make use of the theorem that the limit of the sum is the sum of the limits, for this theorem requires a finite number of functions. However, by use of other devices it is possible to show that, even when n is not a positive integer but is any rational number,

★ $$\text{If } y = x^n, \qquad \text{then } \frac{dy}{dx} = nx^{n-1} \tag{1}$$

If n is irrational, x^n is not an algebraic function, but it is possible to show that the formula still holds. The student will be asked to prove the formula for n a negative integer in Exercises 59 and 60 (page 49). A simple proof making use of the derivative of an exponential function takes care of all cases (including n rational), and this will be given in Art. 51. In the meantime, in order to avoid restricting unduly the functions available for study in this and the next two chapters, *formula* (1) *will be assumed true for all rational values of n.* The following examples illustrate its use:

Example 3. If $y = \frac{1}{x^2} = x^{-2}$, then $\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$.

Example 4. If $y = x^{5/2}$, then $\frac{dy}{dx} = \frac{5}{2}x^{3/2}$.

19. Derivative of a Constant. If y is a *constant*, $y = c$, then, whatever values are assigned to x and Δx, y will remain unchanged. Thus $\Delta y = c - c = 0$ for every value of Δx. Therefore $\frac{\Delta y}{\Delta x} = 0$, and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

Thus

★ $$\frac{dc}{dx} = 0 \tag{2}$$

and *the derivative of any constant is zero.* Geometrically, this expresses the fact that the line $y = c$ has everywhere a slope 0; that is, the line is parallel to the x axis.

20. Derivative of a Sum. Suppose we have two different functions of x, say $u = u(x)$ and $v = v(x)$, which possess derivatives with respect to x throughout the same interval $a \leqq x \leqq b$. Let us call the sum of these functions y, so that $y = u + v$. To calculate the derivative of y with respect to x, we assign an increment Δx to x; then u and v receive increments Δu and Δv. Hence

$$y + \Delta y = u + \Delta u + v + \Delta v \qquad \Delta y = \Delta u + \Delta v$$

Hence
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

and therefore
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

Therefore

★
$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} \tag{3}$$

This formula holds, of course, for all values of x in the interval $a \leqq x \leqq b$.

The extension to the case of more than two functions of x is not difficult. Suppose we have a finite number of functions of x, say $u, v, w, \ldots, s$ all of which possess derivatives with respect to x in the same interval $a \leqq x \leqq b$. If $y = u + v + w + \cdots + s$, then it is clear that corresponding to an increment Δx we have

$$\Delta y = \Delta u + \Delta v + \Delta w + \cdots + \Delta s$$

and finally
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots + \frac{ds}{dx}$$

Therefore, we may state the following **general theorem**: *The derivative of the sum of any finite number of functions is the sum of the derivatives of the separate functions.*

Example. If $y = x^5 + x^{-1/3} + 7$, then $\dfrac{dy}{dx} = 5x^4 - \dfrac{1}{3}x^{-4/3}$.

21. Derivative of a Product. As before, let u and v be two functions of x possessing derivatives in the same interval. If $y = uv$, we have

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\ \Delta y &= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\ \frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v \end{aligned}$$

Making use of the theorems about the limit of a sum and of a product, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta v$$

Now, since v possesses a derivative, v must be a continuous function of x.

Hence, when $\Delta x \to 0$, Δv must approach 0. Therefore $\lim\limits_{\Delta x \to 0} \Delta v = 0$. Consequently, $\lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \lim\limits_{\Delta x \to 0} \Delta v = 0$. This gives at once the result

★
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{4}$$

We may now state the following **general theorem**: *The derivative of the product of two functions is the first times the derivative of the second plus the second times the derivative of the first.*

Example 1. If $y = x^4(x^3 + x + 1)$, then

$$\frac{dy}{dx} = x^4(3x^2 + 1) + (x^3 + x + 1)4x^3 = 7x^6 + 5x^4 + 4x^3$$

The following important special case should be noted. If one of the functions, say v, is a constant c, we have $y = cu$. Hence

$$\frac{dy}{dx} = c\frac{du}{dx} + u\frac{dc}{dx}$$

But

$$\frac{dc}{dx} = 0$$

Therefore

★
$$\frac{d}{dx}(cu) = c\frac{du}{dx} \tag{5}$$

That is, *the derivative of a constant times a function is that constant times the derivative of the function.*

Example 2. If $y = 4x^7 - 5x^3 + 2x^2 + 11x$

$$\frac{dy}{dx} = 28x^6 - 15x^2 + 4x + 11$$

The theorem can be extended to the product of any finite number of differentiable functions. For example, consider three functions u, v, w.

$$y = uvw = (uv)w$$

Thinking of (uv) as a single function, we have

$$\begin{aligned}\frac{dy}{dx} &= (uv)\frac{dw}{dx} + w\frac{d}{dx}(uv) = (uv)\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)\\ &= vw\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}\end{aligned}$$

In general, if we have a finite number of functions of x, say u, v, w, $\ldots$, r, s, each of which possesses a derivative in the same interval, and

if $y = uvw \cdots rs$ then

$$\frac{dy}{dx} = vw \cdots rs\frac{du}{dx} + uw \cdots rs\frac{dv}{dx} + \cdots + uvw \cdots r\frac{ds}{dx}$$

for all values of x within this interval.

Example 3. If $y = x^4(x^2 + 1)(x^3 - 1)$, then

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 1)(x^3 - 1)4x^3 + x^4(x^3 - 1)2x + x^4(x^2 + 1)3x^2\\ &= 9x^8 + 7x^6 - 6x^5 - 4x^3\end{aligned}$$

Let the student verify this by expanding $x^4(x^2 + 1)(x^3 - 1)$ and differentiating the result.

22. Derivative of a Quotient. Again let u and v be functions of x possessing derivatives in the same interval. If $y = u/v$, we have

$$\begin{aligned}y + \Delta y &= \frac{u + \Delta u}{v + \Delta v}\\ \Delta y &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\,\Delta u - uv - u\,\Delta v}{v(v + \Delta v)}\\ \frac{\Delta y}{\Delta x} &= \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}\end{aligned}$$

To evaluate this limit, we use the theorem that the limit of a quotient is the quotient of the limits. The limit of the numerator becomes

$$\lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} - \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} = v\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} - u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = v\frac{du}{dx} - u\frac{dv}{dx}$$

Similarly, in the denominator we have

$$\lim_{\Delta x \to 0} v(v + \Delta v) = v\lim_{\Delta x \to 0}(v + \Delta v) = v^2$$

since $\lim\limits_{\Delta x \to 0} \Delta v = 0$ as explained in Art. 21. Therefore,

★ $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{6}$$

This formula holds for all points in the interval in which u and v both have derivatives except, of course, those points at which $v = 0$. In words, *the derivative of the quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The student will find it very helpful to carry in mind the statements *in words* of formulas (2), (3), (4), (5), and (6).

Example 1. If $y = \dfrac{x^3 + 1}{x^2 + 9}$,

$$\frac{dy}{dx} = \frac{(x^2 + 9)3x^2 - (x^3 + 1)2x}{(x^2 + 9)^2} = \frac{x^4 + 27x^2 - 2x}{(x^2 + 9)^2}$$

Example 2. If $y = \dfrac{4x^2}{3(x - 1)} = \dfrac{4}{3} \cdot \dfrac{x^2}{x - 1}$,

$$\frac{dy}{dx} = \frac{4}{3} \cdot \frac{(x - 1)2x - x^2 \cdot 1}{(x - 1)^2} = \frac{4}{3} \cdot \frac{x^2 - 2x}{(x - 1)^2} = \frac{4x(x - 2)}{3(x - 1)^2}$$

The beginner almost always gives himself unnecessary labor by failing to notice the simplification involved in expressing y as $\frac{4}{3}$ times $\dfrac{x^2}{x - 1}$. In general, any constant factors multiplying the function to be differentiated should be noted, and the fact that $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$ should be used.

EXERCISES

Find the derivatives with respect to x (or t) of the following functions (Ex. 1 to 42):

1. $y = 7x - 11$
2. $y = 3x^2 - x^3$
3. $y = 4x^5 - 3x^3 + 5x + 21$
4. $y = 14x^6 + 3x^4 - x^2 + 12$
5. $y = 3x^5 - 4x^4 + 2x^3 + 7x^2 - 5x + 13$
6. $y = \frac{3}{5}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2$
7. $y = \frac{1}{4}x^4 - \frac{7}{3}x^3 + 6x^2 - 10x + 4$
8. $y = 3x^{-2} - 5x^{-3} + 2x$
9. $y = 6x - \dfrac{2}{3x}$
10. $y = x^3 + \dfrac{1}{x^3}$
11. $y = 9x^3 - 4x^2 + x - \dfrac{1}{x} + \dfrac{4}{x^2} - \dfrac{9}{x^3}$
12. $y = 16x^{1/2} + 12x^{2/3} - 25x^{7/5}$
13. $y = 4x^{-3/4} + 9x^{1/3}$
14. $s = 4\sqrt{t} + 9\sqrt[3]{t}$
15. $s = \dfrac{1}{2t} + \dfrac{3}{5t^2} + 4$
16. $y = \dfrac{11}{25x^4} + \dfrac{25}{11}x^4 + 9$
17. $y = \dfrac{5}{3x^2} - \dfrac{7}{4x^3}$
18. $y = \dfrac{12}{5\sqrt{x}} - \dfrac{7}{9\sqrt{x^3}}$
19. $y = \dfrac{1}{\sqrt{2x}} + 15x^{4/5}$
20. $s = \sqrt{5t} + \dfrac{1}{\sqrt{5t}}$

In Exercises 21 to 27, differentiate as a product; verify by expanding and differentiating.

21. $y = x^4(x^2 + 2x + 3)$
22. $y = (3x^2 + 1)(x^3 + 2x - 5)$
23. $y = (5x^4 - 3x)(2x - 1)$
24. $s = (t^3 - t + 1)\left(\dfrac{1}{t} + \dfrac{1}{t^3}\right)$

25. $s = (2t^2 - 3t + 4)(4t^2 + 3t + 2)$

26. $y = (2x - 1)(x^2 + 2)(5x^3 + 1)$

27. $s = t^4(2t^3 - 3)(3t^2 - 2)(4t - 1)$

28. $y = \dfrac{x + 2}{x - 3}$

29. $y = \dfrac{2x + 7}{5x - 9}$

30. $y = \dfrac{2}{x^2 - 25}$

31. $y = \dfrac{3x^2}{4x + 1}$

32. $s = \dfrac{t^2 + 3t + 2}{t - 5}$

33. $y = \dfrac{x^2 - 16}{x^3 + 27}$

34. $y = \dfrac{x^2 + 5x + 2}{3x - 4}$

35. $s = \dfrac{3t^2 + 1}{2t - 3}$

36. $s = \dfrac{2t^2 + 4t - 1}{t^3 + 8}$

In Exercises 37 and 38, differentiate as a quotient; verify by differentiating the second expression. Which is the easier method?

37. $y = \dfrac{x^2 - 4x - 5}{\sqrt{x}} = x^{3/2} - 4x^{1/2} - 5x^{-1/2}$

38. $s = \dfrac{3t^3 - 5t^2 - 11}{t^2} = 3t - 5 - 11t^{-2}$

39. $y = \dfrac{3(x^3 - 27)}{4(x^3 + 8)}$

40. $y = \dfrac{2(x^4 + 16)}{3x^2 + 6x - 9}$

41. $y = \dfrac{(x + 1)(x + 2)}{(x + 3)}$

42. $s = \dfrac{2(t - 1)(t - 2)}{3(t + 3)(t - 4)}$

In Exercises 43 to 46, find the slopes of the curves at the points specified, and make a rough sketch of each curve:

43. $y = x^3 - 3x^2 + 5$ at $(-1,1)$, $(0,5)$, $(1,3)$, $(2,1)$, $(3,5)$.

44. $y = x^{2/3}$ at $(-8,4)$, $(-1,1)$, $(1,1)$, $(8,4)$. What can you say about the tangent line as $x \to 0$?

45. $y = \dfrac{1}{x^2} - \dfrac{2}{x^3}$ at $(-1,3)$, $(1,-1)$, $(2,0)$, $(3,\frac{1}{27})$. What can you say about the slope of the curve for $x < 3$? For $x > 3$?

46. $y = x^4 - 24x^2 + 25$ at $(1,2)$. Find approximately the angle of inclination of the tangent at this point. Find the values of x that make the derivative zero. What can you say about the tangent line at these points?

In Exercises 47 and 48, find to the nearest degree the angle of inclination of the tangent line to the following curves at the points specified:

47. (*a*) $y = x^{3/2}$ at $(1,1)$ (*b*) $y = x^2$ at $(1,1)$
(*c*) $y = x^3$ at $(1,1)$ (*d*) $y = x^8$ at $(1,1)$

48. $y = \frac{1}{3}x^3 - x^2 + 7$ at $(3,7)$, $(1,\frac{19}{3})$, $(\frac{1}{2},\frac{163}{24})$

49. The distance (s ft.) a ball rolled in t sec. was $s = 8t - 12t^2$. Find the velocity at the time $t = 2$. In what units is the velocity expressed?

50. The velocity (v ft. per second) of a moving object was the following function of the time (t sec.) elapsed since starting: $v = 6t^2 - 2t^3$. Find the rate of change of v (what is this usually called?) at times $t = 1$ and $t = 4$. Interpret the physical meaning of the algebraic sign in each case. In what units is this rate of change expressed?

In Exercises 51 to 54, state whether y is an increasing or a decreasing function at the point specified:

51. $y = 4x^2 - 6x - 11$ at $(4,29)$

52. $y = \dfrac{2x - 1}{3x + 4}$ at $(1,\frac{1}{7})$; at (x,y)

53. $y = \dfrac{x^2}{x^2 + 16}$ at $(-2,\frac{1}{5})$

54. $y = x^2(x^2 + 1)^2$ at $(-1,4)$

In Exercises 55 to 58, find the values of x for which y is (*a*) increasing, (*b*) decreasing:

55. $y = x^2 - 6x + 5$

56. $y = \dfrac{x}{2x - 3}$

57. $y = 2x^3 - 3x^2 - 12x + 4$

58. $y = \dfrac{x}{x^2 + 4}$

59. Prove by the method of increments that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ if n is a *negative* integer. (*Hint:* Let $n = -m$; then $y = 1/x^m$. Now use the method of increments.)

60. Prove the theorem of Exercise 59 by use of the formula for the derivative of a quotient.

23. Derivative of a Function of a Function. We have a convenient means of differentiating a power of x. But if we seek the derivative of such a function as $y = (x^3 + 8)^{2/3}$, we cannot use formula (1), for we have, not a power of x, but a power of *a function of* x. We may, however, overcome the difficulty by introducing an auxiliary variable. If we set $u = x^3 + 8$, then $y = u^{2/3}$ and we have y a function of u (namely, the $\frac{2}{3}$ power) where u is a function of x (namely, $x^3 + 8$). That is, y is *a function of a function* of x. In general, let $y = f(u)$ where $u = g(x)$, and let y have a derivative $\dfrac{dy}{du}$ with respect to u, and let u have a derivative $\dfrac{du}{dx}$ with respect to x. Now, clearly, if a value of x is given, a value of u is determined. This value of u determines a value of y, and therefore y is a function of x through u. A change Δx in x produces a change Δu in u and a change Δy in y. Also, when $\Delta x \to 0$, Δu must approach zero since u is a continuous function of x because we have supposed the existence of the derivative of u with respect to x. We may write the identity

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

which is true for all values of Δu except $\Delta u = 0$.* Taking limits as $\Delta x \to 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

* The resulting formula (7) can be shown to hold even if $\Delta u = 0$, although the proof will not be given here.

whence

★
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{7}$$

Returning to the example $y = (x^3 + 8)^{2/3} = u^{2/3}$, we have

$$\frac{dy}{du} = \frac{2}{3}u^{-1/3} \qquad \text{and} \qquad \frac{du}{dx} = 3x^2$$

from which we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{2}{3u^{1/3}}\cdot 3x^2 = \frac{2x^2}{(x^3 + 8)^{1/3}}$$

It is unnecessary to write down the substitution of u for $x^3 + 8$. We may think of the differentiation as follows: To differentiate the $\frac{2}{3}$ power of any function, we write $\frac{2}{3}$ times that same function to the power $\frac{2}{3} - 1 = -\frac{1}{3}$, then multiply by the derivative of that same function. The result is at once

$$\frac{2}{3}(x^3 + 8)^{-1/3}\cdot 3x^2 = \frac{2x^2}{(x^3 + 8)^{1/3}}$$

Example 1. If $y = \dfrac{x}{\sqrt{x^2 + 4}}$, then

$$\frac{dy}{dx} = \frac{(\sqrt{x^2 + 4})\cdot 1 - x\cdot\dfrac{d}{dx}(\sqrt{x^2 + 4})}{x^2 + 4}$$

Now
$$\frac{d}{dx}\sqrt{x^2 + 4)} = \frac{1}{2}(x^2 + 4)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2 + 4}}$$

Hence
$$\frac{dy}{dx} = \frac{\sqrt{x^2 + 4} - \dfrac{x^2}{\sqrt{x^2 + 4}}}{x^2 + 4} = \frac{x^2 + 4 - x^2}{(x^2 + 4)^{3/2}} = \frac{4}{(x^2 + 4)^{3/2}}$$

Example 2. If $y = (x^3 + 5)^4(x^7 + 13)^5$,

$$\begin{aligned}\frac{dy}{dx} &= (x^3 + 5)^4\cdot\frac{d}{dx}(x^7 + 13)^5 + (x^7 + 13)^5\cdot\frac{d}{dx}(x^3 + 5)^4 \\ &= (x^3 + 5)^4\cdot 5(x^7 + 13)^4\cdot 7x^6 + (x^7 + 13)^5\cdot 4(x^3 + 5)^3\cdot 3x^2 \\ &= (x^3 + 5)^3(x^7 + 13)^4[35x^6(x^3 + 5) + 12x^2(x^7 + 13)] \\ &= x^2(x^3 + 5)^3(x^7 + 13)^4(47x^7 + 175x^4 + 156)\end{aligned}$$

24. Differentiation of a Power of Any Function. We may now write the formula for the derivative of any power of any differentiable function: If $y = u^n$, then

★
$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \tag{8}$$

In words, *the derivative with respect to x of the nth power of a function of x is*

n times the function to the $(n - 1)$ power times the derivative of the function with respect to x. It is to be kept in mind that, although we shall use this formula freely, its *proof* has been given only in case n is a positive integer. The proof for all other values of n will be given in Art. 51.

EXERCISES

Find derivatives with respect to x, t, or z as appropriate (Ex. 1 to 42):

1. $y = \sqrt{2x + 3}$

2. $y = \sqrt{3x^2 + 1}$

3. $y = (x^2 + 9)^{3/2}$

4. $y = (3x^2 - 6x + 11)^{2/3}$

5. $y = (a^2 - x^2)^{5/2}$

6. $y = \sqrt{a/x}$

7. $y = (3x + 4)^{5/2}$

8. $y = (x^3 + 1)^{2/3}$

9. $y = (x^2 + 1)^{4/3}$

10. $y = (x^3 - 8)^{3/4}$

11. $y = (7x - 9)^{3/5}$

12. $y = (6x + 1)^{-2/3}$

13. $y = (x^2 + 4x + 13)^{-3/5}$

14. $y = (ax + b)^{9/5}$

15. $y = \dfrac{1}{\sqrt{ax + b}}$

16. $y = \dfrac{1}{\sqrt{(ax + b)^3}}$

17. $y = \dfrac{1}{\sqrt[3]{ax^2 + bx + c}}$

18. $y = \dfrac{1}{\sqrt[3]{(ax + b)^4}}$

19. $y = (2ax - x^2)^{5/2}$

20. $y = x\sqrt{x^2 - a^2}$

21. $s = t^3(t^2 + 1)^{3/2}$

22. $y = x^2\sqrt{x^2 + 9}$

23. $y = (x^3 + 1)^2(x^2 - 4)^3$

24. $y = (x^2 - 16)^6(3x + 1)^5$

25. $y = (x^2 - 1)^{3/2}\sqrt{x^2 - 4}$

26. $y = (4x^2 - x + 1)^3(x^2 + 2x + 3)^4$

27. $y = (2x^3 + 4x^2 + 5x + 7)^{-5/2}$

28. $y = \dfrac{2x + 1}{\sqrt{x + 2}}$

29. $y = \dfrac{z}{\sqrt{2z + 7}} - \dfrac{\sqrt{2z + 7}}{z}$

30. $y = \dfrac{\sqrt{x^2 - 9}}{2x}$

31. $y = \dfrac{\sqrt{x^2 - a^2}}{x^2}$

32. $y = \dfrac{a^2x^2}{(a^2 - x^2)^{3/2}}$

33. $y = \dfrac{2x}{\sqrt{x^2 - 9}}$

34. $y = \dfrac{a\sqrt{az + b}}{bz}$

35. $s = \dfrac{(t^2 + 1)^{3/2}}{2t + 3}$

36. $s = \dfrac{(2t - 1)^3(t^2 + 4)^2}{3t + 1}$

37. $y = \dfrac{z^2}{(z + 2)^2(4z - 5)}$

38. $y = \sqrt{(z - a)^2 + a^2}$

39. $y = \sqrt{(z - 1)^3 - 1}$

40. $y = \sqrt{1 - \sqrt{1 + z}}$

41. $y = \sqrt{a + \sqrt{z - a}}$

42. $y = (1 - \sqrt{x^2 + 1})^{3/2}$

Find the slope of the curves at the points specified (Ex. 43 to 45):

43. $y = \sqrt{25 - x^2}$ at $(3,4)$, $(-4,3)$

44. $y = x^2(2x - 3)^2$ at $(2,4)$, $(-1,25)$, $(\frac{3}{4},\frac{81}{64})$

45. $y = \dfrac{3x}{\sqrt{x^2 - 9}}$ at $(5,\frac{15}{4})$. The point $(0,0)$ is a point of the graph. What can be said about the derivative at this point?

46. The radius of a circular plate increases at the rate of 0.001 in. per second when the plate is heated. How fast is the area changing when $r = 10$ in.? (*Hint:* A is a function of r where r is a function of t; $\frac{dr}{dt} = 0.001$, and we can calculate $\frac{dA}{dr}$. Now find $\frac{dA}{dt}$.)

47. A solid metal sphere expands when heated. If the radius changes at 0.05 in. per second, find how fast the volume is changing when the radius is 6 in.

25. Inverse Functions and Their Derivatives. If y is a function of x given by the equation $y = f(x)$, then in general x is a function of y. We may find the *inverse* function $x = g(y)$ of the given *direct* function $y = f(x)$ by solving the equation $y = f(x)$ for x.* Thus, if $y = x^3$, then $x = y^{1/3}$ and the *cube root* is the *inverse* of the *cube*. Or again, if $s = \frac{1}{t-1}$, then $t = \frac{s+1}{s}$ is the inverse function.

If $y = f(x)$, the existence of an inverse function, $x = g(y)$, requires proof. For example, if y is a continuous, single-valued, and steadily increasing function of x in the interval $a \leqq x \leqq b$, then it is possible to show that x is a single-valued, continuous, and steadily increasing function of y in the corresponding interval $f(a) \leqq y \leqq f(b)$. An analogous statement holds if y is a decreasing rather than an increasing function of x. If, further, $y = f(x)$ has a finite derivative $f'(x) \neq 0$ at every point of the interval $a \leqq x \leqq b$, then we can show that if y receives an increment Δy, then x receives an increment $\Delta x \neq 0$. Further, if $\Delta y \to 0$, then $\Delta x \to 0$. Proofs of these statements will not be given here, but the student can satisfy himself of their plausibility by drawing the graph of the kind of function described.

We may now write the identity

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$$

Taking limits as Δy and Δx approach 0, we have

$$g'(y) = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}} = \frac{1}{f'(x)}$$

This relation may be written

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

* This can usually be done for functions considered in this book. An example of a function for which it *cannot* be done is $y = 5$. Here, for any given value of x, a value of y (namely, 5) is determined. But for a given value of y no value of x is determined.

or, we may prefer to solve for $f'(x)$, thus

$$\star \qquad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \tag{9}$$

For an example, consider the direct function

$$y = \tfrac{1}{4}x^2 = f(x)$$

(Fig. 17). In the interval $x > 0$, $x = 2\sqrt{y}$ is the inverse of $f(x)$. Note that in this interval the ordinate of $y = \frac{1}{4}x^2$ continually increases as x increases, that at no point is the tangent horizontal or vertical, that for each value of x there corresponds one and only one value of y, and that for each value of y there corresponds one and only one value of x. Hence

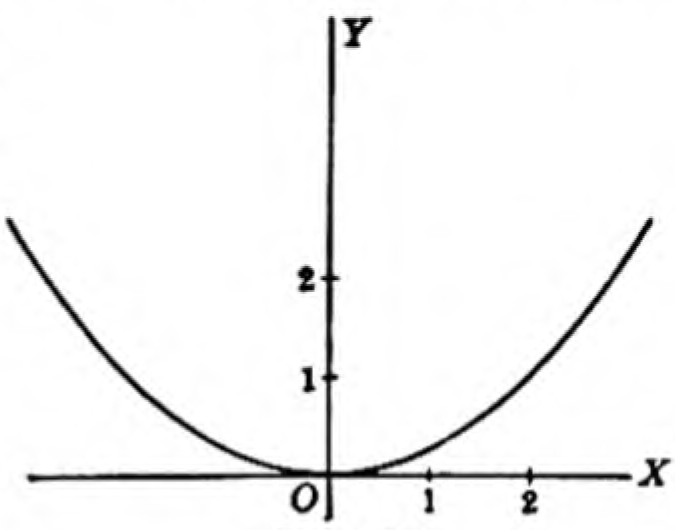


FIG. 17.

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

for all $x > 0$. Since $\dfrac{dx}{dy} = \dfrac{1}{\sqrt{y}}$, we have $\dfrac{dy}{dx} = \sqrt{y}$. It is interesting to observe that, if we calculate $\dfrac{dy}{dx}$ from the direct function $y = \frac{1}{4}x^2$, we obtain

$$\frac{dy}{dx} = \frac{1}{2}x$$

Since $x = 2\sqrt{y}$, the results are the same.

In the interval $x < 0$, $x = -2\sqrt{y}$ is the inverse of $y = \frac{1}{4}x^2$. The graph of this function is the left-hand half of the parabola (Fig. 17). Here the ordinate decreases as x increases, and again

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

(Note that we omit always $x = 0$ where $\dfrac{dx}{dy}$ does not exist.) We find that $\dfrac{dx}{dy} = -\dfrac{1}{\sqrt{y}}$, therefore $\dfrac{dy}{dx} = -\sqrt{y}$. Calculating $\dfrac{dy}{dx}$ from the direct function, we obtain $\dfrac{dy}{dx} = \dfrac{1}{2}x$. But since, for $x < 0$, $x = -2\sqrt{y}$, we have $\dfrac{dy}{dx} = -\sqrt{y}$, and the results are the same.

Note that both branches have in common the point (0,0) but that we cannot find $\frac{dy}{dx}$ at (0,0) by the formula

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

since $\frac{dx}{dy}$ does not exist at that point.

EXERCISES

Find the inverse of each of the following functions. Find $\frac{dy}{dx}$ in terms of $\frac{dx}{dy}$, and verify that the result agrees with that obtained by differentiating the direct function.

1. $y = x^3$
2. $y = x^4$
3. $y = x^2 + 4x + 4$
4. $y = x^2 + 2x + 2$
5. $y = (x^2 - 1)^2$
6. $y = \sqrt{x^2 - 4}$
7. $y = \dfrac{2x}{x - 3}$
8. $y = \dfrac{x^2}{x^2 + 4}$
9. $y = \dfrac{4}{x^3 + 8}$
10. $y = \dfrac{32}{x^4 + 16}$

26. Derivatives of Higher Order. Let us recall the function $y = x^4$. We find that its derivative is $\frac{dy}{dx} = 4x^3$, and we notice that this derivative itself is a function of x which can be differentiated. In general, the derivative of a function of x is another function of x that may be differentiable. The derivative with respect to x of $\frac{dy}{dx}$ is called the *second derivative* of y with respect to x, or the derivative of second order. The derivative of the second derivative is called the *third derivative* of y with respect to x, or the derivative of third order. Similarly, we may speak of the fourth, fifth, sixth, . . . , nth derivative of y with respect to x; $\frac{dy}{dx}$ is called the *first derivative.* The notation most widely used for the successive derivatives is as follows:

$$\frac{d}{dx}(y) = \frac{dy}{dx} \qquad \text{(first derivative)}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \qquad \text{(second derivative)}$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \qquad \text{(third derivative)}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} \qquad (n\text{th derivative})$$

Other notations for the successive derivatives of $y = f(x)$ are also quite common, namely,

$$
\begin{array}{cccccc}
f'(x), & f''(x), & f'''(x), & f^{(4)}(x), & \ldots, & f^{(n)}(x) \\
y', & y'', & y''', & y^{(4)}, & \ldots, & y^{(n)} \\
D_x y, & D_x^2 y, & D_x^3 y, & D_x^4 y, & \ldots, & D_x^n y \\
\dfrac{d}{dx}(y), & \dfrac{d^2}{dx^2}(y), & \dfrac{d^3}{dx^3}(y), & \dfrac{d^4}{dx^4}(y), & \ldots, & \dfrac{d^n}{dx^n}(y)
\end{array}
$$

Sometimes Roman numerals are used for the indices of derivatives of order higher than the third, for example, $f^{\text{IV}}(x)$ for $f^{(4)}(x)$ and y^{IV} for $y^{(4)}$.

Example 1. If $y = x^4$, then

$$\frac{dy}{dx} = 4x^3 \qquad \frac{d^2y}{dx^2} = 12x^2 \qquad \frac{d^3y}{dx^3} = 24x \qquad \frac{d^4y}{dx^4} = 24 \qquad \frac{d^5y}{dx^5} = 0$$

in fact
$$\frac{d^ny}{dx^n} = 0 \qquad \text{if } n > 4$$

Example 2. If $y = 1/x = x^{-1}$,

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{x^2} = -x^{-2} \\
\frac{d^2y}{dx^2} &= \frac{2}{x^3} = 2x^{-3} \\
\frac{d^3y}{dx^3} &= -\frac{2 \cdot 3}{x^4} = -2 \cdot 3x^{-4} \\
\frac{d^4y}{dx^4} &= \frac{2 \cdot 3 \cdot 4}{x^5} = 2 \cdot 3 \cdot 4x^{-5} \\
&\cdots\cdots\cdots\cdots\cdots \\
\frac{d^ny}{dx^n} &= (-1)^n \frac{n!}{x^{n+1}}
\end{aligned}
$$

The value of the nth derivative was inferred from the form of the first few derivatives. The validity of the formula can easily be established by mathematical induction.

The value of $\dfrac{dy}{dx}$ is equal to the rate of change of y; evidently, $\dfrac{d^2y}{dx^2}$ is the rate of change of $\dfrac{dy}{dx}$, $\dfrac{d^3y}{dx^3}$ the rate of change of $\dfrac{d^2y}{dx^2}$ and so on.

EXERCISES

Find the second derivative with respect to x or t as appropriate (Ex. 1 to 10):

1. $y = x^6 - 2x^4 + 4x^2$; also find $\dfrac{d^6y}{dx^6}$

2. $y = 3x^2 + 4x - 11$

3. $y = \dfrac{3}{2x^2}$

4. $y = \sqrt{x^2 - 4}$

5. $y = (4x + 7)^{3/2}$

6. $y = (ax + b)^{2/3}$

7. $y = x^3(7x + 1)^2$

8. $y = (3x - 1)^4(x^2 + 1)^3$

9. $s = \dfrac{2t}{3\sqrt{t^2 + 9}}$

10. $s = \dfrac{\sqrt{3t + 4}}{t}$

11. If $y = \sqrt{x^2 + 16}$, find $\dfrac{d^2y}{dx^2}$.

12. If $y = (ax + b)^{3/2}$, find $\dfrac{d^2y}{dx^2}$.

13. If $y = (x^3 + 8)^{4/3}$, find $\dfrac{d^2y}{dx^2}$.

14. If $s = \dfrac{1}{\sqrt{t}}$, find $\dfrac{d^4s}{dt^4}$.

15. Find an expression for $\dfrac{d^ny}{dx^n}$ if $y = x^n$ (n a positive integer).

16. Find an expression for $\dfrac{d^ny}{dx^n}$ if $y = \dfrac{1 + x}{1 - x}$.

17. Find an expression for $\dfrac{d^ny}{dx^n}$ if $y = \dfrac{x^2}{x - 1}$.

18. Find formulas for $\dfrac{d^2}{dx^2}(uv)$ and $\dfrac{d^3}{dx^3}(uv)$ where u and v are functions of x having derivatives of required orders.

19. Generalize the results of Exercise 18 to find *Leibnitz's formula* for $\dfrac{d^n}{dx^n}(uv)$.

20. Use Leibnitz's formula (Exercise 19) to find $\dfrac{d^4y}{dx^4}$ if $y = x^3(2x - 1)^4$. Verify the result by expanding y and then differentiating four times.

21. Find a formula for $\dfrac{d^2}{dx^2}\left(\dfrac{u}{v}\right)$ where u and v are functions of x.

22. Using the fact that

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

show that

$$\frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}$$

Note particularly that the second derivative of a function is *not* the reciprocal of the second derivative of the inverse function.

27. Differentiation of Implicit Functions. As already mentioned in Art. 5, we may have y given as an *implicit* function of x, say by the equation $F(x,y) = 0$. In order to find the derivative of y with respect to x, we might, if convenient, solve the equation $F(x,y) = 0$ for y and then find the derivative. But it is frequently inconvenient (and may be impossible) first to find y as an explicit function of x. The derivative may be found by differentiating both members of $F(x,y) = 0$ with respect to x, keeping always in mind the fact that y is a function of x. The derivative

of the left-hand member will be equal to the derivative of the right-hand member (namely, 0) for all values of x,y that satisfy the equation $F(x,y) = 0$. We may then solve the resulting equation for $\frac{dy}{dx}$. In general, the expression found for $\frac{dy}{dx}$ will contain both x and y. This method is valid if and only if $F(x,y) = 0$ actually defines y as a differentiable function of x (see Art. 148).

Example 1. If $x^2 + y^2 - 25 = 0$, find $\frac{dy}{dx}$. Since y is a function of x, the derivative of y^2 is $2y\frac{dy}{dx}$ (Art. 24); therefore

$$2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

which holds for every pair of values (x,y) satisfying the equation $x^2 + y^2 - 25 = 0$ (except $x = \pm 5$, $y = 0$). Geometrically, this means that at any point (x,y) of the circle $x^2 + y^2 - 25 = 0$, the tangent line has a slope equal to $-x/y$. When $x \to \pm 5$, $\frac{dy}{dx}$ becomes infinite and the tangent becomes vertical. Note that y is a function of x consisting of two branches, namely, $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$ (upper and lower semicircle); therefore, it is necessary to specify which branch is under discussion in calculating the numerical value of the derivative.

It is instructive to note that the derivative may be found as follows: Since

$$y = \sqrt{25 - x^2}$$

if we take the *upper* branch,

$$\frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}} = -\frac{x}{y}$$

If we take the *lower* branch, $y = -\sqrt{25 - x^2}$ and

$$\frac{dy}{dx} = \frac{x}{\sqrt{25 - x^2}} = -\frac{x}{-\sqrt{25 - x^2}} = -\frac{x}{y}$$

Example 2. If $x^4 - xy^3 + 2y - x + 11 = 0$, find $\frac{dy}{dx}$. We have

$$4x^3 - 3xy^2\frac{dy}{dx} - y^3 + 2\frac{dy}{dx} - 1 = 0$$

$$(2 - 3xy^2)\frac{dy}{dx} = y^3 - 4x^3 + 1$$

$$\frac{dy}{dx} = \frac{y^3 - 4x^3 + 1}{2 - 3xy^2}$$

We may employ a similar method to find higher ordered derivatives of implicit functions. To find the second derivative, first find the first derivative, then differentiate with respect to x, remembering that y is

a function of x. To find the third derivative, first find the second derivative, then differentiate, remembering that y is a function of x. Derivatives of higher order may be found in a similar manner.

Example 3. If $x^2 + y^2 - 25 = 0$, find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. We have already found that $\frac{dy}{dx} = -\frac{x}{y}$. From this, we have at once

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Now substitute the value of $\frac{dy}{dx}$. This gives

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}$$

$$\frac{d^3y}{dx^3} = \frac{75}{y^4}\frac{dy}{dx} = \frac{75}{y^4}\left(-\frac{x}{y}\right) = -\frac{75x}{y^5}$$

We shall return later to the question of finding derivatives of implicit functions (Chap. 17).

EXERCISES

Find $\frac{dy}{dx}$ (Ex. 1 to 18) if

1. $y^2 - 4x = 0$
2. $x^2 + y^2 - 16 = 0$
3. $4x^2 + 9y^2 = 36$
4. $4x^2 - 9y^2 = 36$
5. $xy = 32$
6. $y^2 = 4ax$
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
8. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
9. $x^2 + y^2 + 2ay = 0$
10. $xy = 2a^2$
11. $x^{1/2} + y^{1/2} = a^{1/2}$
12. $x^{2/3} + y^{2/3} = a^{2/3}$
13. $x^3 + y^3 = a^3$
14. $x^2 + 3xy + x - 1 = 0$
15. $x^3 + 2x^2y + xy^2 + y^3 = 0$
16. $ax^2 + 2bxy + cy^2 = 1$
17. $x^2y^3 = 1$
18. $x^4 - 3x^2y^2 + xy + 1 = 0$

19. Find $\frac{d^2y}{dx^2}$ for Exercises 1 to 18.

20. If $as^2 = t^3$, find $\frac{d^2s}{dt^2}$.

21. If $s^3 - at^2 = 0$, find $\frac{d^2s}{dt^2}$.

22. If $x^2 + y^2 = a^2$, find $\frac{d^3y}{dx^3}$.

Find the slope of each of the following curves at the point specified (Ex. 23 to 30):

23. $x^3 + y^3 + 2x^2 - xy - 3x + 4y = 0$ at (0,0)
24. $x^2y^2 - xy^3 + y - 14 = 0$ at (3,2)

25. $y^2 = \dfrac{x+4}{3x-8}$ at $(6,1)$

26. $y^2 = \dfrac{x^3}{x^2+9}$ at $(4, -\frac{8}{5})$

27. $y^2 = \dfrac{36}{x^2(x^2-8)}$ at $(-3,2)$

28. $x^4 + 16y^4 = 32$ (at $2,1$)

29. $x^3 - xy^2 - 3xy - 17 = 0$ at $(-1,3)$

30. $y^4 = \dfrac{3x+64}{x^2+4}$ (at $0,2$)

28. Parametric Representation. It is often very convenient to express the fact that y is a function of x by use of a third variable known as a *parameter*. Suppose that x and y are given as functions of a parameter t by the equations

$$x = \varphi(t) \qquad y = \psi(t) \tag{10}$$

In all the cases with which we shall be concerned, it follows that y is a function of x. We shall not prove this, but it can be seen roughly as follows: If, when a value is assigned to x, the relationship $x = \varphi(t)$ determines a corresponding value of t found from the inverse function, then this value of t serves to determine a value of y because $y = \psi(t)$. Consequently, y is determined when x is given; therefore, y is a function of x, say $y = f(x)$, or $F(x,y) = 0$.

If we eliminate t from equations (10), we find the relationship between y and x. Equations (10) are called the *parametric equations* of the curve whose *cartesian equation* is $y = f(x)$ or $F(x,y) = 0$. We may locate a point on this curve by assigning a value to t; the corresponding values of x and y are then the cartesian coordinates of the point in question. In case the coordinates of a moving point are given in terms of *time* t, equations (10) are the parametric equations of the path of motion. Or the parameter may (or may not) have some other physical or geometrical significance.

As an example, consider equations

$$x = 2t + 1 \qquad y = t^2 - 4$$

as the parametric equations of a curve. The cartesian equation is found by eliminating t, $t = \dfrac{x-1}{2}$; therefore

$$y = \left(\frac{x-1}{2}\right)^2 - 4 = \frac{1}{4}(x^2 - 2x + 1) - 4$$

and

$$x^2 - 2x - 4y - 15 = 0$$

Hence the curve is a parabola with vertex at $(1,-4)$ and axis parallel to the y axis (Fig. 18).

Now suppose that equations (10) imply a relationship $y = f(x)$. Also suppose, as in the above example, that $x = \varphi(t)$ has an inverse function whose derivative $\frac{dt}{dx}$ exists, and that $\frac{dy}{dt}$ exists. Then y is a function of t where t is a function of x, so that (Art. 23)

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Therefore

$$\star \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{\varphi'(t)} \tag{11}$$

Example 1. Find $\frac{dy}{dx}$ where $x = 2t + 1$ and $y = t^2 - 4$. We have

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t \qquad \text{and therefore } \frac{dy}{dx} = \frac{2t}{2} = t$$

The value of $\frac{dy}{dx}$ at the point where $t = 3$ is 3. Hence the slope of the parabola at the point (7,5) is 3. It is instructive to notice that if we eliminate t and then calculate $\frac{dy}{dx}$ by the usual method we have

$$y = \tfrac{1}{4}(x - 1)^2 - 4 \qquad \text{(see above)}$$
$$\frac{dy}{dx} = \frac{1}{2}(x - 1)$$

Since $t = \frac{1}{2}(x - 1)$, we see that the results obtained by the two methods agree.

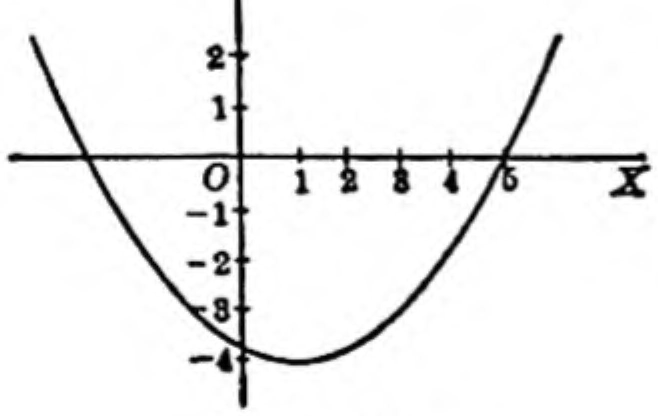


FIG. 18.

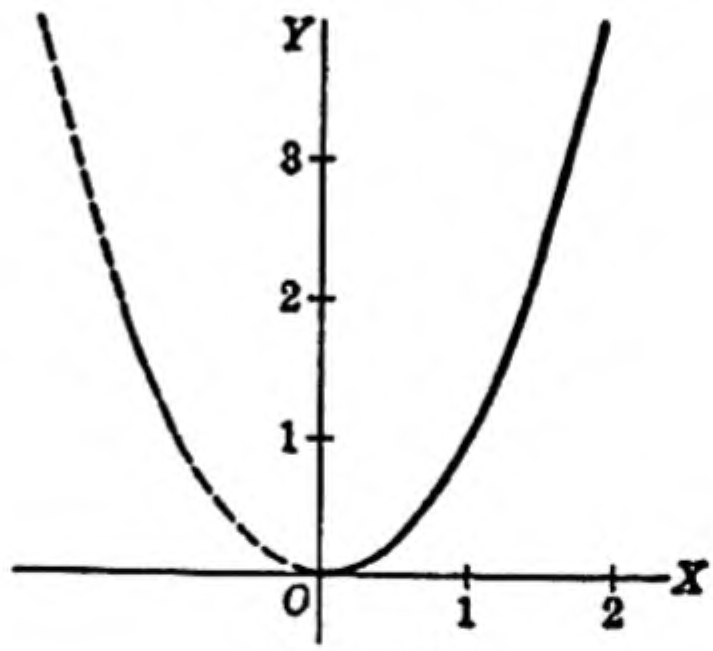


FIG. 19.

Example 2. Find $\frac{dy}{dx}$ if $x = t^2$ and $y = t^4$. We have $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 4t^3$, and $\frac{dy}{dx} = \frac{4t^3}{2t} = 2t^2$. To find the cartesian equation of the curve, we eliminate t, obtaining $y = x^2$. However, the parametric equations give points on only the right-hand half of this parabola (Fig. 19) if only real values of t are admitted, since x must always be

positive or zero. It is also clear that $\frac{dy}{dx}$, and therefore the slope of the curve, is positive or zero for all real values of t.

It is clear from the above examples that $\frac{dy}{dx} = y'$ will be expressed in terms of t. Hence, y' is a function of t where t is a function of x. Consequently

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Similarly, since $\frac{d^2y}{dx^2} = y''$ is a function of t where t is a function of x, we have

$$\frac{d^3y}{dx^3} = \frac{dy''}{dx} = \frac{dy''}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy''}{dt}}{\frac{dx}{dt}}$$

Derivatives of higher order can be found in succession in a similar fashion:

$$\frac{d^ny}{dx^n} = \frac{\frac{dy^{(n-1)}}{dt}}{\frac{dx}{dt}}$$

We can obtain a formula for $\frac{d^2y}{dx^2}$ in terms of the derivatives of x and y with respect to t as follows:

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2}$$

Dividing this by $\frac{dx}{dt}$ gives the final result

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3} \tag{12}$$

Example 3. If $x = t^2$, and $y = t^4$, find $\frac{d^2y}{dx^2}$. We have (from Example 2)

$$\frac{dy}{dx} = y' = 2t^2$$

Therefore $\quad \dfrac{dy'}{dt} = 4t \qquad \dfrac{dx}{dt} = 2t \qquad$ and $\qquad \dfrac{d^2y}{dx^2} = \dfrac{\dfrac{dy'}{dt}}{\dfrac{dx}{dt}} = \dfrac{4t}{2t} = 2$

Here we have found $\dfrac{d^2y}{dx^2}$ by use of the *method* by which formula (12) was obtained rather than by substituting in the formula itself. The formula gives

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 4t^3 \qquad \frac{d^2x}{dt^2} = 2 \qquad \frac{d^2y}{dt^2} = 12t^2$$

$$\frac{d^2y}{dx^2} = \frac{2t \cdot 12t^2 - 2 \cdot 4t^3}{8t^3} = \frac{24 - 8}{8} = 2$$

Example 4. If $x = t^4 - 16$ and $y = t^7 + 1$, find $\dfrac{d^2y}{dx^2}$. We have

$$\frac{dx}{dt} = 4t^3 \qquad \frac{dy}{dt} = 7t^6 \qquad \text{and} \qquad \frac{dy}{dx} = y' = \frac{7t^6}{4t^3} = \frac{7}{4}t^3$$

To find $\dfrac{d^2y}{dx^2}$, we have

$$\frac{dy'}{dt} = \frac{21}{4}t^2 \qquad \text{and} \qquad \frac{d^2y}{dx^2} = y'' = \frac{\frac{21}{4}t^2}{4t^3} = \frac{21}{16t}$$

To find $\dfrac{d^3y}{dx^3}$, we see that $\dfrac{dy''}{dt} = -\dfrac{21}{16t^2}$. Consequently

$$\frac{d^3y}{dx^3} = \frac{\dfrac{dy''}{dt}}{\dfrac{dx}{dt}} = \frac{-\dfrac{21}{16t^2}}{4t^3} = -\frac{21}{64t^5}$$

It is clear from Example 4 how the derivatives of higher than third order can be obtained. It is not particularly convenient to write general formulas for derivatives of higher than second order, although this could be done. It is simpler to carry out the method indicated in Example 4. It is hardly necessary to remark that we have been using letters x, y, t merely for convenience and that other letters would serve as well.

EXERCISES

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for the following functions (Ex. 1 to 7). In each case, find the cartesian equation and sketch the curve. Find the slope at the points specified, and draw the tangent line. Check your calculations by finding $\dfrac{dy}{dx}$ from the cartesian equation.

1. $\begin{cases} x = t - 2 \\ y = t^2 \end{cases}$ at point where curve crosses y axis

2. $\begin{cases} x = 2t \\ y = 16/t \end{cases}$ at $t = 4$

3. $\begin{cases} x = 4a/t^2 \\ y = 4a/t \end{cases}$ at $t = 2$

4. $\begin{cases} x = t^2 - 3 \\ y = 2t + 1 \end{cases}$ at $t = 2$

5. $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$ at $t = 1$

6. $\begin{cases} x = \dfrac{10t}{t^2 + 1} \\ y = \dfrac{5(t^2 - 1)}{t^2 + 1} \end{cases}$ at $t = 3$

7. $\begin{cases} x = 1/t^3 \\ y = t^2 \end{cases}$ at $t = \frac{1}{2}$

8. Find $\dfrac{d^3y}{dx^3}$ for the functions in Exercises 1 to 5.

9. Find $\dfrac{d^4y}{dx^4}$ if $x = t^5 - 3$, $y = 2t^6 + 8$.

10. If $x = \varphi(t)$ and $y = \psi(t)$, then $t = g(x)$ and $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$. Derive formula (12) by differentiating $\dfrac{dy}{dt}\dfrac{dt}{dx}$ with respect to x.

11. If $x = \varphi(t)$, $y = \psi(t)$, find a general formula for $\dfrac{d^3y}{dx^3}$ in terms of the derivatives of x and y with respect to t. Test your result by applying it to Exercise 3.

Summary of Formulas. We have established several *standard formulas* for finding the derivatives of functions. They are

(1) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(2) $\dfrac{d}{dx}(c) = 0$ (constant)

(3) $\dfrac{d}{dx}(u + v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$ (sum)

(4) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (product)

(5) $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$ (constant times a function)

(6) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ (quotient)

(7) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ (function of a function)

(8) $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$ (power of a function)

(9) $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$ (inverse functions)

(10) $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$ (parametric representation)

When using these formulas, we must not forget the nature of the derivative. It is important to keep fixed in mind the fact that the derivative is the limit of the quotient of the increment in y divided by the increment in x, namely, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. The formulas are direct consequences of the definition of the derivative and provide us with convenient means for evaluating this limit.

MISCELLANEOUS EXERCISES

Find the derivatives of the following functions (Ex. 1 to 25):

1. $y = 2x^3 - 4x^2 + 5x + 3 - \dfrac{8}{x}$
2. $z = 4w^3 + 5w - \dfrac{16}{w} + \dfrac{1}{w^2}$
3. $s = 3\sqrt{t} - 14\sqrt[3]{t+1}$
4. $s = 2t^{3/2} + (5t)^{3/2}$
5. $y = (x^2 + 16)(3x^2 - 7)$
6. $y = (2x^4 + 3)(10x^2 - 5x + 1)$
7. $y = \dfrac{17x + 9}{14x - 3}$
8. $y = \dfrac{12z - 1}{3z^2 + 9}$
9. $y = \sqrt{a^2 - x^2}$
10. $y = (ax^2 + 6)^{3/2}$
11. $y = (x^2 + 3x + 11)^{-1/3}$
12. $y = (ax^2 + bx + c)^{-4}$
13. $y = (ax + k)^{-3/4}$
14. $y = \sqrt[4]{ax^2 + b}$
15. $y = (a^2 - x^2)\sqrt{a^2 + x^2}$
16. $y = \dfrac{a^2 - x^2}{\sqrt{a^2 + x^2}}$
17. $s = t^2(t^2 + 4)^{3/2}$
18. $s = \dfrac{t^2}{(t^2 + 9)^{3/2}}$
19. $w = (r^2 + 1)^2(2r - 1)^3$
20. $u = (2v - 1)^4(3v + 2)^3$
21. $z = \dfrac{3t - 4}{\sqrt{t + 1}}$
22. $x = \dfrac{\sqrt{t + 1}}{t^4}$
23. $y = \dfrac{t\sqrt{2t + 3}}{4t - 1}$
24. $y = \dfrac{(x^2 + 1)\sqrt{2x - 1}}{3x + 5}$
25. $y = \dfrac{4\sqrt{x^2 + 9}}{3x}$

In Exercises 26 to 37, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$:

26. $x^2 - y^2 = a^2$
27. $y^3 - x^2 - y^2 = 0$
28. $x^4 + y^4 = a^4$
29. $x^3y^2 = a^5$

30. $xy - 2x - 1 = 0$

31. $x = 2t^3$
$y = 5t^4$

32. $x = 1 + t^2$
$y = 1 - t^2$

33. $x = \alpha^2$
$y = (\alpha - 1)^2$

34. $x = u^2$

$$y = \frac{1}{2u}$$

35. $x = 1/\beta$

$$y = \frac{1}{\beta + 1}$$

36. $$x = \frac{1 - u^2}{1 + u^2}$$

$$y = \frac{2u}{1 + u^2}$$

37. $x = \sqrt{t}$

$$y = \frac{1}{\sqrt{t^2 + 1}}$$

In Exercises 38 to 41, find $\dfrac{d^2y}{dx^2}$:

38. $x^2 - y^2 = a^2$

39. $xy = 2a^2$

40. $x = t^4 + 1$
$y = t^3$

41. $x = t^4$
$y = 1/t$

Find the slope of the curve at the points indicated (Ex. 42 to 49).

42. $2x^3 + 2y^3 - 9xy = 0$ at $(2,1)$

43. $y^2 - 2xy - 27 = 0$ at $(-3,3)$

44. $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^{2/3} = 2$ at (a,b)

45. $x^2y - y^3 + 6 = 0$ at $(1,2)$

46. $\begin{cases} x = 4 - t \\ y = t^2 \end{cases}$ at $t = 2$

47. $\begin{cases} x = 2t^2 - t - 1 \\ y = 5t + 6 \end{cases}$ at $t = -1$

48. $\begin{cases} x = \dfrac{3t}{1 + t^3} \\ y = \dfrac{3t^2}{1 + t^3} \end{cases}$ at $t = 2$

49. $\begin{cases} x = 4t \\ y = 3t - t^3 \end{cases}$ at $t = 0$

50. How fast is the slope changing at the given point in each of Exercises 42 to 49?

CHAPTER 5

SIMPLE APPLICATIONS OF THE DERIVATIVE

We have seen that the derivative of a function $y = f(x)$ can be interpreted as the slope of the tangent line to the graph of the function and as the rate of change of the function. In this chapter we shall illustrate more fully these two interpretations.

29. Tangent and Normal to a Plane Curve. Suppose we have a curve whose equation is known, for example, $y = \frac{1}{8}x^2$ (Fig. 20). Let us find the equation of the tangent line to this curve at some particular point of the curve, for instance, $P(5,\frac{25}{8})$. The slope m_1 of this tangent line is simply the value of $\frac{dy}{dx}$ for $x = 5$. We have at once $\frac{dy}{dx} = \frac{1}{4}x$, and hence $m_1 = \frac{5}{4}$. Now, we need only find the equation of a line with slope $\frac{5}{4}$ passing through $(5,\frac{25}{8})$. This gives

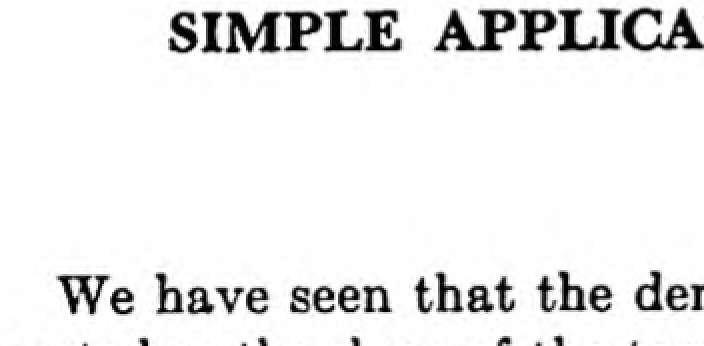


FIG. 20.

$$y - \tfrac{25}{8} = \tfrac{5}{4}(x - 5)$$

that is $$10x - 8y - 25 = 0$$

The line perpendicular to the tangent at the point of contact is called the *normal* to the curve at that point. In our example the slope of the normal to the given curve at $(5,\frac{25}{8})$ is, therefore, $-\frac{4}{5}$, and its equation is

$$y - \tfrac{25}{8} = -\tfrac{4}{5}(x - 5)$$

that is $$32x + 40y - 285 = 0$$

In general (Figs. 21,22), if $f(x,y) = 0$ is the equation of a curve and (x_1,y_1) is a point upon this curve, then the equation of the tangent at (x_1,y_1) is

$$y - y_1 = m_1(x - x_1) \tag{1}$$

where $\left.\frac{dy}{dx}\right]_{x=x_1} = m_1$ is computed from $f(x,y) = 0$. Similarly, the equation of the normal is

$$y - y_1 = -\frac{1}{m_1}(x - x_1) \tag{2}$$

Thus analytic geometry and the idea of the derivative have provided us

Acc No 12395

with a very convenient solution of the *problem of tangents* mentioned in Chap. 1.

30. Subtangent and Subnormal. Consider the curves in Figs. 21 and 22. The tangent at P_1 cuts the x axis at T_1, and the normal cuts the x axis at N_1. Draw P_1Q_1 perpendicular to the x axis. Then T_1P_1 is called the *length of the tangent*, and its projection T_1Q_1 on the x axis the length of the subtangent, or simply the *subtangent*. Similarly, P_1N_1 is called the *length of the normal*, and its projection Q_1N_1 on the x axis the length of the subnormal, or simply the *subnormal*. It is customary to regard the subtangent as positive if T_1Q_1 extends to the right of T_1 as in Fig. 21, negative if to the left of T_1 as in Fig. 22. Similarly, the subnormal is

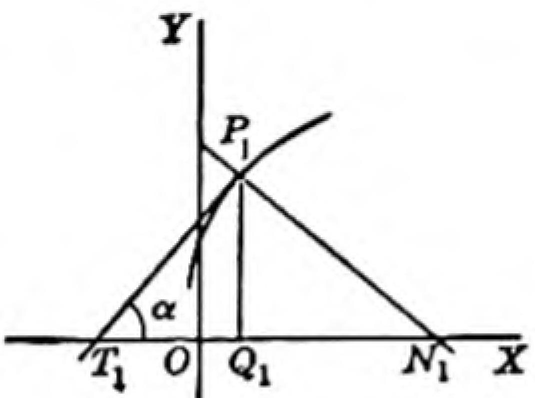


FIG. 21.

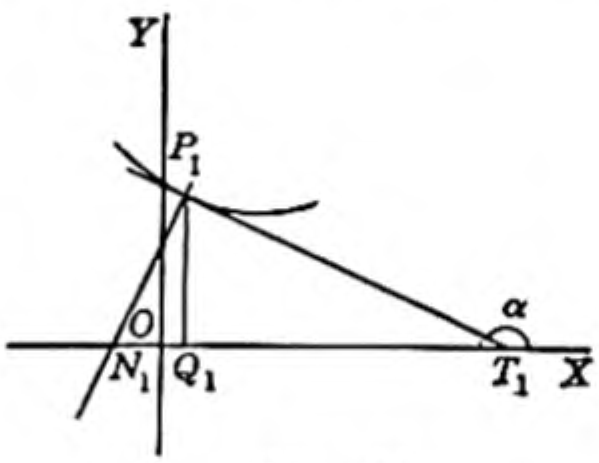


FIG. 22.

regarded as positive if Q_1N_1 extends to the right of Q_1 as in Fig. 21, negative if to the left of Q_1 as in Fig. 22. We may now calculate the length of the subtangent as follows:

From equation (1) the x intercept of the tangent is $x_1 - (y_1/m_1)$ and

therefore $$T_1Q_1 = x_1 - \left(x_1 - \frac{y_1}{m_1}\right) = \frac{y_1}{m_1}$$

The length of the subnormal can be found in the same way:

From equation (2) the x intercept of the normal is $x_1 + m_1y_1$ and

therefore $$Q_1N_1 = m_1y_1$$

In the example of the last paragraph (Fig. 20), we have

$$T_1Q_1 = \tfrac{4}{5} \cdot \tfrac{25}{8} = \tfrac{5}{2} \qquad \text{(length of subtangent)}$$
$$Q_1N_1 = \tfrac{5}{4} \cdot \tfrac{25}{8} = \tfrac{125}{32} \qquad \text{(length of subnormal)}$$

It is easy to find the length of the tangent and of the normal. We have, in the right triangle $T_1Q_1P_1$,

$$T_1P_1 = \sqrt{\overline{T_1Q_1}^2 + \overline{Q_1P_1}^2} = \sqrt{\frac{y_1^2}{m_1^2} + y_1^2} = \left|\frac{y_1}{m_1}\right| \sqrt{1 + m_1^2}$$

(length of tangent)

Similarly, in the right triangle $P_1Q_1N_1$,

$$P_1N_1 = \sqrt{\overline{Q_1N_1}^2 + \overline{Q_1P_1}^2} = \sqrt{m_1^2y_1^2 + y_1^2}$$
$$= |y_1| \sqrt{m_1^2 + 1} \qquad \text{(length of normal)}$$

In the example (Fig. 20), we have

$$T_1P_1 = \tfrac{25}{8} \cdot \tfrac{4}{5} \sqrt{\tfrac{25}{16} + 1} = \tfrac{5}{8} \sqrt{41} \qquad \text{(length of tangent)}$$
$$P_1N_1 = \tfrac{25}{8} \sqrt{\tfrac{25}{16} + 1} = \tfrac{25}{32} \sqrt{41} \qquad \text{(length of normal)}$$

The student will have observed that in developing the formulas of this and the preceding paragraphs the fundamental requirement was the value of the slope of the curve at the given point. This is simply the value of the derivative at that point. When this has been found, elementary analytic geometry is sufficient to enable us to find the equations of the tangent and normal, and from these the lengths of tangent, normal, subtangent, and subnormal are easily obtained.

31. Angle between Two Curves. The angle between two curves that intersect in a point P is defined to be either of the supplementary angles between their tangents at this point of intersection. The slopes of these tangents are given by the values of the derivatives at the point P. The angle between the tangents is then readily found. For example, let us find the angle of intersection of the two curves

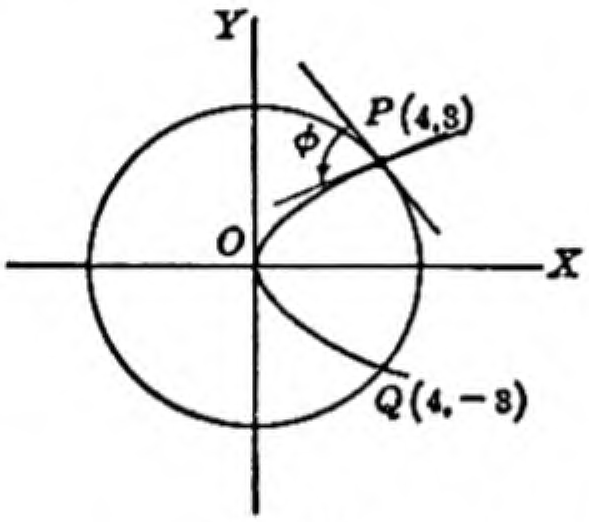


FIG. 23.

$$x^2 + y^2 = 25 \qquad \text{and} \qquad 4y^2 = 9x \qquad \text{(Fig. 23)}$$

By solving the equations simultaneously, the points of intersection are seen to be $P(4,3)$ and $Q(4,-3)$. To find the slope of the circle, we differentiate

$$x^2 + y^2 - 25 = 0$$

obtaining

$$2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

Hence $\left.\dfrac{dy}{dx}\right]_{x=4} = -\dfrac{4}{3} = m_1$, the slope of the circle at P.

To find the slope of the parabola, we differentiate

$$4y^2 - 9x = 0$$

obtaining

$$8y\frac{dy}{dx} - 9 = 0 \qquad \frac{dy}{dx} = \frac{9}{8y}$$

Hence $\left.\dfrac{dy}{dx}\right]_{x=4} = \dfrac{9}{24} = \dfrac{3}{8} = m_2$, the slope of the parabola at P. The angle φ measured in the counterclockwise (positive) direction from the circle's tangent line to the parabola's tangent line is given by

$$\tan\varphi = \frac{m_2 - m_1}{1 + m_1m_2} = \frac{\frac{3}{8} + \frac{4}{3}}{1 - \frac{3}{8}\cdot\frac{4}{3}} = \frac{9 + 32}{24 - 12} = \frac{41}{12}$$
$$= 3.4167 \text{ approximately}$$

Therefore

$$\varphi = 73°41.2' \text{ approximately}$$

Considerations of symmetry show that the angle at Q is the same. Let the student verify this by calculation.

EXERCISES

Find the equations of the tangent and normal to each of the curves at the points indicated; find the lengths of tangent, normal, subtangent, and subnormal. Make a sketch in each case (Ex. 1 to 14).

1. $y = x^2 + 4$ at $(1,5)$
2. $y^2 = 8x - 3$ at $(\frac{1}{2}, -1)$
3. $x^2 + y^2 - 4x - 21 = 0$ at $(5,4)$
4. $x^2 + y^2 = 25$ at $(3,4)$
5. $4x^2 + 9y^2 = 40$ at $(-1,2)$
6. $x^2 - 4y^2 = 5$ at $(-3,-1)$
7. $y = x^3$ at $(-2,-8)$
8. $y^2 = x^3$ at $(4,8)$
9. $xy^2 = 18$ at $(2,3)$
10. $x^2y = 4$ at $(-1,4)$
11. $y^2 = 4x$ where $y = 6$
12. $y = x^2 - 5x$ at points where $y = -4$
13. $x^2 + y^2 = 8$ at points where $\dfrac{dy}{dx} = -1$
14. $y = x^4$ at $(1,1)$

Find the equations of tangents to curves as indicated (Ex. 15 to 23).

15. To $xy = 8$ parallel to $2x + y + 9 = 0$
16. To $x^5 - 16y = 0$ perpendicular to $x + 5y = 10$
17. To $9y^2 - 4x^2 = 36$ parallel to $5x - 2y - 1 = 0$
18. To $x^2 = 20y$ making an angle of 45 deg. with the x axis
19. To $x^2 + 2y = 8$ perpendicular to $2x - 4y + 1 = 0$
20. To $x^2 + y^2 + 6x - 8y + 20 = 0$ parallel to $x - 2y - 7 = 0$
21. To $x^2 - 2xy + 2y^2 - 7x + 6y + 6 = 0$ perpendicular to $6x + 5y - 1 = 0$
22. To $x^4 - 4x^3 - 20x^2 + 104x - 4y - 85 = 0$ with slope 2
23. To $x^3 - x^2 - 2x + y - 4 = 0$ with slope 1

Find the angle between the curves as indicated (Ex. 24 to 32).

24. $x - 9y + 6 = 0$ and $x^2 + y^2 - 2x + 3y - 7 = 0$
25. $x + y - 2 = 0$ and $x^2 + y^2 - 4x + 6y - 4 = 0$
26. $x^2 + y^2 = 5$ and $y^2 = 4x + 8$
27. $y^2 = ax$ and $x^2 = ay$
28. $xy = 2a^2$ and $y^2 = 4ax$
29. $x^2 + y^2 + 2x - 4y + 4 = 0$ and $x^2 + y^2 - x - y - 8 = 0$
30. $x^2 + y^2 - 3x + 2y - 1 = 0$ and $x^2 + y^2 - 4x - y + 4 = 0$
31. $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$
32. $x^2 + y^2 - 8x = 0$ and $y^2 = \dfrac{x^3}{2 - x}$ at all points of intersection

Find the equation of the tangent to the given curve as indicated (Ex. 33 to 36).

33. To $x^2 + y^2 = 40$ and passing through the point $(11,3)$

34. To $y^2 = 16x$ and passing through $(2,6)$

35. To $x^2 - y^2 = 16$ and passing through $(-1,7)$

36. To $x^2 - y^2 = 16$ and passing through $(2,2)$. (Why is there only one such tangent?)

Determine the coefficients so that the curves fulfill the given conditions (Ex. 37 to 39).

37. $y = ax^2 + bx + c$ to be tangent to $5x - y - 3 = 0$ at $(2,7)$ and to pass through $(-1,10)$

38. $y = ax^3 + bx^2 + cx + d$ to be tangent to $8x - y + 8 = 0$ at $(-1,0)$ and to $7x - y - 26 = 0$ at $(2,-12)$

39. $y = ax^4 + bx^3 + cx^2 + dx + e$ to be tangent to $25x - y - 36 = 0$ at $(2,14)$ and to $2x - y + 1 = 0$ at $(1,3)$ and to pass through $(-1,11)$

In Exercises 40 to 45, prove that the tangent to the given curve at the point (x_1,y_1) on the curve is the indicated line.

40. Circle $x^2 + y^2 = a^2$; tangent $x_1x + y_1y = a^2$

41. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; tangent $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$

42. Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; tangent $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$

43. (*a*) Parabola $y^2 = 4ax$; tangent $y_1y = 2a(x + x_1)$
(*b*) Parabola $x^2 = 4ay$; tangent $x_1x = 2a(y + y_1)$

44. Hyperbola $xy = a^2/2$; tangent $x_1y + y_1x = a^2$

45. Any conic $ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0$; tangent
$$ax_1x + h(y_1x + x_1y) + by_1y + f(x + x_1) + g(y + y_1) + c = 0$$

In exercises 46 to 49, prove that the tangent with slope m to the given curve is the indicated line.

46. Circle $x^2 + y^2 = a^2$; tangents $y = mx \pm a\sqrt{1 + m^2}$

47. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; tangents $y = mx \pm \sqrt{a^2m^2 + b^2}$

48. Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; tangents $y = mx \pm \sqrt{a^2m^2 - b^2}$

49. (*a*) Parabola $y^2 = 4ax$; tangent $y = mx + \dfrac{a}{m}$
(*b*) Parabola $x^2 = 4ay$; tangent $y = mx - am^2$

50. Prove that the triangle formed by the coordinate axes and the tangent to the hyperbola $xy = a^2/2$ has a constant area $(= a^2)$.

51. Prove that the triangle formed by a tangent to a hyperbola and the asymptotes has a constant area. Note that Exercise 50 states a special case of this theorem.

52. Prove that the angle between the axis of a parabola and any tangent equals the angle between the tangent and a line drawn from the focus to the point of contact of the tangent.

53. Prove that the projection of any tangent upon the axis of a parabola is bisected by the vertex.

54. Prove that the tangents at the ends of the latus rectum of a parabola meet on the directrix.

55. Prove that the tangents at the ends of a latus rectum of an ellipse meet on a directrix. Prove the analogous theorem for a hyperbola.

56. Prove that the foot of the perpendicular drawn from the focus of a parabola to any tangent lies on the line drawn tangent to the parabola at its vertex.

57. Prove that the distance between the focus of a parabola and the point at which a tangent intersects the axis is equal to the distance from the focus to the point of contact. (Compare Exercise 52.)

58. Prove that the tangent at point (x_1,y_1), on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the tangent at point $x = x_1$, on the major auxiliary circle $x^2 + y^2 = a^2$, meet on the x axis. Devise a ruler and compass construction for a tangent to the ellipse. Generalize for any ellipse.

59. Prove that the sum of the intercepts of the tangent to the arc $x^{1/2} + y^{1/2} = a^{1/2}$ is constant ($= a$).

60. Find the equation of the tangent at point (x_1,y_1) on the hypocycloid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

32. Maximum and Minimum Values of a Function. Let us suppose that $y = f(x)$ is a function of x such as the one whose graph is shown in Fig. 24. As indicated, this function is continuous for all values of x in

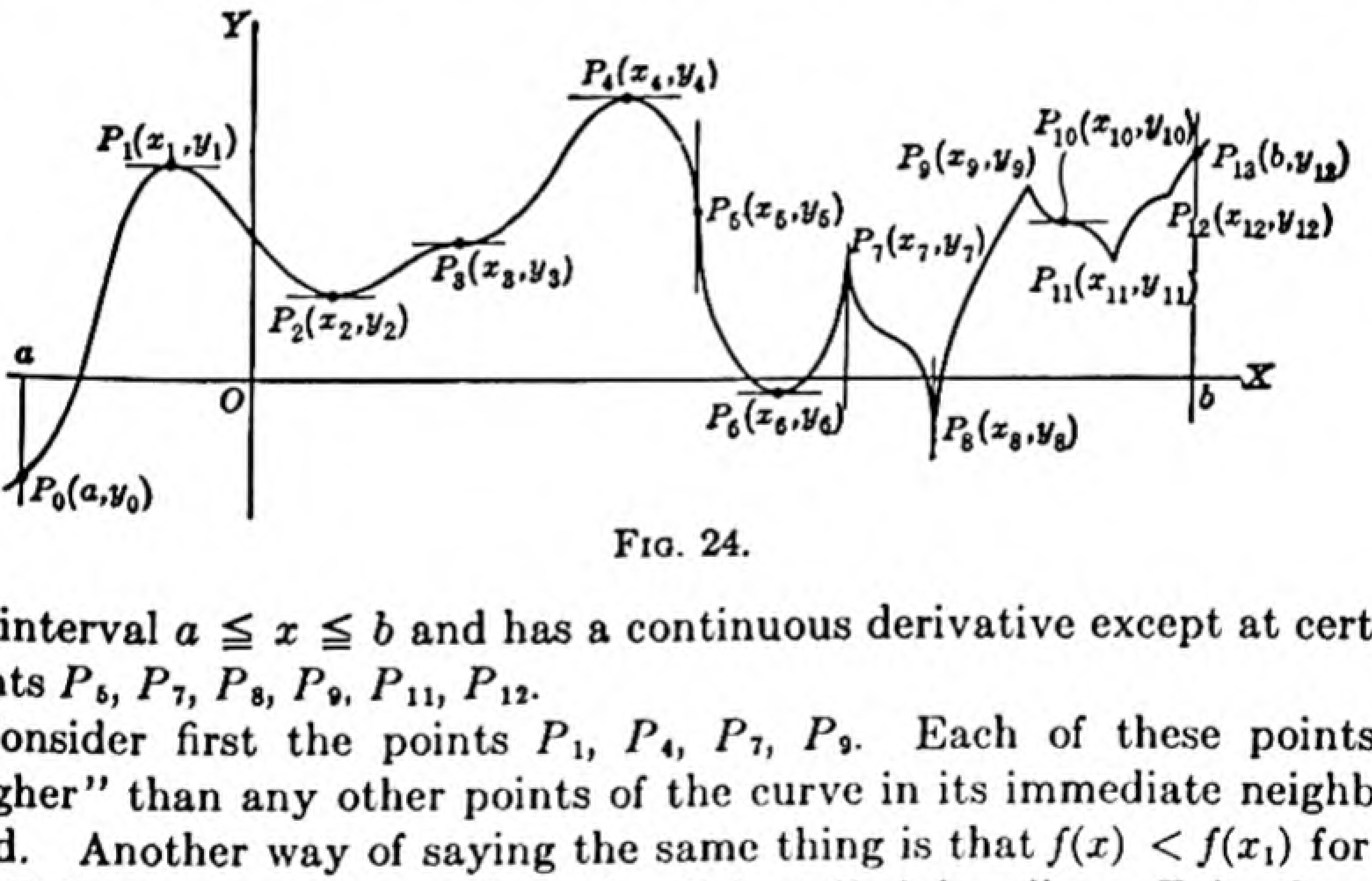


FIG. 24.

the interval $a \leqq x \leqq b$ and has a continuous derivative except at certain points P_5, P_7, P_8, P_9, P_{11}, P_{12}.

Consider first the points P_1, P_4, P_7, P_9. Each of these points is "higher" than any other points of the curve in its immediate neighborhood. Another way of saying the same thing is that $f(x) < f(x_1)$ for all x sufficiently near to x_1. Similarly, $f(x) < f(x_4)$ for all x sufficiently near to x_4, and so on for the other two points. Such points are called *maximum points* of the curve, and the corresponding ordinates $f(x_1)$, $f(x_4)$, $f(x_7)$, $f(x_9)$ are called *maximum values* of the function $f(x)$. Note that the fact that P_1 is not the highest of *all* points on the curve is irrelevant. It is the highest point within some interval of the curve. The same remark applies to P_4, P_7, P_9.

Now consider the points P_2, P_6, P_8, P_{11}. Each of these points is "lower" than any other point of the curve in its immediate neighborhood, that is to say, $f(x) > f(x_2)$ for all x sufficiently near to x_2, and so on for the

other points in question. Such points are called *minimum points*, and the ordinates $f(x_2)$, $f(x_6)$, $f(x_8)$, $f(x_{11})$ are called *minimum values* of the function. Again, note the fact that P_2 is not the lowest point on the curve but that it is nevertheless called a minimum point.

To find a test by which such points can be discovered, we recall that if $f'(x)$ is positive throughout an interval, then $f(x)$ is an increasing function throughout that interval. Similarly, if $f'(x)$ is negative throughout an interval, then $f(x)$ is decreasing throughout that interval. Consequently if $f'(x)$ is positive for all x sufficiently near to, but less than, x_1, then $f(x)$ is an *increasing* function in such an interval and $f(x) < f(x_1)$. If, next, $f'(x)$ is negative for all x sufficiently near to, but greater than, x_1, then $f(x)$ is a *decreasing* function in such an interval and again $f(x) < f(x_1)$. We shall describe such a situation by saying that, as x goes from left to right through x_1, $f'(x)$ changes sign from plus to minus, $f(x)$ changes from increasing to decreasing, and $P_1(x_1,y_1)$ is a *maximum* point of the curve.

If, similarly, as x goes from left to right through x_2, $f'(x)$ changes sign from minus to plus, $f(x)$ changes from decreasing to increasing, and $P_2(x_2,y_2)$ is a *minimum* point of the curve.

In general, we may say that the *change of sign* of $f'(x)$ determines the existence of the maximum or minimum point. Such a change of sign can occur only if $f'(x)$ passes through the value zero (P_1, P_2, P_4, P_6), becomes infinite (P_7, P_8), or is otherwise discontinuous (P_9, P_{11}) at the point in question.

On the other hand, $f'(x)$ may be zero (P_3, P_{10}), become infinite (P_5), or be otherwise discontinuous (P_{12}) at a particular point, but retain the same sign (except, of course, at this point) as x goes from left to right through the abscissa of the point in question. Such a point is, therefore, neither a maximum nor a minimum point.

In Fig. 24 we observe the geometrical interpretation of the foregoing remarks. If $f'(x) = 0$, the tangent line to the graph is horizontal as at points P_1, P_2, P_3, P_4, P_6, P_{10}. If $f'(x)$ becomes infinite, the tangent line is vertical as at points P_5, P_7, P_8. Point P_5 illustrates the case where the left- and right-hand derivatives (see page 36) both become negatively infinite at $x = x_5$. Point P_7 illustrates the case where the left-hand derivative becomes positively infinite while the right-hand derivative becomes negatively infinite at $x = x_7$. Point P_8 illustrates the case where the left- and right-hand derivatives become respectively negatively and positively infinite. Points P_9, P_{11}, and P_{12} each illustrate the case where left- and right-hand derivatives have different finite values so that $f'(x)$ has, in each case, a finite discontinuity.

We may now formulate a general procedure for finding maximum and minimum values of a function $f(x)$ that is continuous in an interval $a \leqq x \leqq b$. First locate all the points in the interval for which $f'(x) = 0$, for which $f'(x)$ has an infinite discontinuity, and for which $f'(x)$ is other-

wise discontinuous. Now consider each point in turn. If the derivative $f'(x)$ changes sign as we pass from left to right through the point, it is a *maximum* point if the change of sign is from plus to minus, a *minimum* point if the change of sign is from minus to plus. If the derivative does not change sign, the point is neither a maximum nor a minimum.

We have seen that if $f'(x)$ changes sign as x goes from left to right through the abscissa of a point, then that point is a maximum or a minimum point. The converse is not necessarily true; that is, a continuous function $f(x)$ may have a maximum or a minimum point, but $f'(x)$ may not possess a constant sign in any interval on either side of the point, so that $f'(x)$ could not "change sign." An example will be given as Exercises 61 and 62, page 137.

Points at which the derivative is zero or has an infinite discontinuity are frequently called *critical points* of the function. Corresponding values of the function are called *critical values* of the function.

It should be remarked that, in case it is desired to find the greatest and smallest values of $f(x)$ in the interval $a \leqq x \leqq b$, we must find all the maximum and minimum points within the interval, then calculate $f(a)$ and $f(b)$. Of all these various values, pick out the greatest and the least. These are often called the *absolute maximum* and *absolute minimum* in the interval, the other values being called *relative* maxima and minima. In Fig. 24, $f(a)$ is the absolute minimum, and $f(x_4)$ the absolute maximum in the interval $a \leqq x \leqq b$.

The student should realize that the rules for finding maximum and minimum points have been justified largely through reliance upon geometrical arguments and "geometrical intuition." Proofs of this nature should be regarded not as rigorous demonstrations, but rather as indications of the lines along which strict analytical proofs can be constructed. Such proofs involve considerable detail and are, as a general rule, better deferred to a more advanced course.

Example 1. Locate any maximum and minimum points of the curve

$$y = x^3 + 3x^2 - 9x - 22$$

We have at once

$$\frac{dy}{dx} = 3x^2 + 6x - 9 = 3(x + 3)(x - 1)$$

Since the derivative is a polynomial, it is continuous for all values of x (see Exercise 9, page 27). We therefore need only find values of x for which $\frac{dy}{dx}$ is zero and investigate each. We have $3(x + 3)(x - 1) = 0$ if $x = -3$ and if $x = 1$.

1. Consider the point where $x = -3$. The corresponding y is 5. Take a value of x close to but less than -3. Evidently for such an x (and, in fact, for all $x < -3$) $x + 3 < 0$ and $x - 1 < 0$; hence $\frac{dy}{dx}$ is positive. For x close to but greater than -3

(and, in fact, for all $-3 < x < 1$), $x + 3 > 0$ and $x - 1 < 0$; hence $\dfrac{dy}{dx}$ is negative. Therefore, the derivative changes sign from plus to minus, and $P_1(-3,5)$ is a *maximum* point.

2. Next consider the point where $x = 1$. The corresponding y is -27. For x close to but less than 1, $\dfrac{dy}{dx} < 0$; for x close to but greater than 1 (in fact, for all $x > 1$), $\dfrac{dy}{dx} > 0$. Since the derivative changes sign from minus to plus, $P_2(1, -27)$ is a *minimum* point. The graph of this function is shown in Fig. 25.

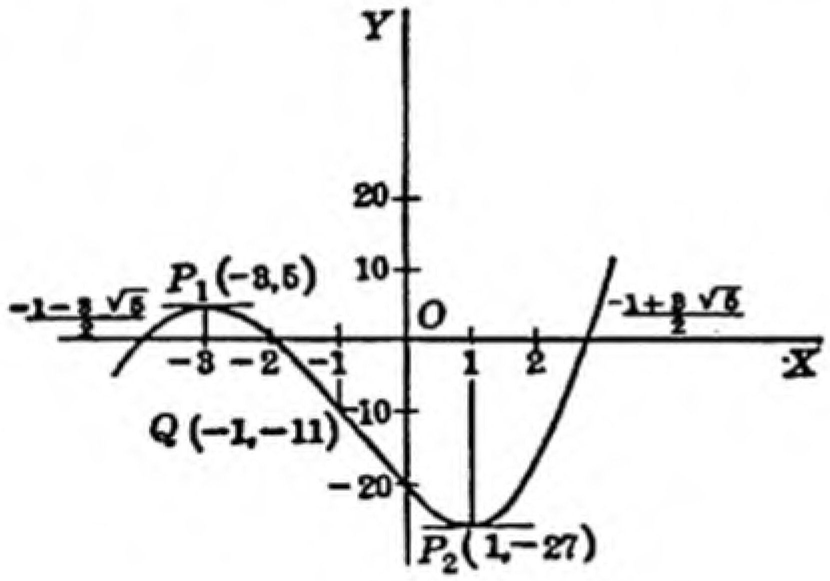


FIG. 25.

Example 2. Locate any maximum and minimum points of the curve

$$y^3 = (x - 1)^2(x + 4)$$

Solving for y, we have

$$y = (x - 1)^{2/3}(x + 4)^{1/3}$$

Differentiating,

$$\frac{dy}{dx} = \frac{1}{3}(x - 1)^{2/3}(x + 4)^{-2/3} + \frac{2}{3}(x - 1)^{-1/3}(x + 4)^{1/3}$$

$$= \frac{x - 1 + 2(x + 4)}{3(x - 1)^{1/3}(x + 4)^{2/3}} = \frac{3x + 7}{3(x - 1)^{1/3}(x + 4)^{2/3}}$$

We observe that the derivative equals zero for $x = -\frac{7}{3}$ and that it becomes infinite for $x = 1$ and $x = -4$. Now consider the point for which $x = -\frac{7}{3}$ with corresponding $y = \frac{1}{3}(500)^{1/3} = 2.65$ approximately. The tangent is horizontal at this point. If x is any value other than -4, we note that $(x + 4)^{2/3}$ is positive. If x is slightly less than $-\frac{7}{3}$, then $\dfrac{dy}{dx} > 0$. If x is slightly greater than $-\frac{7}{3}$, then $\dfrac{dy}{dx} < 0$. Since $\dfrac{dy}{dx}$ changes sign from plus to minus, the point $x = -\frac{7}{3}$ is a maximum.

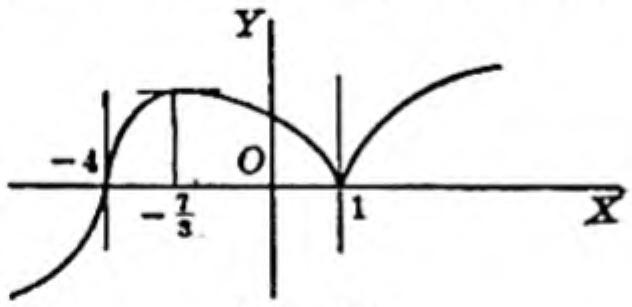


FIG. 26.

We next investigate the two points at which the tangent is vertical. If $x < -4$, $\dfrac{dy}{dx} > 0$. If x is slightly greater than -4, $\dfrac{dy}{dx}$ remains positive. Since the derivative does not change sign, the point $(-4,0)$ is neither a maximum nor a minimum. Finally, we note that, for x slightly less than 1, $\dfrac{dy}{dx}$ is negative. For $x > 1$, $\dfrac{dy}{dx}$ is positive.

Consequently, the derivative changes sign from minus to plus, and the point (1,0) is a minimum. The graph of the function is shown in Fig. 26.

Example 3. Find any maximum and minimum points of the curve $y = (x - 1)^3$. To locate the critical points, we have $\dfrac{dy}{dx} = 3(x - 1)^2$ which is zero for $x = 1$. Hence the point (1,0) is the only point at which the tangent is horizontal. Since the derivative is three times the square of $(x - 1)$, it is positive for all values of x other than $x = 1$ and therefore does not change sign at $x = 1$. Consequently, this point is neither a maximum nor a minimum. The graph is shown in Fig. 27.

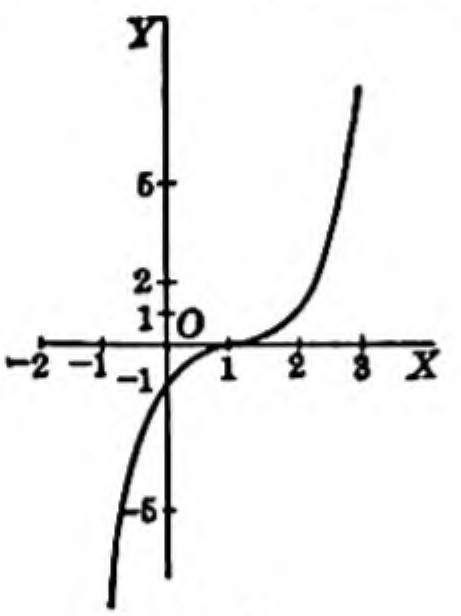


FIG. 27.

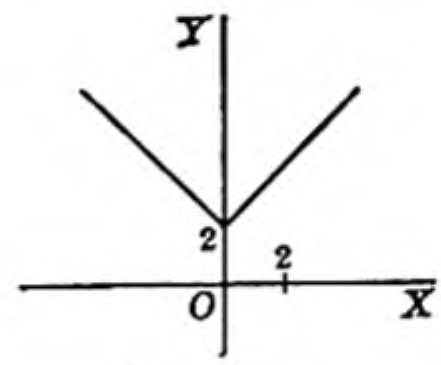


FIG. 28.

Example 4. Consider the function $y = 2 + \sqrt{x^2}$ where we agree always to take the *positive* square root of x^2. We have $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2}}$ which is equal to -1 for all negative x and to $+1$ for all positive x but is undefined (and hence discontinuous) at $x = 0$. The point (0,2) is a minimum point, since the derivative changes sign from minus to plus and has a simple discontinuity at this point. The graph is shown in Fig. 28.

EXERCISES

Find any maximum and minimum points, and make a rough sketch of the curve (Ex. 1 to 28):

1. $y = 2x - x^2$
2. $y = x^2 + 4x + 1$
3. $y = (x + 1)^2$
4. $y = 3x^2 - x^3$
5. $y = 2x^3 + 3x^2 + 4$
6. $y = (4x - 1)^3$
7. $y = x^3 - 3x^5$
8. $y = a^2 - x^2$
9. $y = (a^2 - x^2)^2$
10. $y = x^4 - 8x^2$
11. $y = x^3 - 6x^2 + 9x - 3$
12. $y = 9 + 2x^2 - x^3$
13. $y = 1 + 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3$
14. $y = x^2(3 - x)^2$
15. $y = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + 1$
16. $y = \frac{1}{4}x^4 - x^3 + x^2$
17. $y = \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 - 6x + 1$
18. $y = 2 - (x - 1)^{2/3}$
19. $y = x^{2/3}$
20. $y = x^{4/5}$
21. $y = 4 - (x + 3)^{1/3}$
22. $y = (2x - 3)^{1/3}(x - 3)^{2/3}$
23. $y = x^{2/3} - x^{1/3}$
24. $\nu = x^{2/3}(x - 1)^2$
25. $y = (x - 2)^2(x + 2)^3$
26. $y = x(x - 2)^2(x + 2)^3$
27. $y = \dfrac{x^2 - 4x + 5}{x - 2}$
28. $y = \dfrac{4}{x} + \dfrac{1}{2 - x}$

Determine any maximum and minimum values of the following functions (Ex. 29 to 36):

29. $x^3 - 6x^2 + 9x + 3$
30. $x^4 - 81$
31. $4t^3 - 3t^4$
32. $t(t^2 - 25)^2$

33. $z^3 - 3z^2 + 3z + 6$

34. $3z^4 - 4z^3 + 6z^2 - 12z + 10$

35. $w(2w - 1)^3$

36. $\dfrac{81}{w^2 + 9}$

Find the absolute maximum and minimum of the given function in the interval specified (Ex. 37 to 40):

37. $f(x) = 2x^3 + 3x^2 - 12x + 4 \qquad -3 \leqq x \leqq 3$

38. $f(u) = 9 + 16u - 4u^2 \qquad 0 \leqq u \leqq 3$

39. $f(t) = \dfrac{t^2}{t^2 + 16} \qquad -\infty < t < \infty$

40. $f(r) = 4 + (r + 3)^{3/4} \qquad -4 \leqq r \leqq 0$

33. Concavity; Points of Inflection. We now have a clear picture of the geometrical significance of the sign of the first derivative of a function. What can we say about the meaning of the sign of the second derivative?

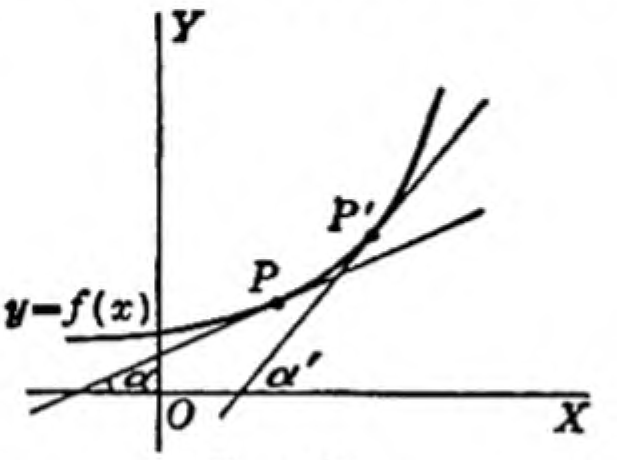


FIG. 29.

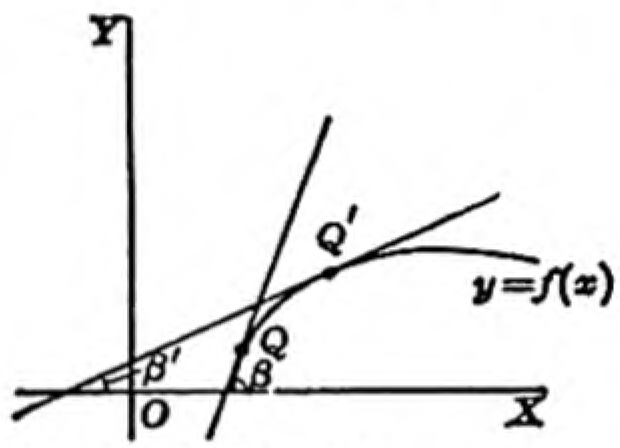


FIG. 30.

Since $\dfrac{d^2y}{dx^2} = f''(x)$ is simply the derivative of $\dfrac{dy}{dx} = f'(x)$, we may say at once that, if $\dfrac{d^2y}{dx^2}$ is *positive*, $f'(x)$ is *increasing;* if $\dfrac{d^2y}{dx^2}$ is *negative*, $f'(x)$ is *decreasing*. Now, to say that $f'(x)$ is an increasing function (as x increases) is to say that as a point P (Fig. 29) moves from left to right along some curve $y = f(x)$ the slope of the curve increases. That is to say, $\tan \alpha$ increases, and therefore α increases. In other words, the tangent turns in a counterclockwise direction as P moves along the curve. Under such circumstances, we say that the curve is *concave upward.*

Similarly, to say that $f'(x)$ is a decreasing function is to say that as point Q (Fig. 30) moves from left to right along the curve $y = f(x)$ the slope of the curve decreases. That is to say, $\tan \beta$ decreases, and therefore β decreases. In other words, the tangent turns in a clockwise direction as Q moves along the curve. Under such circumstances, that is, if $\dfrac{d^2y}{dx^2} < 0$, we say that the curve is *concave downward.*

Now consider the portion of the curve $y = f(x)$ of Fig. 24 which has been redrawn in Fig. 31. Evidently, in the vicinity of the point P_1, the curve is concave downward, whereas in the vicinity of the point P_2 it is concave upward. Therefore, at some point Q_1 between P_1 and P_2 the curve changes from concave downward to concave upward. This will

happen if the second derivative $f''(x)$ changes sign from minus to plus. Such a point is called a *point of inflection*. Such a change of sign can occur only if $f''(x)$ passes through the value zero (Q_1, Q_2, P_3, Q_3, Q_4, Q_5), becomes infinite (P_5), or is otherwise discontinuous at the point in question.

On the other hand, $f''(x)$ may be zero (Example 4, below), become infinite (P_7, Fig. 31), or be otherwise discontinuous at a particular point but retain the same sign (except, of course, at this point) as x goes from left to right through the abscissa of the point in question. Such a point is, therefore, not a point of inflection.

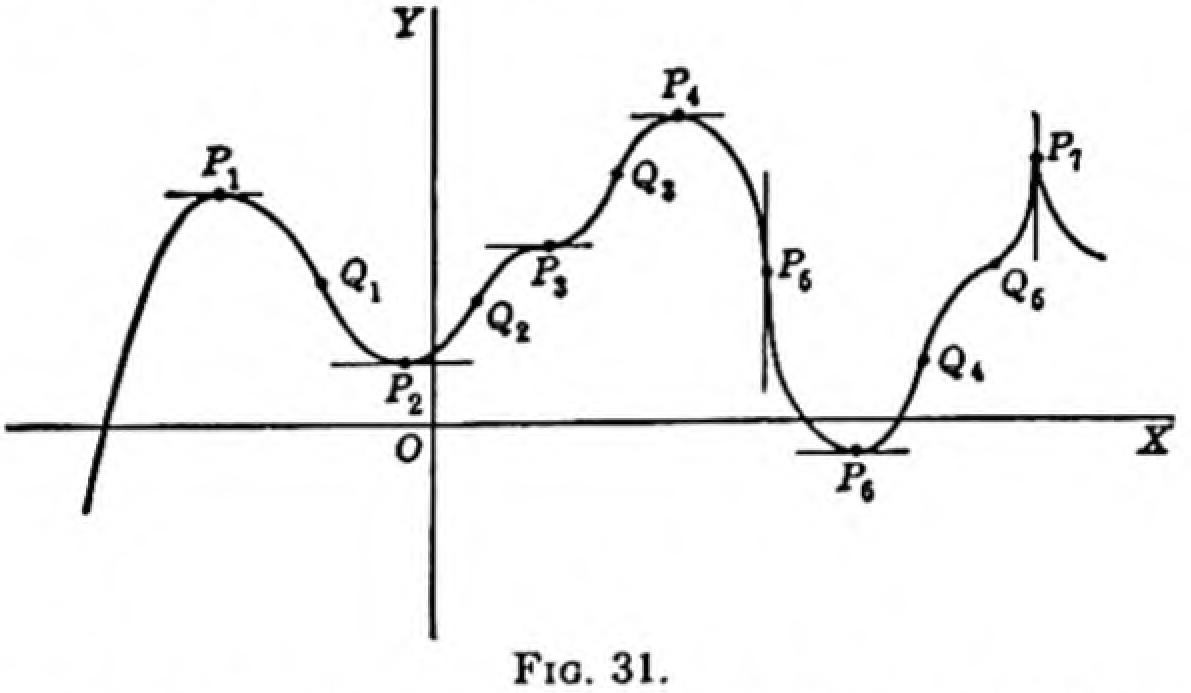


FIG. 31.

34. Second Test for Maximum and Minimum Points. If the first derivative $f'(x)$ of a function $f(x)$ is zero and the second derivative $f''(x)$ is *negative* at a point $x = x_1$, then that point is a *maximum* point. For if $f''(x_1) < 0$, then $f'(x)$ is decreasing in the neighborhood of $x = x_1$; and since $f'(x_1) = 0$, it must change sign from plus to minus. Geometrically, this means that at the point $P_1(x_1,y_1)$ the curve has a horizontal tangent and is concave downward (Fig. 31). Similarly, if $f'(x)$ is zero and $f''(x)$ *positive* at a point $x = x_2$, then that point is a *minimum* point. If both $f'(x)$ and $f''(x)$ are zero at a point, further investigation is required to determine whether the point is a maximum, a minimum, or neither (see Example 4 below).

Attention should again be called to the fact that we are relying largely upon geometrical intuition in reaching the conclusions of the last two sections. However, rigorous analytical proofs can be supplied but are best omitted here (see Chap. 11).

Example 1. Find any points of inflection in the graph of

$$y = x^3 + 3x^2 - 9x - 22$$

(Fig. 25). We have

$$\frac{dy}{dx} = 3x^2 + 6x - 9 = 3(x + 3)(x - 1)$$

$$\frac{d^2y}{dx^2} = 6x + 6 = 6(x + 1)$$

Here $\frac{d^2y}{dx^2}$ is zero if $x = -1$; furthermore, if $x < -1$, $\frac{d^2y}{dx^2} < 0$; and if $x > -1$, $\frac{d^2y}{dx^2} > 0$. Hence, the curve changes from concave downward to concave upward at $x = -1$, and therefore $Q(-1, -11)$ is a point of inflection. Furthermore, it is the only point of inflection; everywhere to the left the curve is concave downward, and everywhere to the right it is concave upward.

Let us also apply our new test for maximum and minimum points. We have $\frac{dy}{dx}$ zero for $x = -3$ and $x = 1$. If $x = -3$, then $\frac{d^2y}{dx^2}$ assumes the value -12. Hence, $P_1(-3,5)$ is a maximum. If $x = 1$, then $\frac{d^2y}{dx^2}$ assumes the value 12, and $P_2(1, -27)$ is a minimum (compare with page 74).

Example 2. Find any points of inflection if $y^3 = (x - 1)^2(x + 4)$ (Fig. 26). We already have

$$\frac{dy}{dx} = \frac{3x + 7}{3(x - 1)^{1/3}(x + 4)^{2/3}}$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{3(x - 1)^{1/3}(x + 4)^{2/3} - (3x + 7)[\frac{2}{3}(x - 1)^{1/3}(x + 4)^{-1/3} + \frac{1}{3}(x - 1)^{-2/3}(x + 4)^{2/3}]}{3(x - 1)^{2/3}(x + 4)^{4/3}}$$

$$= -\frac{50}{9(x - 1)^{4/3}(x + 4)^{5/3}}$$

We see at once that $\frac{d^2y}{dx^2}$ is never zero. However, it has infinite discontinuities for $x = -4$ and $x = 1$. Now the factor $(x - 1)^{4/3}$ is always positive (or zero at $x = 1$). But for all $x < -4$ the factor $(x + 4)^{5/3}$ is negative, and therefore $\frac{d^2y}{dx^2} > 0$. Consequently, the curve is concave upward for all $x < -4$. For all $x > -4$, the factor $(x + 4)^{5/3}$ is positive, and therefore $\frac{d^2y}{dx^2} < 0$. Consequently, the curve is concave downward for all $x > -4$. Hence, the point $(-4,0)$ is a point of inflection (with vertical tangent) and is, in fact, the only point of inflection. Although $\frac{d^2y}{dx^2}$ is infinite at the point $(1,0)$, this is not a point of inflection since $\frac{d^2y}{dx^2}$ does not change sign.

Example 3. Find any points of inflection on the curve (Fig. 27) $y = (x - 1)^3$. We have $\frac{dy}{dx} = 3(x - 1)^2$ and $\frac{d^2y}{dx^2} = 6(x - 1)$. Both the first and second derivatives vanish at $x = 1$. Furthermore, if $x < 1$, $\frac{d^2y}{dx^2} < 0$ and the curve is concave downward; if $x > 1$, $\frac{d^2y}{dx^2} > 0$ and the curve is concave upward. Hence, the point $(1,0)$ is a point (and the only point) of inflection. Note that the tangent is horizontal at this point.

Example 4. Find any maximum, minimum, and inflection points on the curve $y = x^4$. Here $\frac{dy}{dx} = 4x^3$ and $\frac{d^2y}{dx^2} = 12x^2$ are both zero for $x = 0$. Since $\frac{d^2y}{dx^2}$ remains always positive (except when $x = 0$), the point $(0,0)$ is not a point of inflection. Since $\frac{d^2y}{dx^2}$ is zero at this point, our second test for a maximum or minimum fails. But if

$x < 0$, $\frac{dy}{dx} < 0$; if $x > 0$, $\frac{dy}{dx} > 0$. Hence, the first derivative changes sign from minus to plus, and therefore (0,0) is a minimum point. Thus it is clear that the vanishing of the second derivative does not assure us of the presence of a point of inflection. Also, although our second test for a minimum fails, we are able to test definitely by investigating the sign of the first derivative. The graph of $y = x^4$ is shown in Fig. 32. Now $\frac{d^2y}{dx^2}$ is the rate of change of $\frac{dy}{dx}$; and since it is positive except at $x = 0$, $\frac{dy}{dx}$ is an increasing function for all x. But in the immediate neighborhood of $x = 0$ the rate of increase is small. Hence, the tangent turns slowly as we pass from left to right through (0,0), and therefore the curve is "flatter" at this point than is, for example, the curve $y = x^2$.

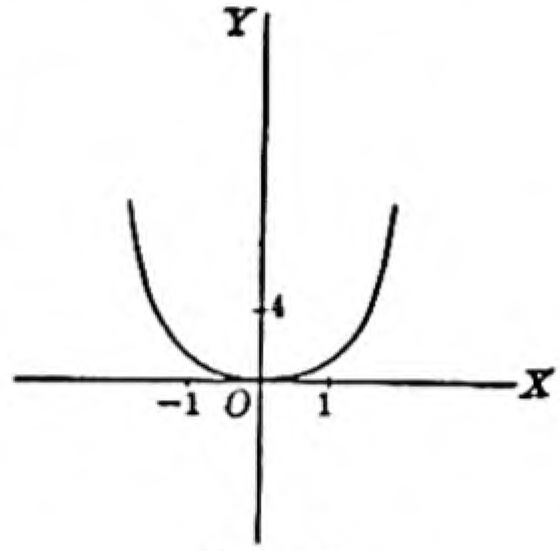


FIG. 32.

35. Simple Curve Tracing. It is often of considerable interest and importance to draw the graph of a given function. In his study of analytic geometry the student has seen how to make a sketch of the curve representing a given equation. It was usually possible to secure very easily such data as the coordinates of the points at which the curve cut the axes (*intercepts*), its possession or lack of *symmetry* with respect to axes or origin, the *sign* of y corresponding to various values of x, values of x for which y is real (and hence which correspond to points on the curve), and horizontal and vertical *asymptotes*. This information was sufficient to indicate the general appearance of the curve. We are now in a position to obtain with comparative ease a more detailed idea of the graph of a given equation; for we may locate maximum, minimum, and inflection points. The general procedure is illustrated in the following example:

Example. Trace the curve $y = \frac{1}{6}x^3 - x^2$ (Fig. 33). Upon writing this equation in the form $y = x^2(\frac{1}{6}x - 1)$, it is clear that the x intercepts are 0 and 6, the y intercept 0. Evidently, y is real for all x, negative for $x < 6$ (except at $x = 0$, where $y = 0$), and positive for all $x > 6$. When $x \to -\infty$, we have $y \to -\infty$; for $x \to +\infty$, $y \to +\infty$ Hence, there are no horizontal asymptotes. Since y is a polynomial in x, it is continuous for all x, and there are no vertical asymptotes. Next we note that

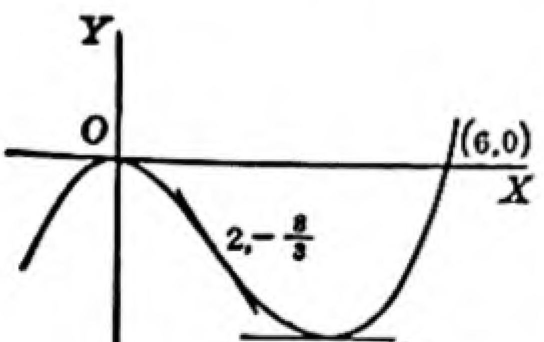


FIG. 33.

$$\frac{dy}{dx} = \frac{1}{2}x^2 - 2x = x\left(\frac{1}{2}x - 2\right) \quad \text{and} \quad \frac{d^2y}{dx^2} = x - 2$$

Since, for $x = 0$, $\frac{dy}{dx}$ is zero and $\frac{d^2y}{dx^2}$ is negative ($= -2$), the point (0,0) is a maximum. Since, for $x = 4$, $\frac{dy}{dx}$ is zero and $\frac{d^2y}{dx^2}$ positive ($= 2$), the point $(4, -\frac{16}{3})$ is a minimum

point. If $x = 2$, then $\dfrac{d^2y}{dx^2}$ is zero and changes sign from minus to plus; hence, the point $(2, -\frac{8}{3})$ is a point of inflection. Evidently, there are no other maximum, minimum, or inflection points. It will be helpful in drawing the curve to sketch the tangent at the point of inflection. Its slope is $\left.\dfrac{dy}{dx}\right]_{x=2} = -2$, and it is easily drawn. Since the curve changes from concave downward to concave upward at the inflection point, it crosses the tangent line and is quite "close" to it in the neighborhood of the point of inflection. We are thus able to draw the graph with a considerable degree of accuracy by plotting only four points and sketching one line.

EXERCISES

Find any maximum, minimum, and inflection points, and sketch the curve (Ex. 1 to 24).

1. $y = x^3 - 3x^2 - 9x + 5$
2. $y = 2x^3 - 9x^2 + 12x - 3$
3. $y = 2x^3 - 3x^2 - 12x + 12$
4. $y = \frac{1}{8}x^4 - 2x^2$
5. $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 2x^2 - 4x + 1$
6. $y = 12x - x^3$
7. $y = 3 - 15x + 9x^2 - x^3$
8. $y = \frac{1}{5}x^5 - \frac{7}{4}x^4 + 2x^3$
9. $y = \dfrac{a^2x}{a^2 + x^2}$
10. $y = \dfrac{8a^3}{x^2 + 4a^2}$
11. $b^2x^2 + x^2y^2 = a^2y^2$
12. $b^2x^2 - x^2y^2 = a^2y^2$
13. $y = x + \dfrac{a^2}{x}$
14. $y = \dfrac{a^3}{x^2} + \dfrac{a^2}{x}$
15. $y = b + (x - a)^{1/3}$
16. $y = b + (x - a)^{2/3}$
17. $y = b + (x - a)^{4/3}$
18. $y = b + (x - a)^{5/3}$
19. $y = (x - 2)(x + 1)^{1/3}$
20. $y = (x - 1)^2(x + 3)^{2/3}$
21. $y = (x - a)^{1/5} + b$
22. $y = (x - a)^{7/5} + b$
23. $y = (x - a)^{3/5} + b$
24. $y = (x - a)^{9/5} + b$

25. Investigate the curves $y = x^n$ for $n = 2, 3, 4, 5, 6$ for possible maximum, minimum, and inflection points. Compare these curves for "flatness" at $x = 0$.

26. Same as Exercise 25 for $n = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}$

27. Investigate for maximum, minimum, and inflection points and horizontal and vertical asymptotes, and sketch the curve $y = \dfrac{x^2 + 1}{x^2 - 9}$.

28. Same as Exercise 27 for $y = \dfrac{x^2 - 1}{x^2 + 9}$

29. Given $y = f(x)$. Suppose $f''(x)$ continuous at $x = x_0$, $f''(x_0) = 0$, whereas $f'''(x_0) = k \neq 0$. Prove that $x = x_0$ is a point of inflection. What can you say about concavity of the curve in the neighborhood of $x = x_0$ if $k > 0$? If $k < 0$?

30. Does the parabola $y = ax^2 + bx + c$ have a point of inflection?

31. Can any conic have a point of inflection?

In Exercises 32 to 40, determine the coefficients so that the given conditions will be satisfied.

32. $y = ax^3 + bx^2 + cx + d$ is to have a maximum at $(-1,10)$ and an inflection at $(1,-6)$.

33. $y = ax^3 + bx^2 + cx + d$ is to have a critical point at $(2,-18)$ and an inflection at $x = \frac{1}{2}$ and is to pass through $(-1,9)$.

34. $y = ax^3 + bx^2 + cx + d$ is to be tangent to $9x - y + 5 = 0$ at $(-1,-4)$ and is to have a critical point at $x = 2$ and an inflection at $x = 1$.

35. $y = ax^4 + bx^3 + cx^2 + dx + e$ is to pass through (1,7) and is to have critical points at $(-2,16)$, $x = 2$, and $x = 0$.

36. $y = ax^4 + bx^3 + cx^2 + dx + e$ is to pass through $(-2,28)$ and is to have a critical point at $(2,-4)$ and inflections at $x = 0$ and $x = 1$.

37. $y = ax^3 + bx^2 + cx + d$ is to have a critical point at $x = 2$ and an inflection with slope $-\frac{27}{2}$ at $(\frac{1}{2}, -\frac{1}{2})$.

38. $y = ax^3 + bx^2 + cx + d$ is to have a critical point at $x = 3$ and an inflection at $(\frac{5}{2}, -\frac{25}{2})$ at which the tangent is $6x + 4y + 35 = 0$.

39. $y = ax^4 + bx^3 + cx^2 + dx + e$ is to have critical points at (1,3) and $(-1,3)$ and is to pass through (2,12).

40. $y = ax^4 + bx^3 + cx^2 + dx + e$ is to have slope -24 at $x = 3$ and inflections at (2,4) and $(-2,-44)$.

36. Derived Curves. If $y = f(x)$ is a function of x, the successive derivatives $f'(x)$, $f''(x)$, $f'''(x)$, . . . are also functions of x, and we may draw their graphs. These graphs are called the first, second, third, . . . *derived curves.* A particularly instructive scheme for exhibiting these curves is to draw them using the same set of coordinate axes. Since confusion would result from having so many curves so close together, it is more convenient to draw as many separate x axes as needed, one below the other, all marked with the same scale. The y, y', y'', y''', . . . axes, with the same or different scales, are then laid off successively along the same vertical line. The process is best made clear by an illustration. Consider the function $y = \frac{1}{6}x^3 - x^2$ of Art. 35 (Fig. 34). Observe that facts previously noted are clearly shown in Fig. 34. Thus, to measure the *slope* of y, we need only measure the ordinate of y'; whenever the y curve is rising (y an increasing function), the y' curve has a positive ordinate; whenever the y curve is falling, the y' curve has a negative ordinate. When the y curve has a horizontal tangent, the y' curve crosses the x axis—from above to below if y has a maximum, from below to above if y has a minimum. Now the slope of the y' curve is y''. Hence, when y is concave downward, the y' curve is falling and thus has negative slope, and the y'' curve has a negative ordinate; when y is concave upward, the y' curve is rising, and the y'' curve has a positive ordinate; when y has a point of inflection at which $y'' = 0$, the y' curve has a minimum if y changes from concave downward to concave upward and the y'' curve crosses the x axis from below to above. Since y''' is the

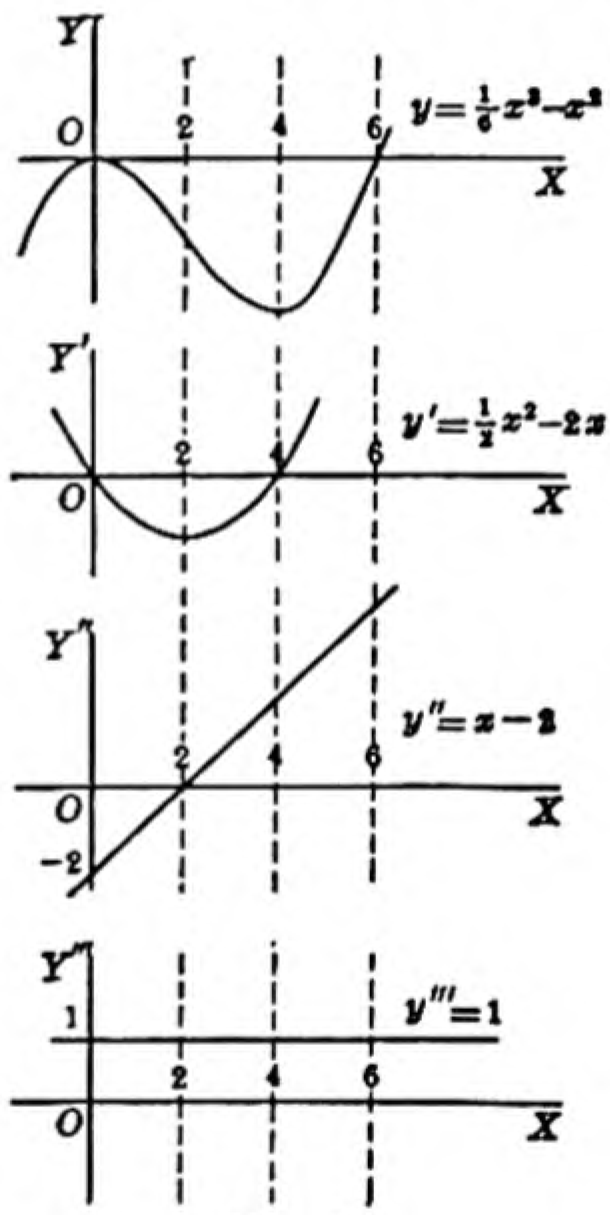


FIG. 34.

slope of the y'' curve, if y'' is represented by a straight line (as in this example), the third derived curve is a horizontal straight line. Since $y^{(4)}$ is the slope of y''', the fourth derived curve in our example is the line $y^{(4)} = 0$ (the x axis). Each succeeding derived curve is then represented by the x axis.

It sometimes happens in making various applications of mathematics to physics, engineering, economics, or other subjects that a function is defined by means of a graph whose equation is not easily obtained. It is still possible to construct with a fair degree of accuracy the first derived curve. From this the second derived curve can be constructed, and so on. The method is best indicated by an example. Suppose the function is the one shown in Fig. 35. To draw the first derived curve, first locate critical points A, B, C, D where the tangent is horizontal. The y' curve will cross or touch the x axis at the corresponding points. Estimate the location of points of inflection E, F, C, G. At each one, sketch a tangent line, and measure its slope. This slope will provide the value of the ordinate of the y' curve at the corresponding point. In case other points are needed to locate the y' curve more accurately, choose any point on y (such as H), sketch the tangent, measure the slope, and plot H'. Now draw a smooth curve through the plotted points. The second derived curve can now be obtained, for it is merely the first derived curve of y'. Further derived curves can be attempted if required. Evidently, since the first derived curve found in this way is at best only an approximation to the true curve, the second, third, and other derived curves may become very seriously inaccurate.

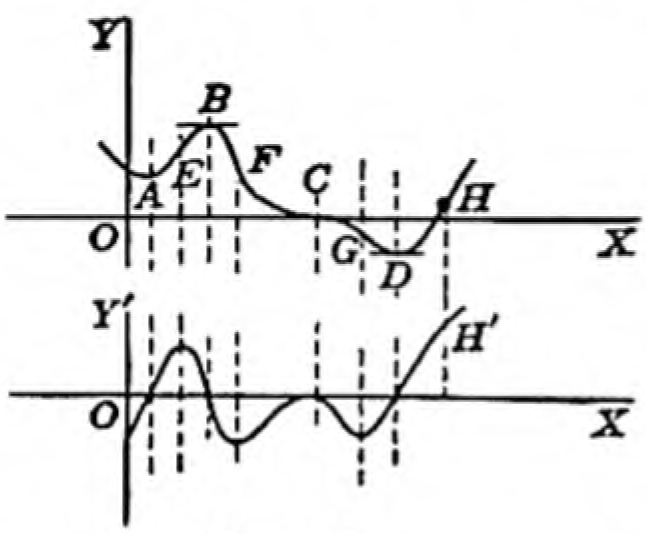


FIG. 35.

EXERCISES

Find y', y'', y''', and draw the original and first three derived curves for the following functions (Ex. 1 to 12):

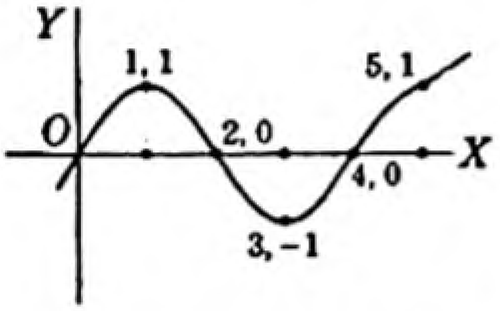
1. $y = \frac{1}{4}x^2$
2. $y = \frac{1}{5}x^3$
3. (a) $y = x^2 + 6x$
   (b) $y = x^2 + 6x + 9$
   (c) $y = x^2 + 6x + 13$
4. $y = 8x - x^2$
5. $y = \frac{1}{3}x^3 - x^2 - 3x + 2$
6. $y = 2x^3 - 3x^2 - 12x$
7. $y = 2x^3 - 9x^2 + 12x - 3$ (see Exercise 2, page 80)
8. $y = \dfrac{4x}{4 + x^2}$ (see Exercise 9, page 80)
9. $y = \dfrac{8}{x^2 + 4}$ (see Exercise 10, page 80)
10. $y = 12x - x^3$ (see Exercise 6, page 80)
11. $y = x^4 - 4x^2$
12. $y = 4x^2 - x^4$

Plot the following curves (Ex. 13 to 18), and construct the first and second derived curves by the method indicated in Art. 36. Compare with the curves plotted for y' and y''.

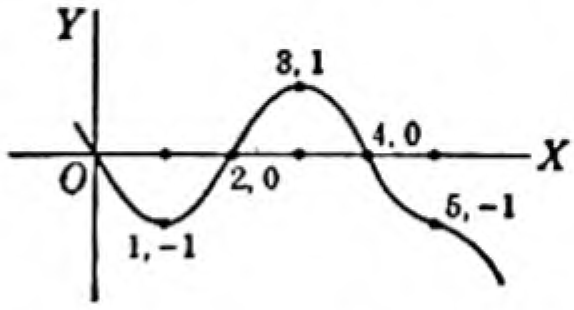
13. $y = \frac{1}{4}x^2$ (Exercise 1)
14. $y = \frac{1}{8}x^3$ (Exercise 2)
15. $y = x^2 + 6x + 9$ (Exercise 3b)
16. $y = 8x - x^2$ (Exercise 4)
17. $y = 12x - x^3$ (Exercise 10)
18. $y = x^4 - 4x^2$ (Exercise 11)

Given the following curves, in each case construct the first and second derived curves. Maximum, minimum, and inflection points are marked in each figure.

19.



20.



21. Plot the curve $y = \sin x$. Using the method of Art. 36, construct the first and second derived curves. Can you guess what functions these curves represent?

22. Draw a smooth curve through the points as given in the table; then construct the first derived curve.

x	-4	-2	0	2	4	6	8	10	12	14	16	18	20
y	0	2	$\frac{5}{2}$	0	-3	-3	-2	1	$\frac{9}{2}$	6	$\frac{9}{2}$	3	0

23. Same as Exercise 22 for

x	-5	-3	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
y	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	0	1	4	1	0	$-\frac{3}{4}$	-1	$-\frac{5}{4}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	2

24. Same as Exercise 22 for

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	10
y	1	3	4	$4\frac{1}{2}$	10	$9\frac{3}{4}$	4	3	0	3	$4\frac{1}{2}$	9

25. Same as Exercise 22 for

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
y	5	2.5	1	0	1	2.5	4	4.8	5	4.8	3.7	2.3	0	-1.4	-2	-1.4	0	2

37. Applications of Maxima and Minima. We may use our methods for locating maximum and minimum points to solve many problems of theoretical interest and practical importance. We frequently have to deal with quantities one of which is a function of another, and we may

exhibit their relationship by a graph. When the equation of the graph is available, the properties—such as maximum and minimum values—may be conveniently studied through use of the methods we have developed. Consider, for instance, the following question:

Example 1. A farmer can afford to buy 400 ft. of wire fencing. He wishes to enclose a rectangular field of the *largest possible area*. What should the dimensions of the field be? Let x and y be the length and breadth of the field (Fig. 36). The area is then $A = xy$. But x and y are not independent of one another; in fact, we are to have $2x + 2y = 400$, and consequently $y = 200 - x$. Hence the area

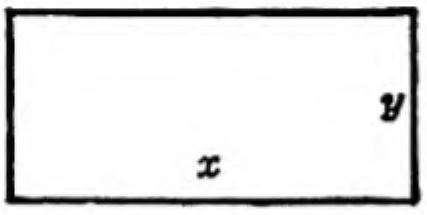


FIG. 36.

$$A = x(200 - x) = 200x - x^2$$

is a function of the single variable x. We can construct the graph of this function and find the value of x that makes A a maximum (Fig. 37). We have $\frac{dA}{dx} = 200 - 2x$. Therefore, $200 - 2x = 0$ or $x = 100$ gives a critical point. Since $\frac{d^2A}{dx^2} = -2$ is negative, this point is a maximum. Hence $x = 100$ and $y = 200 - 100 = 100$ are the dimensions of the field. Note that it is a square.

Example 2. A manufacturer wishes to make an aluminum cup of fixed volume V of (right circular) cylindrical shape open at the top. What proportions will require the least material of uniform thickness? The problem amounts to finding the proportions of a cup of fixed volume and *minimum surface area*. Let the radius of the cylinder be r and the altitude h (Fig. 38). The surface area is then

$$S = 2\pi rh + \pi r^2$$

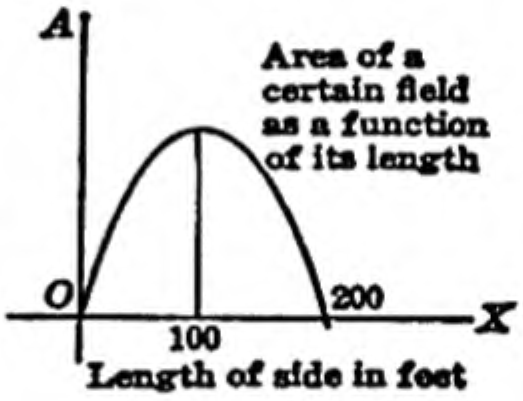


FIG. 37.

But r and h are connected by the fact that the volume is fixed,

$$V = \pi r^2 h$$

whence $h = V/\pi r^2$, and we can express the surface area S as a function of the single variable r,

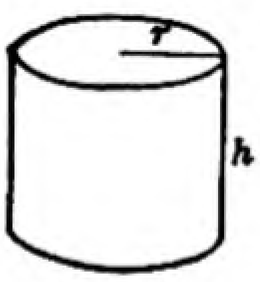


FIG. 38.

$$S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2 = \frac{2V}{r} + \pi r^2$$

A graph of this function could be drawn as in Example 1, but this is really unnecessary. We need only find the value of r that makes S a minimum,

$$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r = \frac{-2V + 2\pi r^3}{r^2}$$

This derivative will be zero if $2\pi r^3 - 2V = 0$, that is, if

$$r^3 = \frac{V}{\pi} \quad \text{or} \quad r = \left(\frac{V}{\pi}\right)^{1/3}$$

The second derivative is

$$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi$$

which assumes the value $6\pi > 0$ when $r^3 = V/\pi$. This value of r therefore gives a minimum value of the function S.

If $r = (V/\pi)^{1/3}$, then

$$h = \frac{V}{\pi(V/\pi)^{2/3}} = \left(\frac{V}{\pi}\right)^{1/3}$$

Consequently the proportions for the cup are $r = h$, radius equals altitude. Or we might use the fact that since $V = \pi r^2 h$, we must have

$$\pi r^3 = \pi r^2 h \qquad \text{or} \qquad r = h$$

Example 3. A rectangular box of fixed volume V is to be twice as long as it is wide. The material in the top and four sides costs three times as much per square foot as that in the bottom. What are the most economical proportions? Our problem is to find the proportions that will make the *cost a minimum*. We shall, therefore, express the cost as a function of a single variable and then find the value of that variable which makes the cost a minimum. First, sketch the box (Fig. 39). Call the width x; the length is then $2x$. Call the depth y. Now, suppose that the cost per square foot of the bottom is c. Then the cost per square foot of the sides and top is $3c$. The *total cost* is, therefore

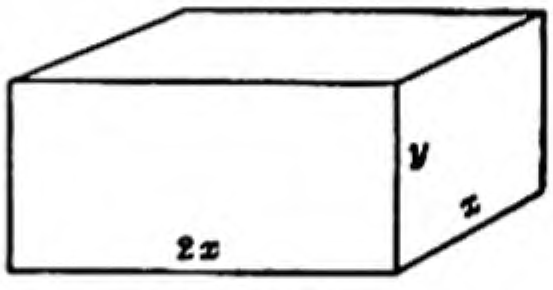


FIG. 39.

$$C = c(2x^2) + 3c(2x^2 + 2xy + 4xy) = 2c(4x^2 + 9xy)$$

This is the function which is to be a minimum. We wish to express C as a function of a single variable. Since $2x^2y = V$,

$$y = \frac{V}{2x^2}$$

Hence
$$C = 2c\left(4x^2 + \frac{9V}{2x}\right)$$

We have
$$\frac{dC}{dx} = 2c\left(8x - \frac{9V}{2x^2}\right) = 2c\left(\frac{16x^3 - 9V}{2x^2}\right)$$

This will be zero if $16x^3 = 9V$ or,

$$x = \left(\frac{9V}{16}\right)^{1/3}$$

To be sure that this gives a minimum, we note that

$$\frac{d^2C}{dx^2} = 2c\left(8 + \frac{18V}{2x^3}\right)$$

which is positive for all $x > 0$.

We may calculate the corresponding value of y and compare it with x, or we may note that, since $2x^2y = V$, $2x^2y = \frac{16}{9}x^3$, so that $y = \frac{8}{9}x$.

In the above examples, we might safely have relied upon "common-sense" considerations to determine whether a maximum or minimum value had been obtained, instead of calculating the second derivative (or observing the behavior of the sign of the first derivative). In the

first example, if we start with a small width and allow it to increase and the length to decrease, it is clear that the area increases for a while, then decreases. Furthermore, there can be no minimum area, for we can make the area as close to zero as we please by making the field narrow enough. Hence, the critical value must give the maximum area. Similarly, in Example 2, if the radius is very small, the altitude must be very great to preserve the constant volume. Hence, the area of the curved surface is large. As the radius is allowed to increase and the altitude to decrease, the area decreases for a while. It must then get larger; for if the altitude is very small, the radius must be very large and the area of the bottom is enormous. Hence there is no maximum area, and the critical value must give a minimum. Similar considerations apply to Example 3. It is hardly necessary to add that caution must be used in applying this rough sort of reasoning. If there is any doubt whatever about a certain critical value, either the sign of the second derivative should be found, or the behavior of the sign of the first derivative investigated.

Example 4. A steamboat whose capacity is 150 passengers is to be chartered for an excursion. The price of a ticket is to be \$10 if 100 people buy tickets, but the operating company agrees to reduce the price of every ticket 6 cents for each ticket sold in

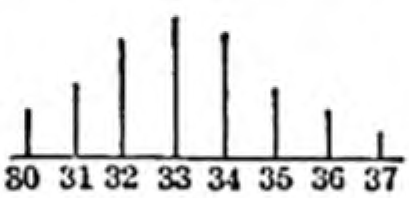


FIG. 40.

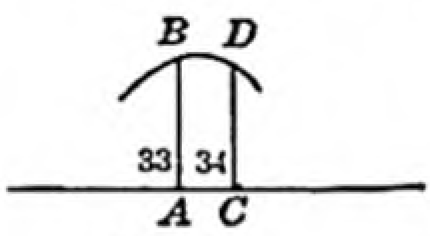


FIG. 41.

excess of 100. What number of passengers will produce the greatest gross income? Let x represent the number of passengers in excess of 100. The price per ticket is then $10 - 0.06x = 10 - \frac{3}{50}x$ dollars. The gross income is

$$I = (100 + x)\left(10 - \frac{3}{50}x\right) = 1000 + 4x - \frac{3x^2}{50} \tag{3}$$

Hence I is a function of a single variable x. But since x must be an integer, it is not a continuous variable, and we cannot differentiate I with respect to x. However, we can plot points and draw ordinates for all integral values of x for $0 \leq x \leq 50$ (Fig. 40). We can then pick out the highest ordinate, and this will represent the maximum I. But suppose we look at equation (3) from a different point of view, namely, let us regard it simply as an algebraic function whose graph (Fig. 41) can be drawn. The ordinates in Fig. 40 are then among the ordinates of this curve. We find its maximum as usual,

$$\frac{dI}{dx} = 4 - \frac{3x}{25}$$

which gives $$3x = 100 \qquad x = 33\tfrac{1}{3}$$

as abscissa of the maximum point. We can now interpret this result in the light of our given conditions. Since $\frac{dI}{dx} > 0$ for all $x < 33\frac{1}{3}$, the ordinate AB at $x = 33$ is greater

than the ordinates for all integral values of x less than 33. Similarly, since $\frac{dI}{dx} < 0$ for all $x > 33\frac{1}{3}$, the ordinate CD at $x = 34$ is greater than the ordinates for all integral values of x greater than 34. We need only find which of AB and CD is the larger. If $x = 33$, $I = 1066.66$; if $x = 34$, $I = 1058.68$. Therefore, the greatest gross income is obtained if 133 passengers are carried.

Example 5. In Example 4, suppose that the boat can carry only 125 passengers. What number yields maximum gross income? Here x must not be greater than 25. But since we found $\frac{dI}{dx}$ positive for all $x < 33\frac{1}{3}$, the income for $x = 25$ is greater than that for any smaller value of x. Hence 125 passengers produce maximum gross income. In other words, the maximum is attained at the end point of the interval in which x lies. This illustrates the important fact that the student must not blindly assume that in any problem involving maxima and minima it is sufficient to pay attention only to zero values of the derivative.

Examples 4 and 5 illustrate a point of fundamental importance, namely, that a mathematical formula may represent much more than the data of some given problem; hence, results must be interpreted carefully. No amount of mathematical discussion of equation (3) would justify us in stating that $133\frac{1}{3}$ passengers will produce the greatest gross income. It is exceedingly useful to regard x temporarily as a continuous variable, for we quickly see that only two values of I need be computed and compared to ascertain the maximum. Thus, in general, in making applications of mathematics we must, evidently, have a clear idea of just what our methods can and cannot tell us. Otherwise we may obtain erroneous results, not because our calculations are faulty, but because we read into our results a meaning that the original data do not justify.

We may now formulate a general plan of attack on maximum and minimum problems:

1. Make a sketch illustrating the conditions of the problem. Label carefully the various parts of the figure.

2. Express the function which is to be a maximum or minimum in terms of the variables used in the figure. Use the conditions of the problem to reduce this expression to terms of a single variable.

3. Differentiate the function with respect to this variable, and use the methods of Arts. 32 and 34 to locate the maximum or minimum.

The student should now reread Examples 1 to 5 and observe that the above plan has been followed in each case.

38. Alternative Method of Solution. In each of Examples 1 to 3 of the last section the function that was to be a maximum or a minimum was expressed in terms of two variables. The conditions of the problem were then used to reduce this expression to terms of one variable. In many problems, especially those which, like Examples 2 and 3, do not require the actual numerical value of the variable rendering a function a maximum or minimum, it is very convenient to vary slightly our method of attack. We proceed to solve Examples 1 to 3 of Art. 37 as follows:

Example 1. The function to be made a maximum is $A = xy$. We also have $2x + 2y = 400$ or $x + y = 200$. Hence, in reality, A is a function of a single variable, say x. We differentiate A with respect to x, remembering that y is itself a function of x,

$$\frac{dA}{dx} = x\frac{dy}{dx} + y \tag{4}$$

For A to be a maximum, we should find x and y so that this derivative will be zero. But first we must find $\frac{dy}{dx}$ in terms of x and y. We note that the relation

$$x + y = 200 \tag{5}$$

gives y as a function of x; hence $1 + \frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -1$. Using this in equation (4), we have

$$\frac{dA}{dx} = x(-1) + y = y - x$$

Therefore $\frac{dA}{dx}$ will be zero if $y - x = 0$, that is, if

$$y = x \tag{6}$$

To test whether this is a maximum or minimum, we have

$$\frac{d^2A}{dx^2} = \frac{dy}{dx} - 1 = -1 - 1 = -2$$

$\left(\text{since } \frac{dy}{dx} = -1 \text{ for all values of } x\right)$; hence $y = x$ makes A a maximum. In other words, the field must be a square. We can find the dimensions by solving simultaneously equations (5) and (6); thus $x = 100$, $y = 100$.

Example 2. Here the function that was to be a minimum was S, the surface of the cup,

$$S = 2\pi rh + \pi r^2$$

Since the volume V is given, S is in reality a function of the single variable r. Consequently, to find the minimum we equate to zero the derivative of S with respect to r,

$$\frac{dS}{dr} = 2\pi\left(r\frac{dh}{dr} + h\right) + 2\pi r = 2\pi\left(r\frac{dh}{dr} + h + r\right) \tag{7}$$

To find $\frac{dh}{dr}$, we note that $\pi r^2h = V$ gives h as a function of r. Differentiating, we have

$$\pi\left(r^2\frac{dh}{dr} + 2rh\right) = 0 \tag{8}$$

since the derivative of the given constant V is zero. It is very important to realize that $\frac{dV}{dr}$ is zero because V is a *given constant*, whereas $\frac{dS}{dr}$ is set equal to zero so that we may find values of r and h which make the variable surface area S a minimum. In this problem, V plays a role similar to that played by the value 400 (length of available fence wire) in Example 1. From equation (8), we have

$$\pi r\left(r\frac{dh}{dr} + 2h\right) = 0$$

whence
$$\frac{dh}{dr} = -\frac{2h}{r}$$

The possibility that $r = 0$ is discarded, for this would require V to be zero, which it is not. Substituting this value of $\frac{dh}{dr}$ into (7), we have

$$\frac{dS}{dr} = 2\pi(-2h + h + r) = 2\pi(r - h)$$

whence $r = h$ makes $\frac{dS}{dr}$ zero. To test for a minimum, we have

$$\frac{d^2S}{dr^2} = 2\pi\left(1 - \frac{dh}{dr}\right) = 2\pi\left(1 + \frac{2h}{r}\right)$$

which is positive because both h and r are positive. Hence, $r = h$ gives a minimum S.

Example 3. Here $C = 2c(4x^2 + 9xy)$ is to be a minimum. Therefore

$$\frac{dC}{dx} = 2c\left(8x + 9x\frac{dy}{dx} + 9y\right)$$

must be zero. But we have $2x^2y = V$, a given constant. Hence

$$2\left(x^2\frac{dy}{dx} + 2xy\right) = 0 \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{2y}{x}$$

(since x cannot be zero). Consequently

$$\frac{dC}{dx} = 2c(8x - 18y + 9y) = 2c(8x - 9y)$$

which is zero if $9y = 8x$, or $y = \frac{8}{9}x$. The reader may test for a minimum by finding the second derivative.

EXERCISES

1. A rectangle has a given perimeter. What shape gives maximum area?
2. A rectangle has a fixed area. What shape gives minimum perimeter?
3. Find two numbers whose sum is a and whose product is a maximum.
4. Find two positive numbers whose product is a^2 and whose sum is a minimum.
5. Find two numbers whose sum is $a > 0$ if the product of one by the square of the other is a maximum.
6. Find two numbers whose sum is $a > 0$ if the product of one by the cube of the other is a maximum.
7. What positive number when added to its reciprocal gives the minimum sum?
8. Find the shape of the rectangle of maximum area inscribed in a circle.
9. Find the shape of the rectangle of maximum perimeter inscribed in a circle.
10. A long strip of sheet iron 28 in. wide is to be made into a gutter by turning up equal widths vertically along the two edges. How many inches should be turned up at each side to give maximum carrying capacity?

11. A rectangular yard is fenced off, an existing stone wall being used as one side. If the area of the yard is to be 7200 sq. ft., what dimensions will require the least amount of new fencing?

12. A rectangular field of area 28,800 sq. yd. is to be fenced off along a straight road. The front fencing costs 75 cents per yard; that for the sides and back costs 25 cents per yard. Find the dimensions giving minimum cost, and find the cost.

13. A rectangular piece of tin is 8 by 5 in. An open box is to be made by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be so made.

14. A rectangular field of 25,000 sq. yd. is to be enclosed and divided into four lots by parallels to one of the sides. What dimensions of the field will make the amount of fencing a minimum?

15. A rectangular box whose base is twice as long as it is wide has a volume of 256 cu. in. Material for the top costs 10 cents per square inch; that in the sides and bottom costs 5 cents per square inch. Find the dimensions that will make the cost a minimum, and find the cost.

16. The base of an open rectangular box is three times as long as it is wide. The volume is 144 cu. in. Find the dimensions giving a minimum surface area.

17. A sheet of paper is to contain 18 sq. in. of printed matter. The margins at top and bottom are 2 in. each and at the sides 1 in. each. Find the dimensions of the sheet of smallest area.

18. Find the most economical proportions for a circular cylindrical can (with top) to hold 1 gal.

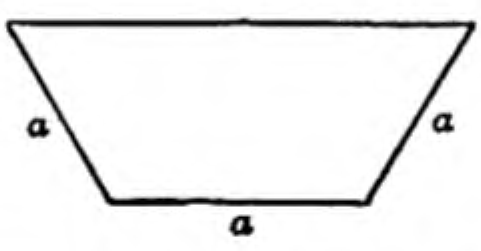
19. An open circular cylindrical tank of given volume stands with vertical axis. The material in the bottom costs twice as much per unit area as that in the sides. Find the most economical proportions.

20. Solve Exercise 19 if the tank has a cover made of the same material as the sides.

21. A closed water tank of given volume consists of a hemisphere surmounted by a cylinder. Find the most economical proportions.

22. Solve Exercise 21 if the steel in the hemisphere costs twice as much per unit area as that in the cylinder.

23. A gutter is to be made of a strip of sheet iron $3a$ in. wide, the cross section being an isosceles trapezoid as shown. Find the width across the top giving maximum carrying capacity.



24. The load that can be supported by a beam of rectangular cross section is proportional to the breadth and square of the depth. Find the shape of the strongest beam that can be cut from a given circular log.

25. The stiffness of a beam of rectangular cross section is proportional to the breadth and cube of the depth. What is the shape of the stiffest beam that can be cut from a given circular log?

26. Formerly a parcel-post package could have the sum of its girth and length not greater than 100 in. What are the dimensions of the rectangular package with square ends of greatest volume that could be sent?

27. A Norman window is in the shape of a rectangle surmounted by a semicircle. What proportions give minimum perimeter for a given area?

28. Find the proportions of the right circular cone of maximum volume and fixed slant height.

29. Find the altitude h of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of altitude H.

30. Find the radius r of the right circular cylinder of maximum convex surface area inscribed in a given right circular cone of radius of base R.

31. Find the altitude h of the right circular cone of maximum volume inscribed in a given sphere of radius a.

32. Find the altitude h of the right circular cone of minimum volume circumscribed about a given sphere of radius a.

33. Show that the isosceles triangle of maximum area inscribed in a given circle is equilateral.

34. A line is drawn tangent to the circle $x^2 + y^2 = a^2$. The distance between its x and y intercepts is l. Find the minimum value of l. (*Hint*: l is a minimum if l^2 is a minimum.)

35. Same as Exercise 34 if the line is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

36. Find the point of the parabola $y^2 = 6x$ nearest to (5,0).

37. A cruiser is anchored 9 miles from the nearest point of a straight beach. It is necessary to send a messenger to a military camp 15 miles along the beach from this point. If he can ride 40 m.p.h. in the ship's motorboat and can be driven at 50 m.p.h. in an automobile on a road along the shore, where should he land to reach the camp in the shortest possible time?

38. Solve Exercise 37 if the camp is 20 miles along the beach.

39. It is found that the total cost of producing x manufactured articles is T dollars where $T = 50 + 14x + 0.02x^2$. How many articles will give the lowest cost per article?

40. Solve Exercise 39 if $T = 50 + 15x + 0.02\,x^2$.

41. A manufacturing concern finds that it makes a profit of \$20 on each article if 800 or fewer are made per week. The profit decreases 2 cents per article over 800. How many articles should be made per week to give maximum profit?

42. Busses are to be chartered for an excursion. The price of a ticket is to be \$15 if not more than 150 people buy tickets, but the operating company agrees to reduce the price of every ticket 5 cents for each passenger in excess of 150. What number of passengers will produce the maximum gross income?

43. In Exercise 42, what number of passengers will produce the maximum profit if expenses are \$800 for 150 or fewer passengers and if expenses increase \$2 per passenger over 150?

44. Solve Exercise 42 if the reduction per ticket is 7 cents.

45. A gardener has a ft. of wire fencing. He wishes to fence off a grass plot in the form of a sector of a circle. What radius should he use to yield the greatest area, and what is this area?

46. The feet of two vertical poles are 20 ft. apart. The poles are, respectively, 10 and 15 ft. high. They are to be stayed by guy wires fastened to a single stake on the ground and running to the tops of the poles. Where should this stake be placed to use the least amount of wire? Solve this problem using plane geometry only.

47. A right circular cone of altitude h is inscribed in a fixed right circular cone of altitude H, the vertex of the inside cone being at the center of the base of the outside cone. Find h so that the volume of the inscribed cone shall be a maximum.

48. The fuel consumed per hour by a certain steamship is proportional to the cube of the velocity that would be imparted to the ship in still water. It is required to run the ship k miles against a current flowing r m.p.h. What is the most economical rate?

49. A silo of fixed volume consists of a right circular cylinder surmounted by a hemisphere. The hemispherical roof costs twice as much per unit area as the floor and walls. Find the most economical proportions.

50. The intensity of light varies inversely as the square of the distance from its source. If two searchlights are 600 yd. apart and one light is eight times as strong as the other, where should a man cross the line between them in order to be illuminated as little as possible?

51. One ship, A, was 60 miles directly north of another, B, at noon. B was sailing east at 10 m.p.h., and A was sailing south at 20 m.p.h. When were they nearest together? How near?

52. In the corner of a field bounded by two perpendicular roads a spring is situated a yd. from one road and b yd. from the other. A straight road is to be run across the corner and by the spring. Where should it intersect the bounding roads to cut off the least area from the field?

53. It is known that a wire bent in the form of a circle of radius r units exerts upon a particle h units directly above the center of the circle a force of attraction proportional to $\dfrac{h}{(r^2 + h^2)^{3/2}}$. If r is fixed, find h so that the attraction shall be a maximum.

54. A lot has the form of a right triangle with perpendicular sides 90 and 120 ft. Find the dimensions of the rectangular building of largest floor space that can be built fronting on the hypotenuse of the triangle.

55. In statistical work, if $a_1, a_2, a_3, \ldots, a_n$ are n constants, then

$$\sqrt{\frac{1}{n}(a_1^2 + a_2^2 + \cdots + a_n^2)}$$

is called their *root mean square.* If x is some number, then

$$f(x) = \sqrt{\frac{1}{n}[(a_1 - x)^2 + (a_2 - x)^2 + \cdots + (a_n - x)^2]}$$

is called their *root-mean-square deviation from* x. Show that $f(x)$ is a minimum if $x = \dfrac{1}{n}(a_1 + a_2 + \cdots + a_n)$, the *arithmetic mean* (denoted by $\bar{x}$). The expression $f(\bar{x})$ is called the *standard deviation* of the n constants (denoted by σ).

56. A conical wineglass is a in. deep, and the angle at the vertex is 2α. If a spherical ball of radius r is carefully lowered into a full glass, show that the greatest overflow results if $r = \dfrac{a \sin \alpha}{\cos 2\alpha + \sin \alpha}$. [*Hint:* The volume of a segment of one base and of height h cut from a sphere of radius r is $\frac{1}{3}\pi h^2(3r - h)$.]

39. Rate Problems. We have several times referred to the fact that the derivative is the rate of change of a function. A case of great practical importance occurs when the independent variable represents time. We may, for instance, have $y = f(t)$ and wish to find the rate of change of y with respect to t where t represents time. It is sufficient to calculate $\dfrac{dy}{dt} = f'(t)$ directly. Again, we may have y a function of x where x is itself a function of t. If, as is often the case in physical problems, we have given $\dfrac{dx}{dt}$, the "time rate of change of x," we can calculate $\dfrac{dy}{dt}$ from

the formula

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

(see Art. 23).

The method of solving problems of this type is illustrated in the following examples:

Example 1. Water is flowing into a conical vessel 18 in. deep and 10 in. across the top at the rate of 4 cu. in. per minute. Find the rate at which the surface is rising when the water is 12 in. deep. Let h be the depth and r the radius of the surface of the water at time t (Fig. 42). The volume of the water at time t is then

$$V = \tfrac{1}{3}\pi r^2 h$$

We have h, r, and therefore V all functions of t. We know that, for all t,

$$\frac{dV}{dt} = 4 \text{ in.}^3/\text{min.}$$

and we wish to find $\dfrac{dh}{dt}$ at the instant when $h = 12$ in.

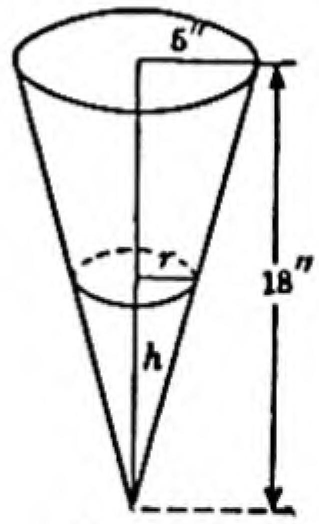


FIG. 42.

Now r and h are connected by the relation (from similar triangles) $r/h = \frac{5}{18}$. Since we wish to discover something about the depth h, it will be convenient to find r in terms of h and then express V as a function of h. Thus

$$r = \tfrac{5}{18}h \qquad \text{and} \qquad V = \tfrac{1}{3}\pi \cdot \tfrac{25}{324}h^3$$

It is most important to observe that this relation is true *at any time*. Since V is a function of h and h is a function of t,

$$\frac{dV}{dt} = \pi \cdot \frac{25}{324}\,h^2\,\frac{dh}{dt} \tag{9}$$

Since we have given that $\dfrac{dV}{dt} = 4$ in.^{3}/min. and wish to find $\dfrac{dh}{dt}$ at the instant when $h = 12$ in., we may substitute these values for $\dfrac{dV}{dt}$ and h in (9), obtaining

$$4 = \pi \cdot \frac{25}{324} \cdot 144 \left[\frac{dh}{dt}\right]_{12}$$

Hence $$\left[\frac{dh}{dt}\right]_{12} = \frac{9}{25\pi} = 0.11 \text{ in./min. approximately}$$

Example 2. One of two intersecting railroad tracks runs east and west, the other north and south. At noon a train is 20 miles west of the crossing and traveling east at 40 m.p.h. on the first track. At the same time a train is 50 miles south of the crossing and traveling north at 60 m.p.h. on the second track. How fast are the trains separating at 1:00 P.M.? When will they be nearest together, and what is their minimum distance apart? Assuming the tracks to be straight and of indefinite length, how fast are the trains separating after a very long time? We may represent this situation as in Fig. 43. Let the coordinate axes represent the tracks, let A with abscissa x represent the first train, let B with ordinate y represent the second train,

and let z represent the distance between the trains. Suppose we let t represent time in hours elapsed since noon. Then $\frac{dx}{dt} = 40$ m.p.h., $\frac{dy}{dt} = 60$ m.p.h.; and when $t = 0$, $x = -20$ and $y = -50$. Evidently

$$x = -20 + 40t \qquad \text{and} \qquad y = -50 + 60t \quad (10)$$

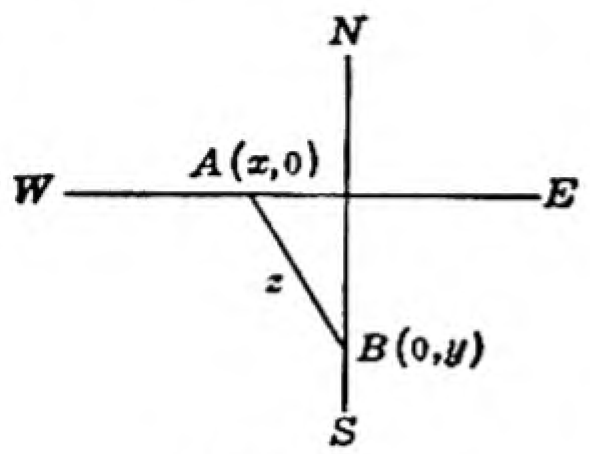


FIG. 43.

We may now write down a relation that is true *at any time*, $x^2 + y^2 = z^2$, whence

$$\begin{aligned} z &= \sqrt{x^2 + y^2} = \sqrt{(-20 + 40t)^2 + (-50 + 60t)^2} \\ &= 10\sqrt{52t^2 - 76t + 29} \qquad \text{(miles)} \qquad (11) \end{aligned}$$

From this we find that, at any time,

$$\frac{dz}{dt} = \frac{20(26t - 19)}{\sqrt{52t^2 - 76t + 29}} \qquad \text{(miles per hour)} \quad (12)$$

To find $\frac{dz}{dt}$ at time $t = 1$, we need only substitute $t = 1$ in (12), obtaining

$$\left[\frac{dz}{dt}\right]_{t=1} = \frac{20(26 - 19)}{\sqrt{52 - 76 + 29}} = \frac{140}{\sqrt{5}} = 62.6 \text{ m.p.h. approximately}$$

To find when z is a minimum, we first observe that z is never zero, for the sum of squares $(-2 + 4t)^2 + (-5 + 6t)^2$ could be zero only if both terms were simultaneously zero (that is for the same value of t), which they are not. The minimum of z will occur for the value of t that makes $\frac{dz}{dt} = 0$, namely, for $26t - 19 = 0$, that is, for

$$t = \tfrac{19}{26} \text{ hr.} = \tfrac{19}{26} \cdot 60 \text{ min.} = 43.8 \text{ min. approximately}$$

Hence z will be a minimum at 12:43.8 P.M.

To calculate the minimum distance apart, we set $t = \frac{19}{26}$ in equation (11), obtaining

$$\begin{aligned} z &= 10\sqrt{(-2 + 4 \cdot \tfrac{19}{26})^2 + (-5 + 6 \cdot \tfrac{19}{26})^2} = \tfrac{10}{13}\sqrt{12^2 + 8^2} \\ &= \tfrac{10}{13}\sqrt{208} = \tfrac{40}{13}\sqrt{13} = 11.1 \text{ miles approximately} \end{aligned}$$

To find how fast the trains are separating after a very long time, we find $\lim_{t \to \infty} \frac{dz}{dt}$. To do this, in the right-hand member of equation (12) we divide numerator and denominator by t, obtaining

$$\frac{dz}{dt} = \frac{20\left(26 - \frac{19}{t}\right)}{\sqrt{52 - \frac{76}{t} + \frac{29}{t^2}}}$$

If t is allowed to increase indefinitely, the limiting value of this expression is

$$\frac{20(26)}{\sqrt{52}} = \frac{260}{\sqrt{13}} = 72.0 \text{ m.p.h. approximately}$$

Hence, after a long time, the trains will be separating at about 72 m.p.h.

This problem might have been solved by differentiating $x^2 + y^2 = z^2$ with respect to t:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

This relation between x, y, z, and their time derivatives could then have been used together with equations (10) to obtain the results sought. A method similar to this will be used to solve Example 3.

Example 3. A man is walking at the rate of 4 ft. per second across a bridge 30 ft. above a river. A boat traveling 12 ft. per second at right angles to the roadway of the bridge passes directly beneath him. How fast is the distance between the man and the boat increasing 5 sec. later? Let A represent the position of the man directly above the boat C, and let M and B be their positions t sec. later (Fig. 44). We then have

$$\frac{dx}{dt} = 4 \text{ ft./sec.} \qquad \frac{dz}{dt} = 12 \text{ ft./sec.}$$

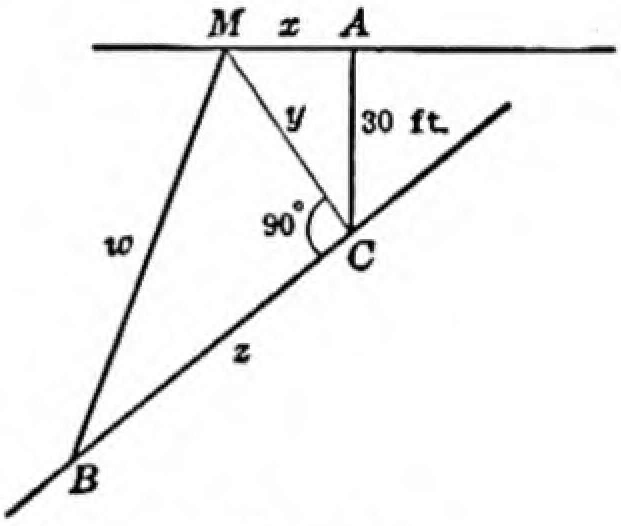


FIG. 44.

and we wish to find $\frac{dw}{dt}$ at $t = 5$. We have

$$\begin{aligned} y^2 &= x^2 + 900 \\ w^2 &= z^2 + y^2 = z^2 + x^2 + 900 \\ w &= \sqrt{z^2 + x^2 + 900} \end{aligned} \tag{13}$$

for all values of t. Hence

$$\frac{dw}{dt} = \frac{z\frac{dz}{dt} + x\frac{dx}{dt}}{\sqrt{z^2 + x^2 + 900}}$$

At the time $t = 5$ sec., $x = 20$ ft., $z = 60$ ft. Therefore

$$\left.\frac{dw}{dt}\right]_{t=5} = \frac{60 \cdot 12 + 20 \cdot 4}{\sqrt{60^2 + 20^2 + 900}} = \frac{6 \cdot 12 + 2 \cdot 4}{\sqrt{6^2 + 2^2 + 9}}$$
$$= \frac{80}{7} = 11\tfrac{3}{7} \text{ ft./sec.}$$

An alternative method of solution would be to use the fact that $x = 4t$ and $z = 12t$ in equation (13) to express w in terms of t; $\frac{dw}{dt}$ would then be obtained by differentiation (compare Example 2).

The general plan of attack on problems involving time rates may be summarized as follows:

1. Make a sketch illustrating the conditions of the problem. Let x, y, z, w, . . . represent the quantities that vary with time.
2. List the given and required quantities.
3. Express by means of an equation a relationship among the variables that is *true at any time*.
4. Differentiate with respect to time.
5. Now, *and not before*, substitute values of the variables for the particular instant in question, and solve the resulting equation for the required quantity.

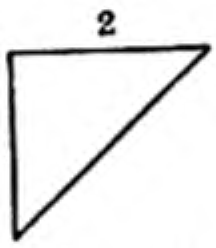
It is most important *not* to substitute values of the variables that hold only at a given instant *before* differentiating. For instance, in Example 3, it would obviously be wrong to substitute $x = 20$ in equation (13). For x is equal to 20 only at a particular instant, and at this instant z is 60; hence w is 70, and the derivative of w is simply zero. The student should reread Examples 1, 2, and 3 and observe that the above plan has been followed in each case.

EXERCISES

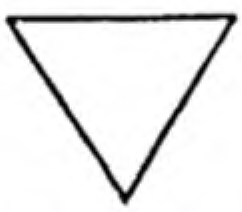
1. Water is flowing into a conical vessel 24 in. deep and 12 in. across the top at the rate of 100 cu. in. per minute. How fast is the surface rising when the water is 10 in. deep?

2. A rectangular trough is 3 ft. wide and 15 ft. long. Water flows in at the rate of 2 cu. ft. per minute. How fast is the surface rising?

3. A trough has for cross section an isosceles right triangle as shown. It is 2 ft. across the top and 10 ft. long. Water flows in at the rate of 3 cu. ft. per minute. How fast is the surface rising when the water is 6 in. deep?



4. A trough has for cross section an isosceles triangle as shown. It is 3 ft. across the top, 2 ft. deep, and 12 ft. long. Water flows in at the rate of 5 cu. ft. per minute and is being pumped out at the rate of 2 cu. ft. per minute. How fast is the surface rising when the water is 6 in. deep?



5. A man 6 ft. tall walks directly away from a lamppost 15 ft. high at the rate of 3 m.p.h. (*a*) How fast does his shadow lengthen? (*b*) How fast does the tip of his shadow move?

6. A spherical toy balloon is being filled with gas at the rate of 2 cu. in. per second. How fast is the radius increasing when the balloon is 1 ft. in diameter?

7. Gas is expelled from a spherical toy balloon by decreasing the radius at the rate of $\frac{1}{4}$ in. per second. At what rate is the gas escaping when the radius is 4 in.?

8. A launch whose deck is 6 ft. below the level of a wharf is pulled toward the wharf by a rope attached to a ring in the deck. If a windlass on the wharf hauls in the rope at 4 ft. per minute, how fast does the launch move through the water when there are 10 ft. of rope out? Does the speed of the launch increase or decrease as it approaches the wharf? Can the windlass maintain this constant rate?

9. A balloon leaves the ground 50 ft. from an observer. If it rises at the rate of 6 ft. per second, how fast is it receding from the point of observation when it is 120 ft. above the ground?

10. A kite flies horizontally directly away from the man who is flying it at the rate of 5 m.p.h. If it is 160 ft. high when there are 200 ft. of string out, how fast is the string being payed out at this instant? (Assume, for simplicity, that the string is in a straight line.)

11. Sand is poured onto the ground at the rate of 3 cu. ft. per minute. It forms a pile whose shape is a right circular cone whose altitude is half the radius of the base. How fast is the altitude increasing when the radius of the base is 4 ft.?

12. A man hoists a bucket of cement to a scaffold 40 ft. above the level of his hand by means of a rope passed through a pulley on the scaffold. If he pulls in the rope at the rate of 10 ft. per minute at the same time that he walks away from a point directly below the pulley at the rate of 50 ft. per minute, how fast is the bucket rising at the end of 36 sec.?

13. A light is placed on the ground 30 ft. from a building. A man 6 ft. tall walks from the light toward the building at the rate of 5 ft. per second. Find the rate at which his shadow on the wall is shortening when he is 15 ft. from the wall.

14. A ladder 20 ft. long leans against a vertical wall. If the top slides downward at the rate of 2 ft. per second, find the rate at which the lower end moves on a horizontal floor when it is 12 ft. from the wall.

15. In Exercise 14, find the rate at which the slope of the ladder changes.

16. A light is placed on the ground 40 ft. from a house and 15 ft. from the pathway leading from the house to the street. A man walks along the path at 5 ft. per second. How fast is his shadow moving on the wall of the house when he is 15 ft. from it?

17. As an automobile runs across a bridge at 40 ft. per second, a train passes directly beneath it, traveling 80 ft. per second on a track at right angles to the bridge roadway and 30 ft. below it. How fast are the automobile and train separating after 2 sec.?

18. A man walking at 5 ft. per second on a bridge 40 ft. above the water crosses the course (perpendicular to the direction of the bridge) of a motorboat approaching at 20 ft. per second at the moment when the boat is 50 ft. from a point in the water directly below the center of the bridge. How fast are the man and boat separating 4 sec. later?

19. In Exercise 18, find when the man and boat are closest together, and find the shortest distance.

20. A ship sails at a constant speed of 12 knots. If it travels from its starting point directly north for 2 hr. and then turns north 60° east, how fast is it leaving the starting point after 3 hr.? *Note:* A knot is one nautical mile per hour; assume the ocean surface to be a plane.

21. A ship sails with a constant speed of 10 knots. If it travels from its starting point directly north for 1 hr., then turns east 30° south, (*a*) how fast is it leaving its starting point after 3 hr.? (*b*) When is it nearest its starting point?

22. A trough whose cross section is an isosceles trapezoid is 2 ft. across the top, 1 ft. wide at the bottom, 18 in. deep, and 8 ft. long. Water flows in at the rate of 3 cu. ft. per minute and is pumped out at 1 cu. ft. per minute. How fast is the surface rising when the water is 8 in. deep?

23. A boat is anchored with its deck 45 ft. above the point where the anchor is fast on the bottom. It drifts at the rate of 3 m.p.h. How fast does the anchor chain slip over the edge of the deck when there are 75 ft. of chain out? (Assume that the chain forms a straight line from deck to anchor.)

24. Two roads, AB and BC, intersect at B, the angle ABC is 60 deg., and the distance AB is 28 yd. A man starts to ride a bicycle at 4 yd. per second from A toward B at the instant another man passes B on a bicycle going 8 yd. per second along BC. How fast is the distance between them changing after 3 sec.?

MISCELLANEOUS EXERCISES

Find the equation of the tangent and of the normal at the point indicated (Ex. 1 to 4).

1. $x^3 + y^3 = 9$ at $(1,2)$
2. $x^2y^2 - y^2 = 32$ at $(3,-2)$
3. $y = x^3 + 4x^2 - x + 1$ at $(1,5)$
4. $y^2 = x^3 - 2x^2 + 3x + 3$ at $(2,3)$

Find the equation of the tangent to the given curve as indicated (Ex. 5 to 8).

5. To the circle $x^2 + y^2 = 13$ and passing through $(5,-1)$
6. To the curve $y = x^3 + 3x^2 + 1$ and parallel to the line $9x - y + 3 = 0$
7. To the parabola $y = 3x^2 - 4x + 5$ and having slope 2
8. To the curve $y^2 = x^3 - 7$ and perpendicular to the line $x - 6y - 1 = 0$

Find the angle between the curves (Ex. 9 to 12).

9. $y^2 = x^3,\ y^2 = \dfrac{x^3}{5 - x}$ 10. $y^2 = 4x + 4,\ y^2 = -8x + 16$

11. $y^2 = 4ax + 4a^2,\ y^2 = -4bx + 4b^2$. Also show that these parabolas have the same focus.

12. $xy = 9,\ y^2 = \dfrac{x^3}{18 - x}$

In Exercises 13 to 16, determine the coefficients so that the curves fulfill the given conditions.

13. $y^2 + ay + bx + c = 0$ to be tangent to $3x + 2y + 1 = 0$ at $(-1,1)$ and to pass through $(7,-2)$

14. $y = ax^2 + bx + c$ to be tangent to $x + y - 2 = 0$ at $(2,0)$ and to pass through $(0,4)$

15. $y = ax^3 + bx^2 + cx + d$ to be tangent to $6x - y - 2 = 0$ at $(1,4)$, and to pass through $(-1,0)$ and $(2,15)$

16. $y = ax^4 + bx^3 + cx^2 + dx + e$ to be tangent to $25x - y - 41 = 0$ at $(2,9)$ and to pass through $(1,-2)$, $(-1,6)$, and $(-2,37)$

17. Show that the tangent to the parabolic arc $x^{1/2} + y^{1/2} = a^{1/2}$ at the point (x_1,y_1) is $\dfrac{x}{x_1^{1/2}} + \dfrac{y}{y_1^{1/2}} = a^{1/2}$.

18. Prove that the segment of the tangent to the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant in length $(= a)$.

19. Show that in the parabola $y^2 = 4ax$ the subnormal is constant in length. Generalize this statement for *any* parabola.

20. Let any two lines parallel to the axis of a parabola cut the parabola in points A and B. Let a third parallel midway between these two cut the parabola at M. Prove that the tangent at M is parallel to the chord AB.

21. Prove that the product of the distances from the foci of a hyperbola to any tangent line is constant. Prove the analogous theorem for an ellipse.

22. Prove that the equilateral hyperbolas $x^2 - y^2 = h$ and $xy = k (hk \neq 0)$ intersect at right angles. Sketch several members of each family.

23. Show that the normal at any point of an ellipse bisects the internal angle formed by the focal radii to the point. State and prove an analogous theorem for a hyperbola.

Find any maximum and minimum points and points of inflection (Ex. 24 to 33).

24. $y = x^3 - 9x^2 + 24x - 16$ (Sketch the curve.)
25. $y = 12x - x^3$ (Sketch the curve.)
26. $y = x^2(8 - x^2)$ (Sketch the curve.)
27. $(y - 3)^3 = x - 4$ (Sketch the curve.)
28. $x + 1 = (y + 2)^3$
29. $y = x^4 - 2x^2$
30. $y = x^3 - 3x^2 - x + 3$
31. $y = 1 + 6x - 3x^2 - 2x^3$
32. $y = (x - 1)^{3/5}$
33. $y = (x - 1)^{4/5}$

In Exercises 34 to 37, determine the coefficients so that the given conditions will be satisfied.

34. $y = ax^3 + bx^2 + cx + d$ is to have a minimum at $(3, -20)$ and an inflection at $(1, -4)$.

35. $y = ax^3 + bx^2 + cx + d$ is to have a point of inflection with horizontal tangent at $(1,0)$ and is to pass through $(2,1)$.

36. $y = ax^4 + bx^3 + cx^2 + dx + e$ is to have critical points at $(2, -\frac{2}{3})$, $(-2, -\frac{34}{3})$ and an inflection at $x = -1$.

37. $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is to have points of inflection with horizontal tangents at $(1,1)$ and $(-1, -15)$.

38. Draw a smooth curve through the points indicated in the table, and construct the first derived curve.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
y	-5	-3	-1.5	-0.7	-0.2	0	0.2	0.6	1.7	2.9	3.7	4	3.4	2.2	0.6	-1.5

39. Same as Exercise 38 for

x	-3	-2	-1	0	1	2	3	4	5	6	7	8
y	2.5	0.8	0.3	0	0.3	1	2.7	8	3.6	2	0	-5

40. Suppose the y' curve is tangent to the x axis at a minimum point, $x = x_0$. Describe the behavior of the y curve in the neighborhood of $x = x_0$. The same if the y' curve is tangent to the x axis at a maximum point. Suppose the y curve has a vertical tangent at $x = x_1$; discuss the behavior of the y' curve in the neighborhood of $x = x_1$.

41. What number exceeds its square by the greatest amount?

42. The sum of two numbers is 12. The sum of twice one number and the square of the other is a minimum. Find the numbers.

43. A rectangular bin 100 cu. ft. in volume has a square base. The material in the bottom and back costs 10 cents per square foot; that in the top, front, and sides costs 30 cents per square foot. Find the most economical dimensions.

44. An open box is to be made from a rectangular piece of cardboard 8 by 3 in. by cutting equal squares out of the corners and folding up the sides. Find the dimensions of the box of maximum volume that can be made in this way.

45. A cylindrical glass jar of given volume has a metal top. The metal costs two-thirds as much per square inch as the glass. Find the most economical proportions.

46. Find the proportions of the right circular cone of given volume and minimum convex surface area.

47. A gutter is to be made from a strip of sheet iron 16 in. wide by bending up the sides at an angle of 45 deg. Find the width of the base for the maximum carrying capacity.

48. Find the altitude of the cone of maximum convex surface that can be inscribed in a given sphere of radius R.

49. A powerhouse stands on one side of a (straight) river a miles wide, and a factory stands on the opposite side b miles downstream. Find the most economical route for the connecting cable if it costs l dollars per mile on land and w dollars per mile under water. Does it matter how far down the river the factory is? Discuss fully.

50. A tent with given wall surface area is to be constructed in the form of a regular quadrangular pyramid. Find the ratio of its altitude to the side of the base if the air space inside the tent is to be a maximum.

51. The lower corner of a page whose width is a is folded over so as just to reach the inner edge of the page. Find the width of the part folded over (*a*) when the length of the crease is a minimum; (*b*) when the area folded over is a minimum.

52. It is known that the *bending moment* at a point in a beam of length l at a distance x units from the end of the beam, uniformly loaded, is given by the formula

$$M = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2$$

where w is the load per unit length. Show that the maximum bending moment is at the center of the beam.

53. A rectangular strip of sheet iron a ft. wide is to be bent to form a gutter whose cross section is a semicircle surmounted by a rectangle (U-shaped). What dimensions yield maximum cross-sectional area if the drain is (*a*) open on the top? (*b*) Closed on the top?

54. If a lens has a focal length f_0, there will be a correct focus when $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{f_0}$ where x is the distance from object to lens and y is the distance from lens to image. Find the least possible distance between object and image for a correct focus.

55. One end of a ladder 15 ft. long leans against a vertical wall; the other end is on a horizontal pavement. If the foot of the ladder is pulled away from the wall at the rate of 4 ft. per second, how fast is the top descending when it is 12 ft. above the pavement?

56. One ship was sailing north at 6 knots, another west at 8 knots. At 3:00 P.M. the second crossed the course of the first at the point where the first was at 1:00 P.M. At what rate was the distance between the ships changing at (*a*) 2:00 P.M.? (*b*) 4:00 P.M.? *Note:* A knot is one nautical mile per hour. Assume the ocean surface to be a plane.

57. A ship is anchored in 20 ft. of water. The anchor chain passes through an opening in the bow 10 ft. above the surface of the water. If the chain is pulled in at the rate of 6 in. per second, how fast is the ship moving when there are 50 ft. of chain out?

58. A solution is poured into a conical filter 12 cm. across the top and 24 cm. deep at the rate of 2 cc. per second. It filters out at the rate of 1 cc. per second. How fast is the level of the solution rising when it is 8 cm. deep?

59. A light is 30 ft. above the center of a street 72 ft. wide. A man 6 ft. tall walks along the sidewalk at 4 ft. per second. How fast is his shadow lengthening when he is 48 ft. up the street? How fast is the tip of his shadow moving?

60. A spherical balloon is being filled with gas at the rate of 100 cu. ft. per minute. (*a*) How fast is the surface area increasing when the radius is 6 ft.? (*b*) How fast is the radius increasing at this instant?

CHAPTER 6

DIFFERENTIATION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Of the elementary transcendental functions the *trigonometric*, or *circular*, functions are perhaps the most familiar to the student. A brief review of the definitions of these functions and the construction of their graphs will indicate to him that $\sin x$ and $\cos x$ are continuous for all values of x, while $\tan x$ and $\sec x$ are continuous for all values of x except odd multiples of $\pi/2$, for which values of x these functions have infinite discontinuities. Similarly, $\cot x$ and $\csc x$ are continuous for all x except multiples of π, for which values of x these functions have infinite discontinuities. The trigonometric functions are *periodic*, $\sin x$, $\cos x$ and their reciprocals $\csc x$, $\sec x$ with period 2π, and $\tan x$ and its reciprocal $\cot x$ with period π.

40. Evaluation of an Important Limit. Before we can find the derivative of $\sin x$, it is necessary to evaluate

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$$

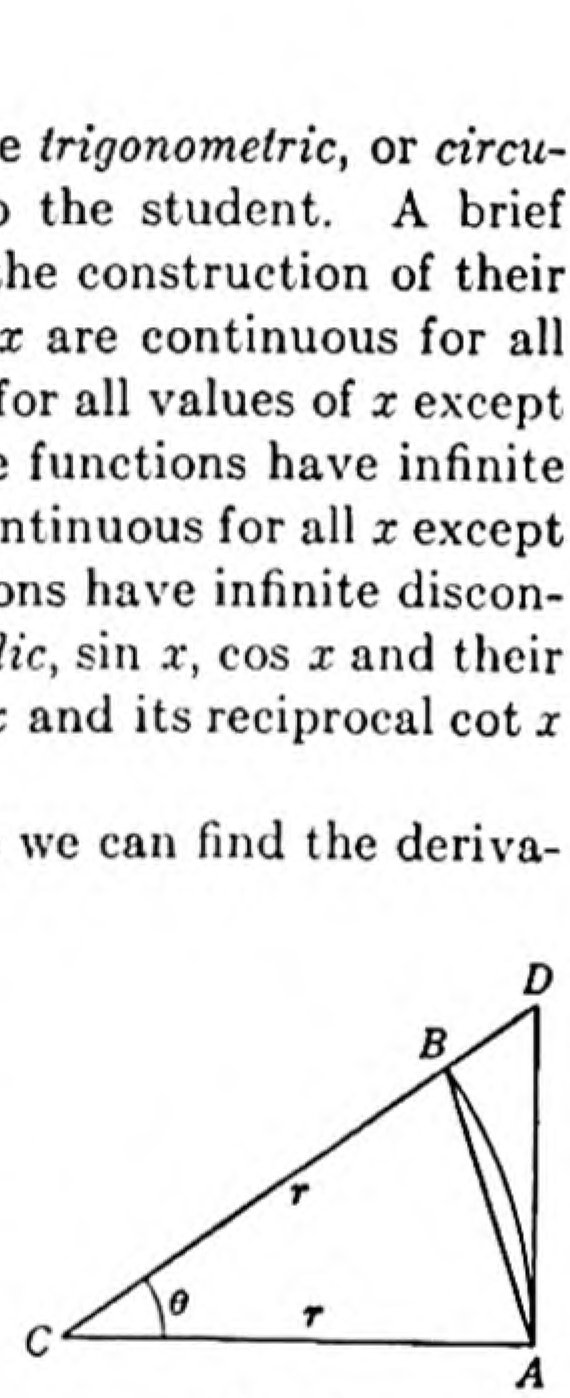


FIG. 45.

where θ is measured in *circular*, that is to say *radian*, measure. To this end, consider a circular arc AB of radius r (Fig. 45) which subtends a positive acute angle θ (measured in radians) at the center C. Draw the chord AB; draw the tangent at A, and extend it until it meets CB at D. Then AD is perpendicular to CA. We have at once

$$\begin{aligned}
&\text{Area of } \triangle CAB = \tfrac{1}{2}CA \cdot CB \sin \theta = \tfrac{1}{2}r^2 \sin \theta \\
&\text{Area of sector } CAB = \tfrac{1}{2}r^2\theta \\
&\text{Area of } \triangle CAD = \tfrac{1}{2}CA \cdot AD = \tfrac{1}{2}CA \cdot CA \tan \theta = \tfrac{1}{2}r^2 \tan \theta
\end{aligned}$$

We have area of $\triangle CAB <$ area of sector $CAB <$ area of $\triangle CAD$, that is, $\frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan \theta$. If each member of an inequality is divided by a positive number, the sense of the inequality remains unchanged. Therefore, dividing by $\frac{1}{2}r^2 \sin \theta$ ($\sin \theta$ is positive since θ is a positive acute angle),

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

that is

$$1 > \frac{\sin \theta}{\theta} > \cos \theta \tag{1}$$

So far we have required θ to be positive. However, if θ' is a negative acute angle, say $\theta' = -\theta$, we have

$$\sin \theta' = \sin(-\theta) = -\sin \theta$$

hence

$$\frac{\sin \theta'}{\theta'} = \frac{-\sin \theta}{-\theta} = \frac{\sin \theta}{\theta}$$

Furthermore

$$\cos \theta' = \cos(-\theta) = \cos \theta$$

and therefore

$$1 > \frac{\sin \theta'}{\theta'} > \cos \theta'$$

That is, (1) holds also for negative acute angles, that is to say for all $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, except $\theta = 0$ for which $\frac{\sin \theta}{\theta}$ is not defined. Consequently, no matter how θ is made to approach zero, $\frac{\sin \theta}{\theta}$ is always contained between 1 and $\cos \theta$. Since $\cos \theta$ is continuous, $\lim\limits_{\theta \to 0} \cos \theta = \cos 0 = 1$ (see Exercise 16, page 28), and therefore $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta}$ must also be 1.*

Thus

★
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \tag{2}$$

If the angle θ is measured in degrees instead of radians, a slight modification is necessary. Let the circular (radian) measure of the angle be θ, and let its measure in degrees be α. Since 1 radian equals $180/\pi$ deg., we have $\alpha = \frac{180}{\pi}\theta$. Note that $\sin \alpha = \sin \theta$. Therefore

$$\frac{\sin \alpha}{\alpha} = \frac{\sin \theta}{\frac{180}{\pi}\theta} = \frac{\pi}{180}\frac{\sin \theta}{\theta}$$

* To apply directly the definition of a limit, we have, subtracting 1 from each member of the inequality (1),

$$0 > \frac{\sin \theta}{\theta} - 1 > \cos \theta - 1$$

and therefore

$$0 < \left| \frac{\sin \theta}{\theta} - 1 \right| < |\cos \theta - 1|$$

Now $|\cos \theta - 1|$ can be made arbitrarily small by taking θ close enough to zero. Hence $\left| \frac{\sin \theta}{\theta} - 1 \right|$ can be made arbitrarily small by taking θ close enough to zero. Therefore $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

and
$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = \lim_{\theta \to 0} \frac{\pi}{180} \frac{\sin \theta}{\theta} = \frac{\pi}{180}$$

41. Derivative of the Sine. If $y = \sin x$, where x is in circular measure, we may calculate the derivative by the method of Art. 14. We have

$$\begin{aligned} y &= \sin x \\ y + \Delta y &= \sin (x + \Delta x) \\ \Delta y &= \sin (x + \Delta x) - \sin x \\ \frac{\Delta y}{\Delta x} &= \frac{\sin (x + \Delta x) - \sin x}{\Delta x} \end{aligned} \tag{3}$$

We must now evaluate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, but the quotient $\frac{\Delta y}{\Delta x}$ is in a form that requires further reduction before the limit can be conveniently found. As a means of making such a reduction, recall the trigonometric identity

$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta)$$

In (3), let $\alpha = x + \Delta x$, and let $\beta = x$. This gives

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{2 \cos \frac{1}{2}(2x + \Delta x) \sin \frac{1}{2} \Delta x}{\Delta x} \\ &= \cos \left(x + \frac{\Delta x}{2} \right) \frac{\sin \frac{1}{2} \Delta x}{\frac{1}{2} \Delta x} \end{aligned}$$

whence

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \cos \left(x + \frac{\Delta x}{2} \right) \cdot \lim_{\Delta x \to 0} \frac{\sin \frac{1}{2} \Delta x}{\frac{1}{2} \Delta x} \\ &= \cos \lim_{\Delta x \to 0} \left(x + \frac{\Delta x}{2} \right) \cdot 1 \qquad (4) \\ &= \cos x \end{aligned}$$

In case $y = \sin u$ where u is a function of x having a derivative $\frac{du}{dx}$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

whence

★
$$\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$$

Note that, if x were measured in degrees, then in (4),

$$\lim_{\Delta x \to 0} \frac{\sin \frac{1}{2} \Delta x}{\frac{1}{2} \Delta x} = \frac{\pi}{180} \qquad \text{and} \qquad \frac{d}{dx} (\sin x) = \frac{\pi}{180} \cos x$$

In order to avoid the cumbersome factor $\pi/180$, we shall understand that x is *always expressed in circular measure* whenever trigonometric functions of x are in question.

42. Derivatives of the Other Trigonometric Functions. Since

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

and $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ for all values of x, we have

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \cos\left(\frac{\pi}{2} - x\right)\cdot(-1) = -\sin x$$

Or if u is a function of x with derivative $\frac{du}{dx}$,

★ $$\frac{d}{dx}(\cos u) = -\sin u\,\frac{du}{dx}$$

Since $\tan x = \frac{\sin x}{\cos x}$, we may apply the formula for the derivative of the quotient of two functions, thus:

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Or, in general

★ $$\frac{d}{dx}(\tan u) = \sec^2 u\,\frac{du}{dx}$$

Since $\cot x = \frac{1}{\tan x} = (\tan x)^{-1}$, we may apply the formula for the derivative of a power of a function of x,

$$\frac{d}{dx}(\cot x) = -(\tan x)^{-2}\sec^2 x = -\frac{\sec^2 x}{\tan^2 x} = -\csc^2 x$$

or, in general

★ $$\frac{d}{dx}(\cot u) = -\csc^2 u\,\frac{du}{dx}$$

Since $\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$, we have

$$\frac{d}{dx}(\sec x) = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x\tan x$$

or

★ $$\frac{d}{dx}(\sec u) = \sec u\tan u\,\frac{du}{dx}$$

Since $\csc x = \frac{1}{\sin x} = (\sin x)^{-1}$, we have

$$\frac{d}{dx}(\csc x) = -(\sin x)^{-2}\cos x = -\frac{\cos x}{\sin^2 x} = -\csc x\cot x$$

or

$$\star \qquad \frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

Example 1. Find $\frac{dy}{dx}$ if $y = \sin\left(\frac{3}{5}x + \frac{\pi}{4}\right)$. Here $u = \frac{3}{5}x + \frac{\pi}{4}$ so that $\frac{du}{dx} = \frac{3}{5}$; therefore $\frac{dy}{dx} = \frac{3}{5}\cos\left(\frac{3}{5}x + \frac{\pi}{4}\right)$.

Example 2. Find $\frac{dy}{dx}$ if $y = \cos^3 x$. Here $y = v^3$ where $v = \cos x$. The derivative of the cube of a function is three times the square of the function times the derivative of the function, or, in symbols,

$$\frac{dy}{dx} = 3v^2\frac{dv}{dx} = 3\cos^2 x\frac{d}{dx}(\cos x)$$

Thus

$$\frac{dy}{dx} = 3\cos^2 x\,(-\sin x) = -3\cos^2 x \sin x$$

Example 3. Find $\frac{dy}{dx}$ if $y = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax$ where a is a constant. We have

$$\frac{dy}{dx} = \frac{1}{a}\frac{d}{dx}(\sin ax) - \frac{1}{3a}\frac{d}{dx}(\sin^3 ax)$$

This yields at once $\frac{d}{dx}\sin ax = (\cos ax)a = a\cos ax$;

$$\begin{aligned}\frac{d}{dx}(\sin^3 ax) &= (3\sin^2 ax)\cdot\frac{d}{dx}(\sin ax)\\ &= 3\sin^2 ax\cdot(\cos ax)\cdot a\\ &= 3a\sin^2 ax\cdot\cos ax\end{aligned}$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \cos ax - \sin^2 ax\cdot\cos ax\\ &= \cos ax\,(1 - \sin^2 ax)\\ &= \cos^3 ax\end{aligned}$$

Example 4. If $y = x^3\tan^2\frac{x}{4}$, find $\frac{dy}{dx}$. This is the product of two functions of x, namely, x^3 and $\tan^2(x/4)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= x^3\frac{d}{dx}\left(\tan^2\frac{x}{4}\right) + \tan^2\frac{x}{4}\frac{d}{dx}(x^3)\\ &= x^3\left(2\tan\frac{x}{4}\right)\cdot\left(\sec^2\frac{x}{4}\right)\cdot\left(\frac{1}{4}\right) + \left(\tan^2\frac{x}{4}\right)\cdot 3x^2\\ &= \frac{1}{2}x^2\tan\frac{x}{2}\left(x\sec^2\frac{x}{4} + 6\tan\frac{x}{4}\right)\end{aligned}$$

EXERCISES

Find the derivative of each of the following functions with respect to the variable indicated (Ex. 1 to 36):

1. $y = \sin 4x$

2. $y = \sin\left(2x + \frac{\pi}{3}\right)$

3. $y = \tan(ax^3)$

4. $y = \sec(x/2)$

5. $y = \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)$

6. $y = \cot 6x$

7. $y = \csc(x + k)$

8. $y = \csc x^3$

9. $y = x \cos x$

10. $z = u^2 \sin 3u$

11. $w = v \cot(v/2)$

12. $h = \sin^2 r$

13. $z = \cos^3 4y$

14. $w = x^2 \tan^2 x$

15. $y = \tan^2 x + \cot^2 x$

16. $z = \cos^2 5x$

17. $y = \tan^4 3\theta$

18. $s = \cot^3\left(\frac{t}{3} + \alpha\right)$

19. $r = \theta + \cot\theta$

20. $r = \tan 3\theta - 3\theta$

21. $r = \sin\theta \tan\theta$

22. $s = \cos 3t \cot 3t$

23. $s = \sec\left(\frac{\pi}{4} - t\right)$

24. $r = \theta^2 \sin^3 4\theta$

25. $y = \frac{1}{x}\cot^2 x$

26. $y = \frac{\cos 3x}{4x}$

27. $y = \frac{\sin^2 x}{x^2}$

28. $y = \sqrt{1 + \sin x}$

29. $y = x(1 + \cot x)^3$

30. $y = \frac{1 - \tan^2 2x}{\tan 2x}$

31. $y = (1 - \cos 4z)^{3/2}$

32. $y = \frac{\sin 3z}{1 + \sin 3z}$

33. $y = \frac{\cos^2 x}{1 + \sin^2 x}$

34. $z = \sin^3 w^2$

35. $y = \sqrt{\frac{\sec x + \tan x}{\sec x - \tan x}}$ (*Hint:* Rationalize the denominator.)

36. $y = \sqrt{\frac{\csc x + \cot x}{\csc x - \cot x}}$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (Ex. 37 to 46).

37. $x = a\cos\theta$
$y = b\sin\theta$

38. $x = a\sin(2t + 3)$
$y = a\cos(2t + 3)$

39. $x = a\cos^3\theta$
$y = a\sin^3\theta$

40. $x = 3\sin t$
$y = 2\sin t + \cos t$

41. $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$

42. $x = a\cos^2\varphi$
$y = a\sin^2\varphi$

43. $x = a\sec\varphi$
$y = b\tan\varphi$

44. $x = a\csc\varphi$
$y = b\cot\varphi$

45. $x = a\cos(kt - \alpha)$
$y = b\sin(kt - \alpha)$

46. $x = a(\cos\theta + \theta\sin\theta)$
$y = a(\sin\theta - \theta\cos\theta)$

Find any maximum or minimum values of the functions (Ex. 47 to 51).

47. $y = \sin^2 x$; also sketch the curve, and find points of inflection.

48. $y = 3 \sin x + 4 \cos x$

49. $y = a \sin x + b \cos x$. Verify the result by expressing y in the form

$$y = A \cos (x - \alpha)$$

(*Hint:* Multiply and divide $a \sin x + b \cos x$ by $\sqrt{a^2 + b^2}$.)

50. $y = x \sin x + \cos x$ in the interval $0 \leq x < 2\pi$

51. $y = \dfrac{\sin (x + \alpha)}{\sin (x + \beta)}$

Find the angle of intersection of the following pairs of curves (Ex. 52 to 55):

52. $y = \sin 2x,\ y = \cos 2x$

53. $y = \sin x,\ y = \sin 3x$

54. $y = \cos x,\ y = \cos\left(x + \dfrac{\pi}{3}\right)$

55. $y = \tan x,\ y = \cot x$

56. Discuss the increasing and decreasing character and concavity of the curves $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \csc x$.

57. From the formula for $\sin (x + k)$, find the formula for $\cos (x + k)$ by differentiation.

58. From the formula for $\tan 2x$, find the formula for $\cos 2x$ by differentiation.

59. From the formula for $\cot 2x$, find the formula for $\sin 2x$ by differentiation.

60. From the formula for $\cos (x + k)$, find the formula for $\sin (x + k)$ by differentiation.

61. Derive the formula for $\dfrac{d}{dx} (\cos x)$ by the use of increments as was done for $\dfrac{d}{dx} (\sin x)$ in Art. 41.

62. Derive the formula for $\dfrac{d}{dx} (\tan x)$ by the use of increments.

63. Derive the formula for $\dfrac{d}{dx} (\sec x)$ by the use of increments.

64. In addition to the six trigonometric functions whose derivatives have been found, there are four functions less frequently encountered but of some practical importance. They are the *external secant*, the *versed sine*, the *coversed sine*, and the *haversine* and are defined, respectively, by

$$\begin{aligned} \text{exsec } x &= \sec x - 1 \\ \text{vers } x &= 1 - \cos x \\ \text{covers } x &= 1 - \sin x \\ \text{hav } x &= \tfrac{1}{2} \text{ vers } x = \tfrac{1}{2}(1 - \cos x) \end{aligned}$$

Obtain formulas for the derivatives of these functions.

65. (*a*) Show that the function defined by $f(x) = x \sin (1/x)$ for $x \neq 0$ and $f(0) = 0$ is continuous at $x = 0$, but that $f'(0)$ does not exist. [*Hint:* Use the "increment method" to find $f'(0)$.]

(*b*) Show that, on the other hand, if $g(x) = x^2 \sin (1/x)$ for $x \neq 0$ and $g(0) = 0$, then $g'(0) = 0$.

43. Applications to Maximum and Minimum Problems. Many problems involving maxima and minima can be solved very conveniently by expressing the function whose critical value is required in terms of

trigonometric functions of a variable angle. The following example will indicate the method of attack.

Example. A wall 7 ft. high is 5 ft. from a building. Find the length of the shortest ladder that will rest with one end on the ground and the other on the building (Fig. 46). Evidently the ladder should touch the top of the wall at B; let A be the point of rest on the ground, C the point of rest on the building, and θ the angle that the ladder makes with the horizontal. If l is the length of the ladder, it may be expressed in terms of θ. For

$$\begin{aligned} l &= AB + BC = BD \csc \theta + BF \sec \theta \\ &= 7 \csc \theta + 5 \sec \theta \qquad \text{(ft.)} \end{aligned} \tag{5}$$

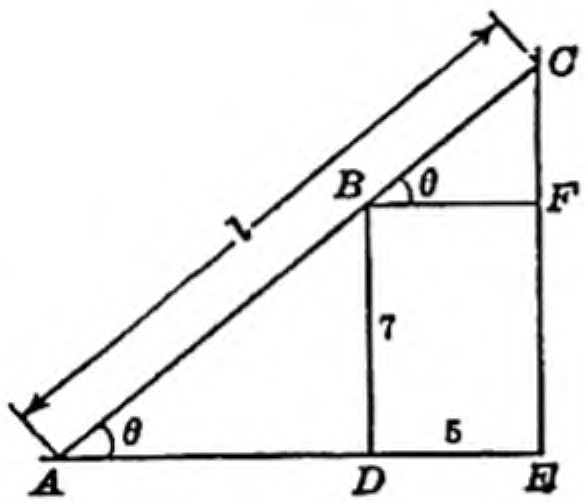


FIG. 46.

We must find θ so that l will be a minimum. We have

$$\frac{dl}{d\theta} = -7 \csc \theta \cot \theta + 5 \sec \theta \tan \theta$$

This will be zero if

$$-7 \csc \theta \cot \theta + 5 \sec \theta \tan \theta = 0$$

$$-7 \frac{\cos \theta}{\sin^2 \theta} + 5 \frac{\sin \theta}{\cos^2 \theta} = 0$$

$$-7 \cos^3 \theta + 5 \sin^3 \theta = 0$$

that is, if

$$\tan^3 \theta = \tfrac{7}{5} \qquad \text{or} \qquad \tan \theta = (\tfrac{7}{5})^{1/3}$$

Evidently, l has no maximum, and the angle θ whose tangent is $(\frac{7}{5})^{1/3}$ gives a minimum. The second derivative can be evaluated to test this conclusion if desired. It is now necessary to find the value of l corresponding to this value of θ. If $\tan \theta = (\frac{7}{5})^{1/3}$,

$$\sec \theta = \sqrt{1 + \left(\frac{7}{5}\right)^{2/3}} = \frac{\sqrt{5^{2/3} + 7^{2/3}}}{5^{1/3}}$$

$$\cot \theta = (\tfrac{5}{7})^{1/3}$$

$$\csc \theta = \sqrt{1 + \left(\frac{5}{7}\right)^{2/3}} = \frac{\sqrt{7^{2/3} + 5^{2/3}}}{7}$$

Substituting these values into (5) gives

$$\begin{aligned} l &= 7 \cdot \frac{\sqrt{7^{2/3} + 5^{2/3}}}{7^{1/3}} + 5 \cdot \frac{\sqrt{5^{2/3} + 7^{2/3}}}{5^{1/3}} \\ &= \sqrt{7^{2/3} + 5^{2/3}}\,(7^{2/3} + 5^{2/3}) = (7^{2/3} + 5^{2/3})^{3/2} \end{aligned} \tag{6}$$

This is the exact minimum value of l. It is easily approximated by use of an ordinary table of squares, square roots, cubes, and cube roots. From such a table, we find that

$$\begin{aligned} 7^{2/3} &= 49^{1/3} = 3.66 \\ 5^{2/3} &= 25^{1/3} = 2.92 \\ 7^{2/3} + 5^{2/3} &= 6.58 \\ (6.58)^{3/2} &= (2.57)^3 = 17.0 \end{aligned}$$

The shortest ladder is, therefore, approximately 17 ft. long. If only an approximation to the length of the shortest ladder is required, the angle θ could be found, first by

getting $(\frac{7}{5})^{1/3} = (1.400)^{1/3} = 1.119$, then using a table of tangents, secants, and cosecants to calculate l from (5). However, we cannot, in general, obtain the exact value of l in this way. For this reason the method resulting in expression (6) is to be preferred.

EXERCISES

1. A rectangular strip of sheet iron is bent in the middle to form a gutter. What angle between the sides gives maximum carrying capacity?

2. Find the altitude and radius of the right circular cone of maximum volume with given slant height s.

3. Solve Exercise 23, page 90.

4. Solve Exercise 24, page 90.

5. Solve Exercise 25, page 90.

6. Find the altitude h of the right circular cone of maximum convex surface inscribed in a given sphere of radius a.

7. Find the altitude h of the right circular cone of maximum volume inscribed in a given sphere of radius a (Exercise 31, page 91).

8. Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius a.

9. Find the altitude of the right circular cylinder of maximum convex surface that can be inscribed in a given sphere of radius a.

10. One side of a triangle is 20 ft. and the opposite angle is 50 deg. Find the other angles so that the area will be a maximum.

11. A wall a ft. high is b ft. from a building. Find the length of the shortest ladder that will clear the wall and rest with one end on the ground and the other on the building.

12. A steel girder 30 ft. long is moved horizontally along a passageway 10 ft. wide and around the corner into a hall at right angles to the passageway. Neglecting the width of the girder, find how wide the hall must be to permit this.

13. Two hallways are at right angles to one another, one being 10 ft. and the other 15 ft. wide. Find the length of the longest steel girder that can be moved horizontally around the corner (neglect width of girder).

14. Solve Exercise 13 if the hallways are, respectively, a and b ft. wide.

15. Show that the isosceles triangle with fixed perimeter and maximum area is equilateral.

16. The lower edge of a picture 7 ft. high is 9 ft. above the level of an observer's eye. How far should the observer stand from the wall in order to see the picture to the best advantage? (*Note:* He should stand so that the angle between the lines from his eye to the top and bottom of the picture is a maximum.) (*Hint:* An acute angle is a maximum if its tangent is a maximum.)

17. An isosceles trapezoid has its equal sides 10 in. long, and the shorter of the two parallel sides 17 in. long. Find the base angles and the longer of the two parallel sides when the area is a maximum.

18. An arc light hangs above the center of a circular enclosure of radius a. How high should the light be hung if a narrow path around the edge of the enclosure is to be most brightly lighted? It is known that the illumination varies inversely as the square of the distance from the source of light and directly as the cosine of the angle of incidence.

19. A rectangular strip of sheet iron a in. wide is to be bent to form a gutter whose cross section is an arc of a circle. What radius will give maximum carrying capacity?

20. Two sides and the included angle of a triangle are, respectively, x, y, and ϕ.

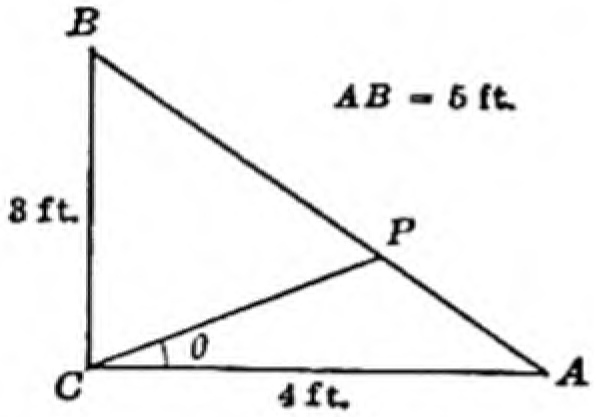
If x is increasing at the rate of 2 ft. per minute, y is increasing at 3 ft. per minute, and θ is increasing at 1 deg. per minute, find how fast the area is changing when $x = 40$ ft., $y = 75$ ft., $\theta = 30$ deg.

21. Solve Exercise 20 if x is increasing at 2 ft. per minute, y decreasing at 3 ft. per minute, and θ decreasing at 1 deg. per minute.

22. The slant height of a right circular cone is 4 in., and the angle at the vertex is 2θ. If θ is increasing at the rate of 2 deg. per second, how fast is the volume changing when $\theta = 30$ deg.?

23. The hypotenuse of a right triangle is 20 in. If one of the acute angles decreases at the rate of 5 deg. per second, how fast is the area decreasing when this angle is 30 deg.?

24. In the figure, θ is decreasing at the rate of 2 deg. per second. How fast is P moving down AB when $\theta = 30$ deg.?



44. The Inverse Trigonometric Functions. Suppose we have y a function of x defined by the equation

$$\sin y = x \tag{7}$$

This is a brief way of stating that y is the measure of an angle whose sine is x. If we wish to find the inverse function (Art. 25), we must solve this equation for y. This gives

$$y = \arcsin x \tag{8}$$

which is simply another way of saying that y is the measure of an angle whose sine is x.* The notation $\sin^{-1} x$ is frequently used for $\arcsin x$, but we shall avoid it because of possible confusion with $(\sin x)^{-1}$ and to preserve consistency in exponential notation. Since equation (8) is merely another way of writing equation (7), both equations have the same graph, which is shown in Fig. 47. Note that the graph can be obtained by "reflecting" the curve $y = \sin x$ in the line $y = x$, for by interchanging x and y we obtain the given equation $x = \sin y$. Now it is obvious that for any given x, between -1 and $+1$ inclusive, y may have infinitely many values. For example, if $x = \frac{1}{2}\sqrt{2}$, $y = \pi/4, 3\pi/4, 9\pi/4, 11\pi/4, \ldots, -5\pi/4, -7\pi/4, \ldots$. Since we prefer to deal with single-valued functions, we shall classify the values of $\arcsin x$ as follows: For a given x the value of $\arcsin x$ between $-\pi/2$ and $\pi/2$ inclusive shall be desig-

* If we draw a circle with center at the vertex of the angle whose measure is y, the arc subtended by this angle also has measure y. Therefore y is the *arc whose sine is x*. Hence the notation.

nated as the *principal value* and as belonging to the *principal branch* of the function; the values of arcsin x between $\pi/2$ and $3\pi/2$, $3\pi/2$ and $5\pi/2$, etc., between $-\pi/2$ and $-3\pi/2$, $-3\pi/2$ and $-5\pi/2$, etc., shall be designated as belonging to *other branches* of the function (see Art. 3). Thus, if $x = \frac{1}{2}\sqrt{2}$, then $y = \pi/4$ is the principal value. Our function

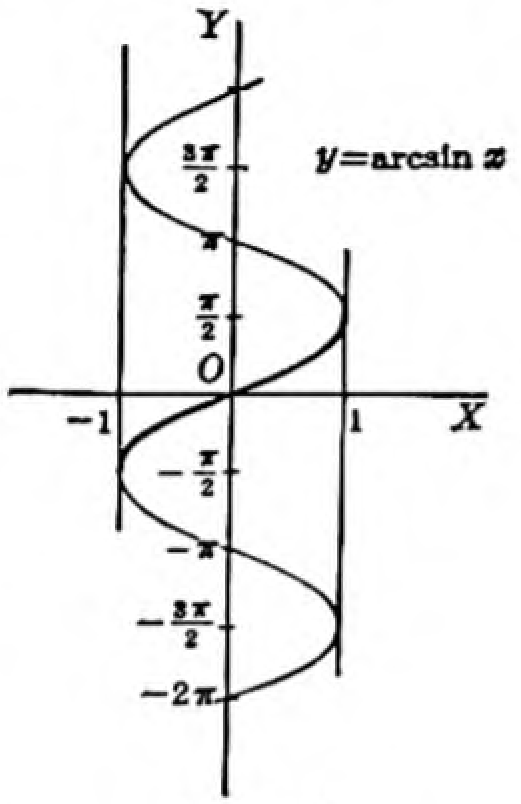


FIG. 47.

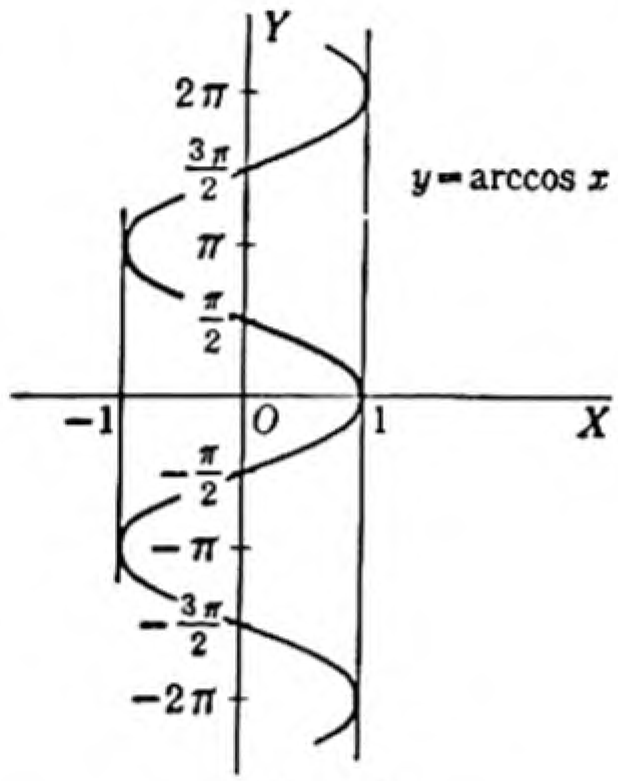


FIG. 48.

may now be thought of as consisting of infinitely many *branches*, each one of which (and in particular the principal branch) is, by itself, a single-valued function of x. The principal branch of arcsin x is shown by the heavy line in Fig. 47.

In the same way, $y = \text{arccos } x, y = \text{arctan } x, y = \text{arccot } x, y = \text{arcsec } x$, $y = \text{arccsc } x$ consist of infinitely many branches. Their graphs are

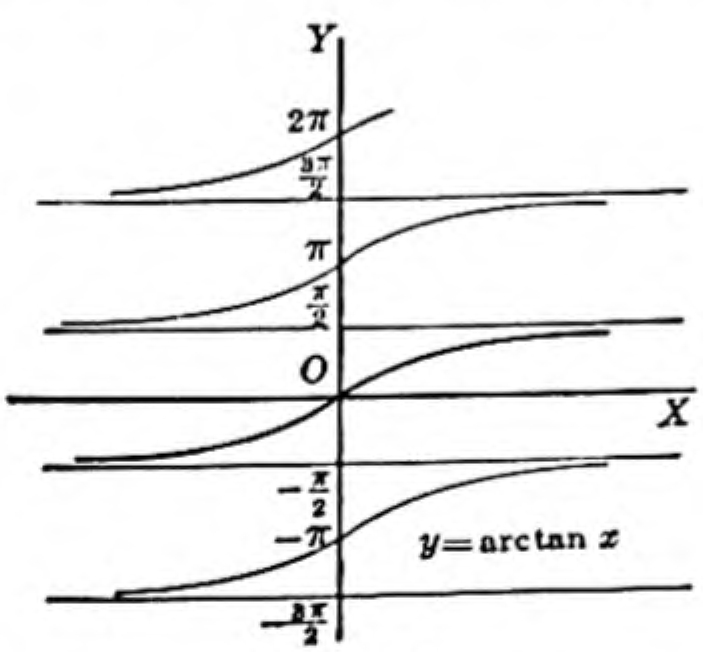


FIG. 49.

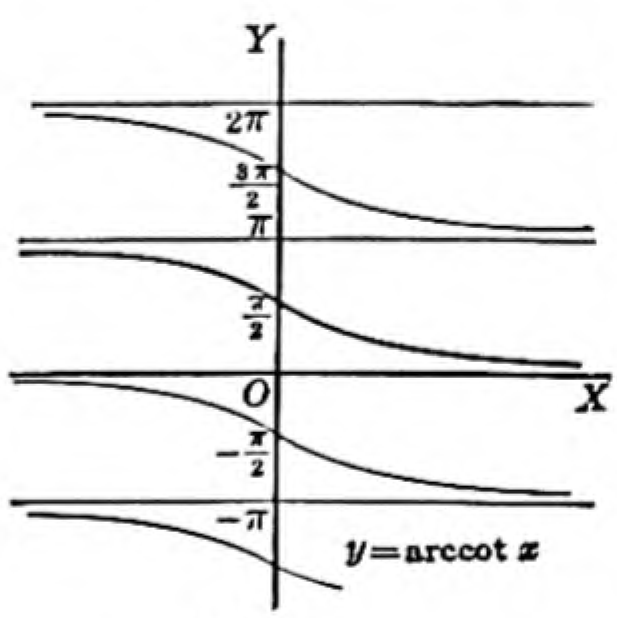


FIG. 50.

shown in Figs. 48 to 52, the principal branch being indicated in each case by a heavy line.

For future work, it is most important for the student to keep clearly in mind the fact that we shall, in general, restrict our attention to the principal values of the inverse trigonometric functions. For convenience, these are listed below.

$-\pi/2 \leqq \arcsin x \leqq \pi/2$ that is, the measure of the angle of smallest numerical value whose sine is x

$0 \leqq \arccos x \leqq \pi$ that is, the measure of the smallest positive angle whose cosine is x

$-\pi/2 < \arctan x < \pi/2$ that is, the measure of the angle of smallest numerical value whose tangent is x

$0 < \operatorname{arccot} x < \pi$ that is, the measure of the smallest positive angle whose cotangent is x

$-\pi \leqq \operatorname{arcsec} x < -\pi/2$ for $x \leqq -1$

$0 \leqq \operatorname{arcsec} x < \pi/2$ for $x \geqq 1$

$-\pi < \operatorname{arccsc} x \leqq -\pi/2$ for $x \leqq -1$

$0 < \operatorname{arccsc} x \leqq \pi/2$ for $x \geqq 1$

Thus, $\arcsin \frac{1}{2} = \pi/6$, not $5\pi/6$, etc.; $\arccos(-1) = \pi$, not 3π, etc.; $\arctan 1 = \pi/4$, not $5\pi/4$, etc.; $\operatorname{arccot} \sqrt{3} = \pi/6$, not $7\pi/6$, etc.:

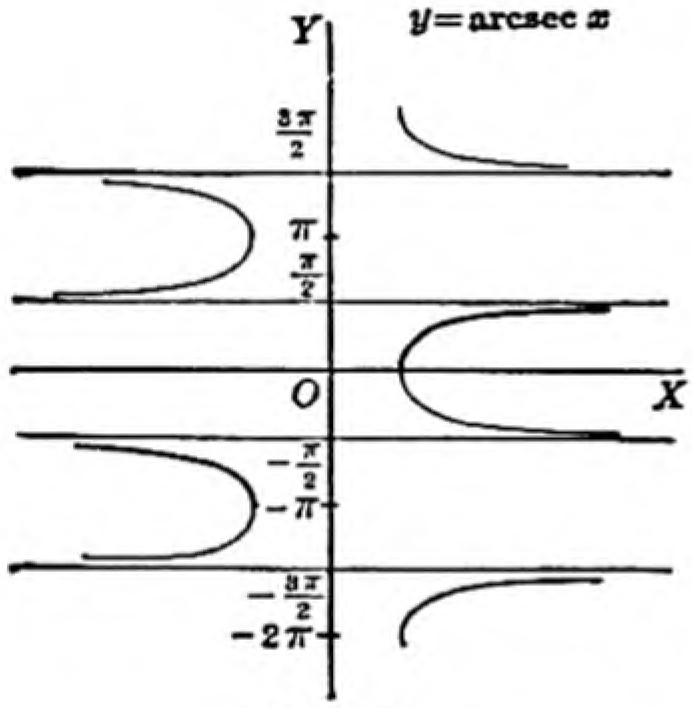


FIG. 51.

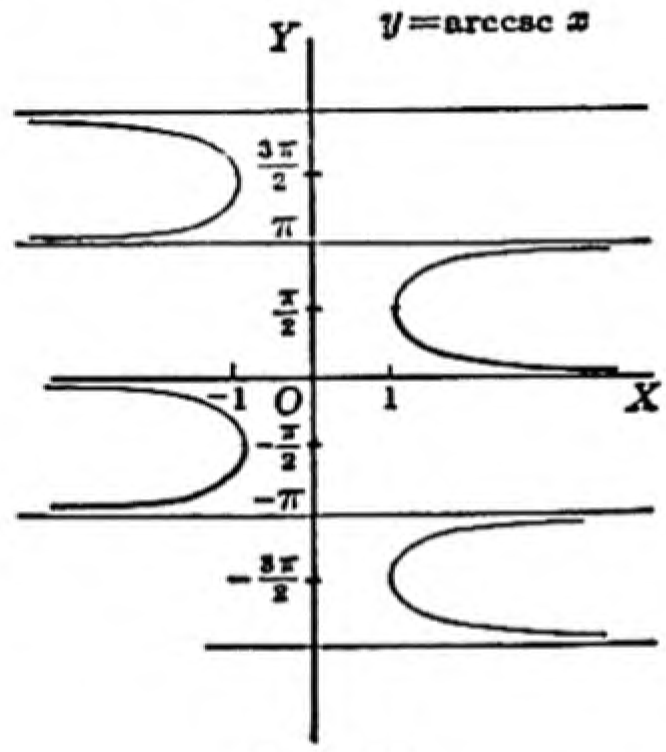


FIG. 52.

$\operatorname{arcsec}(-\sqrt{2}) = -3\pi/4$, not $3\pi/4$, etc.; $\operatorname{arccsc}(-\sqrt{2}) = -3\pi/4$, not $-\pi/4$, etc. The student should compare these statements very carefully with the graphs of the functions involved.

We shall designate as *principal values* of the inverse trigonometric functions those shown in the above list. However, it is not unusual to find a somewhat different classification, as follows: The principal value of $\arcsin x$ is defined to be the measure of the numerically smallest angle which has the same sign as x, whose sine is x. Similar definitions are given for the principal values of $\arctan x$, $\operatorname{arccot} x$, $\operatorname{arccsc} x$. These values then all lie between $-\pi/2$ and $\pi/2$. The principal value of $\arccos x$ is defined to be the measure of the smallest positive angle whose cosine is x; a similar definition is given for the principal value of $\operatorname{arcsec} x$. These values, therefore, all lie between 0 and π.* These definitions agree

* See, for example, E. W. Hobson, *A Treatise on Plane Trigonometry*, 4th ed., Cambridge University Press, New York, 1918, pp. 32–33.

with ours for arcsin x, arccos x, arctan x. For our purpose, they would have the following disadvantages in the case of the other three inverse trigonometric functions: arccot x would be discontinuous at $x = 0$; the graphs of arcsec x and arccsc x would have, respectively, a positive slope and a negative slope. The reader should check these statements with Figs. 47 to 52. Since the choice of the principal branch of any one of these functions is quite arbitrary, we make the choice that proves to be most convenient for our use.

If the student is not already well acquainted with the elementary properties of these functions, he should refer to any standard trigonometry text for review.

45. Differentiation of the Inverse Trigonometric Functions. To find the derivative of arcsin x, we use the fact that, if $y = \arcsin x$, the inverse function is $\sin y = x$. Since

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

(Art. 25), we have

$$\frac{dx}{dy} = \cos y \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

where the plus sign is taken for the square root since we shall use the principal value of y, namely, $-\pi/2 \leqq y \leqq \pi/2$, and therefore $\cos y$ is positive. Hence,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

If u is a function of x with derivative $\dfrac{du}{dx}$, we have

★ $$\frac{d}{dx}(\arcsin u) = \frac{\dfrac{du}{dx}}{\sqrt{1 - u^2}}$$

The derivatives of the other inverse trigonometric functions are found in a similar manner using principal values, as follows:

If $y = \arccos x$, then $\cos y = x$. Hence

$$\frac{dx}{dy} = -\sin y \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - x^2}}$$

Here, since $0 \leqq y \leqq \pi$, $\sin y = \sqrt{1 - x^2}$ must be positive; therefore, we take the plus sign for the square root. In general

★ $$\frac{d}{dx}(\arccos u) = -\frac{\dfrac{du}{dx}}{\sqrt{1 - u^2}}$$

If $y = \arctan x$, then $\tan y = x$,

$$\frac{dx}{dy} = \sec^2 y \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}$$

In general

★ $$\frac{d}{dx}(\arctan u) = \frac{\dfrac{du}{dx}}{1 + u^2}$$

If $y = \operatorname{arccot} x$, then $\cot y = x$,

$$\frac{dx}{dy} = -\csc^2 y \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1 + x^2}$$

In general

★ $$\frac{d}{dx}(\operatorname{arccot} u) = -\frac{\dfrac{du}{dx}}{1 + u^2}$$

If $y = \operatorname{arcsec} x$, then $\sec y = x$,

$$\frac{dx}{dy} = \sec y \tan y \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Now, if $x \geqq 1$, then $0 \leqq y < \pi/2$ and

$$\tan y = +\sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$$

If $x \leqq -1$, then $-\pi \leqq y < -\pi/2$ and again

$$\tan y = +\sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$$

Therefore, in all cases

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

In general

★ $$\frac{d}{dx}(\operatorname{arcsec} u) = \frac{\dfrac{du}{dx}}{u\sqrt{u^2 - 1}}$$

If $y = \operatorname{arccsc} x$, then $\csc y = x$,

$$\frac{dx}{dy} = -\csc y \cot y \qquad \text{and} \qquad \frac{dy}{dx} = \frac{-1}{\csc y \cot y}$$

Now, if $x \geqq 1$, then $0 < y \leqq \pi/2$ and

$$\cot y = +\sqrt{\csc^2 y - 1} = \sqrt{x^2 - 1}$$

If $x \leqq -1$, then $-\pi < y \leqq -\pi/2$ and again

$$\cot y = +\sqrt{\csc^2 y - 1} = \sqrt{x^2 - 1}$$

Therefore, in all cases

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

In general

$$* \qquad \frac{d}{dx}(\operatorname{arccsc} u) = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}$$

It is now clear why the principal branches of these various functions were chosen as explained above. The choice enables us to express the derivative of each function with no ambiguity of sign. The student should recall that $\frac{dy}{dx}$ is the slope of the curve and should again inspect Figs. 47 to 52.

Example 1. Find $\frac{dy}{dx}$ if $y = \arcsin\frac{x^2}{3}$.

$$\frac{dy}{dx} = \frac{2x/3}{\sqrt{1-\frac{x^4}{9}}} = \frac{2x}{\sqrt{9-x^4}}$$

Example 2. Find $\frac{dy}{dx}$ if $y = \arctan\sqrt{1+x}$.

$$\frac{dy}{dx} = \frac{1/(2\sqrt{1+x})}{1+(1+x)} = \frac{1}{2(2+x)\sqrt{1+x}}$$

Example 3. Find $\frac{dy}{dx}$ if $y = x^2 \operatorname{arcsec} x^2$.

$$\frac{dy}{dx} = x^2\cdot\frac{2x}{x^2\sqrt{x^4-1}} + 2x\operatorname{arcsec} x^2 = 2x\left(\frac{1}{\sqrt{x^4-1}} + \operatorname{arcsec} x^2\right)$$

Example 4. Find $\frac{dy}{dx}$ if

$$y = \frac{1}{2}\left[(x-a)\sqrt{2ax-x^2} + a^2\arcsin\frac{x-a}{a}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[(x-a)\cdot\frac{a-x}{\sqrt{2ax-x^2}} + \sqrt{2ax-x^2} + a^2\cdot\frac{1/a}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{-(a-x)^2+2ax-x^2}{\sqrt{2ax-x^2}} + \frac{a^2}{\sqrt{a^2-(x-a)^2}}\right]$$

$$= \frac{1}{2}\left(\frac{-a^2+2ax-x^2+2ax-x^2+a^2}{\sqrt{2ax-x^2}}\right) = \sqrt{2ax-x^2}$$

Example 5. A ladder 20 ft. long leans against a vertical wall. If the top slides downward at the rate of 4 ft. per second, how fast is the angle between the ladder

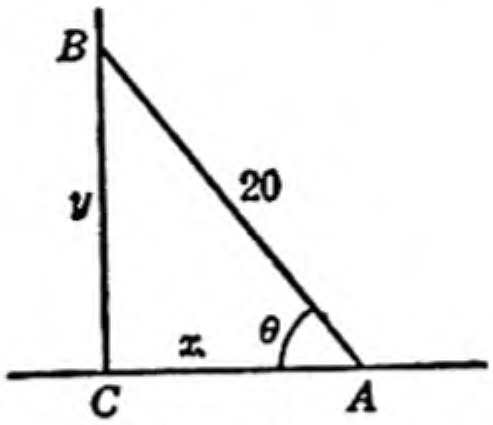


FIG. 53.

and the floor changing when the bottom end of the ladder is 16 ft. from the wall? In Fig. 53, we have given $\frac{dy}{dt} = -4$ ft. per second. We wish to find $\frac{d\theta}{dt}$ at the instant when $x = 16$. We have $\theta = \arcsin (y/20)$. Therefore

$$\frac{d\theta}{dt} = \frac{\frac{1}{20}\frac{dy}{dt}}{\sqrt{1 - \frac{y^2}{400}}} = \frac{\frac{dy}{dt}}{\sqrt{400 - y^2}} = \frac{1}{x}\frac{dy}{dt}$$

at any time t. In particular, when $x = 16$,

$$\left.\frac{d\theta}{dt}\right]_{x=16} = \frac{-4 \text{ (ft./sec.)}}{16 \text{ (ft.)}} = -\frac{1}{4} \text{ radian/sec.}$$

Hence, the angle is decreasing at the rate of $\frac{1}{4}$ radian per second at the instant in question.

EXERCISES

Find the derivative of each of the following functions (Ex. 1 to 42):

1. $y = \arcsin 4x$
2. $z = \arcsin (x/5)$
3. $z = \arccos (3x/4)$
4. $y = \arccos \sqrt{x}$
5. $y = \arctan x^2$
6. $y = \frac{1}{a} \arctan \frac{x}{a}$
7. $y = \operatorname{arccot} \sqrt{x/3}$
8. $\theta = \operatorname{arccot} (1/x)$
9. $\varphi = \operatorname{arcsec} (1/x)$
10. $\theta = \operatorname{arcsec} (3x/5)$
11. $z = \operatorname{arccsc} x^3$
12. $z = \operatorname{arccsc} (a/x)$
13. $y = \arcsin \sqrt{x^2 - 1}$
14. $y = \arcsin x^4$
15. $y = \operatorname{arcsec} (3x - 5)$
16. $y = \arcsin \frac{2}{x - 1}$
17. $y = \operatorname{arccot} (x^2/a^2)$
18. $z = \operatorname{arcsec} \frac{a}{\sqrt{a^2 - y^2}}$
19. $s = \arcsin (2t - 1)$
20. $z = \arctan (\tan y)$
21. $y = \arcsin \frac{x - a}{a}$
22. $y = \frac{1}{a} \operatorname{arccot} \frac{a}{x}$

23. $y = \sqrt{\frac{2}{7}} \arctan \frac{\sqrt{7}}{\sqrt{2}} x$

24. $z = \arctan \frac{2y}{1 - y^2}$

25. $z = \arctan \frac{y + k}{1 - ky}$

26. $z = \arctan \frac{y^2 - a^2}{2ay}$

27. $y = \arcsin \left(x - \frac{1}{x} \right)$

28. $y = \operatorname{arccsc} \left(x + \frac{1}{x} \right)$

29. $y = \arcsin \frac{x^2 - a^2}{x^2 + a^2}$

30. $y = x^3 \arcsin 3x$

31. $w = u^3 \arctan (u/3)$

32. $y = \arcsin \sqrt{x/a}$

33. $y = 2x \arcsin 2x + \sqrt{1 - 4x^2}$

34. $s = k \arcsin (t/k) + \sqrt{k^2 - t^2}$

35. $y = (x^2 + a^2) \arctan \frac{x}{a} - ax$

36. $y = \sqrt{x^2 - a^2} + a \arcsin (a/x)$

37. $y = x \sqrt{a^2 - x^2} + a^2 \arcsin (x/a)$

38. $y = a \arcsin \sqrt{x/a} + \sqrt{ax - x^2}$

39. $y = 2 \sqrt{x} - 2 \arctan \sqrt{x}$

40. $w = v \sqrt{1 - v^2} + \arcsin v$

41. $y = (\arctan 3x)^2$

42. $y = (\arcsin x^2)^3$

43. Explain what modifications in the differentiation formulas of Art. 45 are needed if the inverse trigonometric functions are not restricted to their principal values.

44. A man is walking at the rate of 6 ft. per second away from the base of a tower 60 ft. high standing in a horizontal field. A sentry on top of the tower keeps a rifle pointed at the man. At what rate is the angle between the rifle and the vertical changing when the man is 80 ft. away from the base of the tower?

45. A balloon, leaving the ground 600 ft. from an observer, rises at the rate of 100 ft. per minute. At what rate is the angle of elevation of the observer's line of sight increasing after 8 min.?

46. A ladder 15 ft. long standing on a horizontal floor leans against a vertical wall. If the top slides downward at the rate of 18 in. per second, find the rate at which the angle between the floor and the ladder is decreasing when the lower end of the ladder is 9 ft. from the wall.

47. A kite is 90 ft. high with 150 ft. of string out. It is moving horizontally at 5 m.p.h. directly away from the man flying it. Assuming the string to be in a straight line, find the rate at which the angle between the string and the horizontal is changing.

48. A rotating beacon light is 300 ft. from a fence upon which it casts a spot of light. If the beacon makes two revolutions per minute, how fast is the spot moving when at the point nearest to the light? How fast when 60 ft. down the fence?

49. Solve Exercise 16, page 109.

50. A man on a wharf 20 ft. above the water pulls in, at the rate of 4 ft. per second, a rope to which is attached a small boat. Find the rate at which the angle between the rope and the surface of the water is changing when there are 25 ft. of rope out.

CHAPTER 7

DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

46. The Exponential Function. In our work so far we have made use of only *rational* powers of a number. In fact, in algebra the number a^x $(a > 0)$ has a meaning only for rational values of x. Thus, if $x = p/q$, then a^x means the qth root of the pth power of a. It becomes important to give a^x a definition for x irrational. Now it can be shown, although the proof will not be given here, that, if x takes on any sequence of rational values whose limit is an irrational number x_0, then a^x approaches a limit b. We shall define $a^{x_0} = b$. The function a^x (for a a positive constant) is now defined for all real values of x. It is continuous and can be shown to obey the familiar laws of exponents previously associated only with rational exponents, namely,

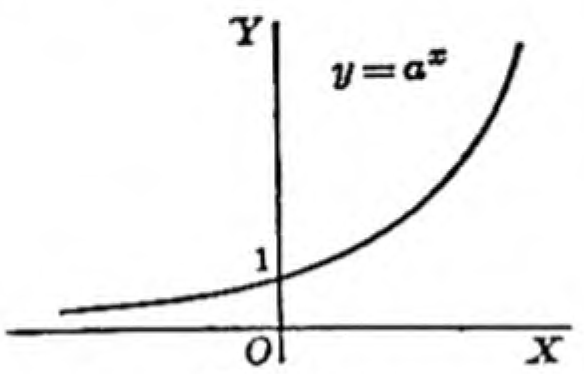


FIG. 54.

$$a^x \cdot a^z = a^{x+z} \qquad \text{and} \qquad (a^x)^z = a^{xz}$$

The graph of $y = a^x$ is shown in Fig. 54 for $a > 1$. Evidently the curve passes through the points $(0,1)$ and $(1,a)$ whatever a may be.

47. The Number e. There is a certain expression whose limit plays a very important role in mathematical analysis. This expression is $z = (1 + t)^{\frac{1}{t}}$. If values of t are assigned, z will have values as indicated in the following table:

t	1	0.1	0.01	0.001	0.0001	0.00001
z	2	2.594	2.705	2.717	2.7182	2.71827

It looks as if z might approach a limiting value whose decimal representation begins with the four digits 2.718 when $t \to 0$. This is indeed the case, although we shall not present the proof here. In fact, it can be shown that

$$\lim_{t\to 0} (1 + t)^{\frac{1}{t}} = 2.718281828459 \ldots$$

This number is denoted by e.*

It has been shown that e is a transcendental number; that is, it is not the root of an algebraic equation with rational coefficients. Since it is, therefore, not a rational number, it is not represented by a *repeating* decimal. It is also possible to show that

$$\lim_{t\to 0} (1 + xt)^{\frac{1}{t}} = e^x \dagger$$

The special case of the exponential function of Art. 46 in which we set $a = e$, giving $y = e^x$, is of great importance in our future work.

Example 1. A function of considerable importance in the theory of probability and statistics is of the form $y = ke^{-x^2}$ (k a positive constant). Note that its graph is symmetrical to the y axis, that it passes through the point $(0,k)$, and that y decreases as x increases from zero. This curve (Fig. 55) is called the *probability curve* or *normal frequency curve.*

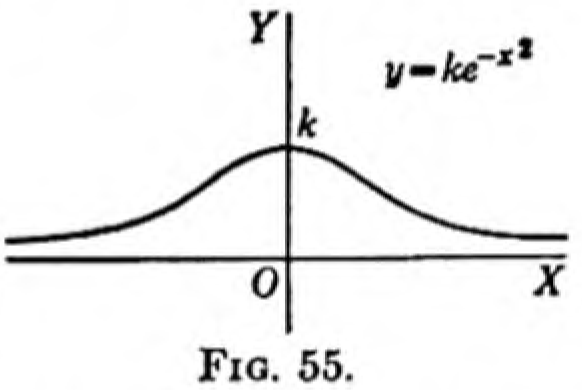


FIG. 55.

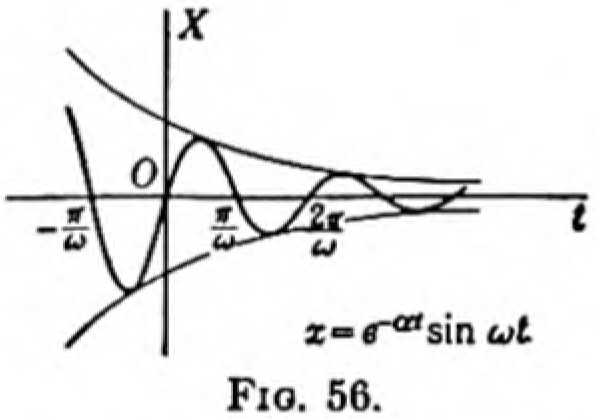


FIG. 56.

Example 2. The function $x = e^{-\alpha t} \sin \omega t$ (where α and ω are positive constants) is of importance in mechanics. To draw the graph (Fig. 56), first sketch lightly the curves $x = e^{-\alpha t}$ and $x = -e^{-\alpha t}$. Values of x corresponding to given values of t are obtained by multiplying the ordinates of $x = e^{-\alpha t}$ by $\sin \omega t$.

Since $-1 \leq \sin \omega t \leq 1$, points on our required graph lie on or between the two curves just sketched. When t is a multiple of π/ω, x is zero. When t is an odd multiple of $\pi/2\omega$, x is equal either to $e^{-\alpha t}$ or to $-e^{-\alpha t}$.

Example 3. If $y = e^{\frac{1}{x}}$, we note that for $x = 0$ the function is not defined. When $x \to -\infty$, $y \to 1$ (remaining always less than 1); and as x increases through *negative* values toward zero, y decreases and approaches zero as $x \to 0^-$. (Consider, for instance, $x = -\frac{1}{100}$; then $1/x = -100$ and $y = e^{-100}$, which is very small.)

On the other hand, when $x \to 0^+$, $y \to +\infty$. As x increases through positive values, y decreases; and as $x \to +\infty$, $y \to 1$ (remaining always greater than 1). The graph is shown in Fig. 57.

* It is possible to calculate e to as many places of decimals as desired. It has, for instance, been calculated to 346 places by J. M. Boorman, *Mathematical Magazine*, vol. 1, p. 204, 1884. Modern computing machines provide a means of making such calculations readily.

† Proofs of this and of most of the other statements in Arts. 46 and 47 are readily accessible, for instance, in Lloyd L. Smail, *Elements of the Theory of Infinite Processes*, McGraw-Hill Book Company, Inc., New York, 1925, pp. 38ff.

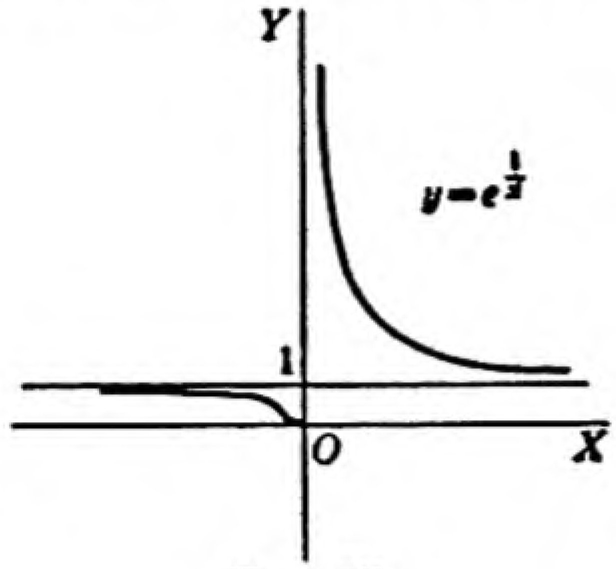


FIG. 57.

EXERCISES

In Exercises 1 to 25, sketch the graphs of the functions.

1. $y = x^2$ and $y = 2^x$ on the same scale; compare the appearance of the two graphs.
2. $y = 2^{-x}$
3. $y = 2^{x^2}$
4. $y = 3^{2x}$; compare with $y = 2^{3x}$.
5. $y = 10^x$; compare with $y = 2^x$.
6. $y = (\frac{1}{3})^x$; compare with $y = 3^x$.
7. $y = x \cdot 2^x$
8. $y = \frac{1}{2}(e^x - e^{-x})$
9. $y = \frac{1}{2}(e^x + e^{-x})$
10. $x = e^{-t} \sin t$
11. $x = e^{-t} \cos t$
12. $x = e^t - \sin t$
13. $y = 10^{\frac{1}{x}}$
14. $y = 10^{-\frac{1}{x}}$
15. $y = x \cdot e^{-x}$
16. $y = x^2 \cdot e^{-x}$
17. $y = e^{1-x}$
18. $y = e^{x-1}$
19. $y = e^{\frac{1}{1-x}}$
20. $y = e^{\frac{1-x}{1+x}}$
21. $y = e^{\frac{1+x}{1-x}}$
22. $y = \dfrac{20}{1 + e^x}$
23. $y = \dfrac{20}{1 + e^{-x}}$
24. $y = \dfrac{20}{1 + e^{\frac{1}{x}}}$
25. $y = \dfrac{20}{1 + e^{-\frac{1}{x}}}$

26. On the same set of axes, draw graphs of $y = a^x$ for $a = \frac{1}{2}$, $a = \frac{1}{4}$, $a = 1$, $a = 2$, $a = 4$. Describe the effect of the value of a upon the appearance of the curve.

48. Logarithms. The student has become acquainted in earlier courses with the use of *logarithms* for the purpose of making calculations. We shall define the logarithmic function to be the *inverse* of the exponential function. Thus, if

$$x = a^y$$

then

$$y = \log_a x$$

For convenience we shall require a, called the *base*, to be greater than 1. Thus, the *logarithm of x to the base a* is the *exponent* of the power to which a must be raised to produce x. Since $a^y = x$, it is at once clear that, if y is to be real, x must be positive. Other properties are also evident:

Negative numbers do not have real logarithms.

If $0 < x < 1$, $\log_a x < 0$.

If $x = 1$, $\log_a x = 0$.
If $x > 1$, $\log_a x > 0$.
As $x \to 0^+$, $\log_a x \to -\infty$.
As $x \to +\infty$, $\log_a x \to +\infty$.

The function is single-valued and continuous for all positive x. The graph is shown in Fig. 58.

There are only two bases ordinarily used for systems of logarithms, namely, 10 and e. Logarithms to the base 10, called *common logarithms*, are extraordinarily handy for making computations, for the fractional part of the logarithm (mantissa) depends only upon the sequence of digits in the given number and not upon the position of the decimal point. However, logarithms to the base e, called *natural* or *napierian logarithms*, are much more convenient in mathematical analysis for many reasons, some of which will presently appear. Consequently, we shall make use of natural logarithms in the major part of our work and shall always write $\ln x$ instead of $\log_e x$. We shall ordinarily write $\log x$ instead of $\log_{10} x$ for the common logarithm of x.

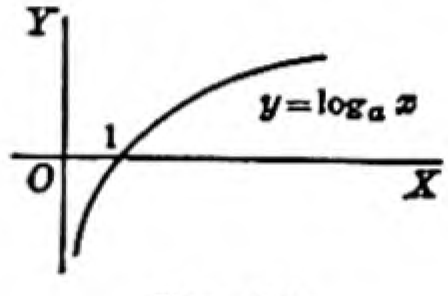


FIG. 58.

In case we have available a table of common logarithms and wish to find the natural logarithm of a given number, we may use the formula for change of base:

$$\log_b x = (\log_b a) \log_a x \tag{1}$$

Hence
$$\ln x = (\ln 10) \log x$$
Similarly
$$\log x = (\log e) \ln x$$

Note that if in (1) we take $x = b$, we have

$$1 = \log_b b = (\log_b a) \log_a b$$

so that
$$\log_a b = \frac{1}{\log_b a} \tag{2}$$

The number $M = \log e = 0.43429$ (to five places of decimals) is called the *modulus* of the system of common logarithms; we shall follow the usual custom and denote it by M. Note that, from (2),

$$\ln 10 = 1/M = 2.30259$$

(to five places).

49. Derivative of the Logarithm of x. In order to find the derivative of $\log_a x$, we proceed by the familiar method, as follows:

$$y + \Delta y = \log_a (x + \Delta x)$$
$$\Delta y = \log_a (x + \Delta x) - \log_a x$$
$$= \log_a \frac{x + \Delta x}{x} = \log_a \left(1 + \frac{\Delta x}{x}\right)$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$

We cannot evaluate this limit directly, and so we resort to a special device. Multiplying numerator and denominator by x, we have

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{1}{x}\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)\\ &= \frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\end{aligned}$$

and therefore

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{1}{x}\lim_{\Delta x\to 0}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

To evaluate this limit, we note that, if we set $t = \frac{\Delta x}{x}$, then, as $\Delta x \to 0$, $t \to 0$. Hence the expression on the right-hand side becomes

$$\frac{1}{x}\lim_{t\to 0}\log_a(1+t)^{\frac{1}{t}}$$

Since the logarithm is a continuous function, this is equal to

$$\frac{1}{x}\log_a\lim_{t\to 0}(1+t)^{\frac{1}{t}} = \frac{1}{x}\log_a e$$

Therefore $\frac{dy}{dx} = \frac{1}{x}\log_a e$. In general, if u is a differentiable function of x,

★ $$\frac{d}{dx}(\log_a u) = \frac{1}{u}(\log_a e)\frac{du}{dx}$$

If we take e as our base of logarithms, $\log_a e = \ln e = 1$ and

★ $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

This last result indicates the convenience of using e as a base of logarithms, for the cumbersome factor $\log_a e$ reduces to unity. The student will also notice that we have not made use of the restriction that $a > 1$. The formula for the derivative of the logarithm is therefore true for any $a > 0$ except $a = 1$.

Example 1. Find $\frac{dy}{dx}$ if $y = \log(x^2 + 9)$.

$$\frac{dy}{dx} = \frac{2x}{x^2+9}\log e = \frac{2Mx}{x^2+9}$$

Example 2. Find $\frac{dy}{dx}$ if $y = \ln 57x^3(x^2+4)^2(4x-1)^4$. The differentiation will be very much simplified if the expression on the right-hand side is first reduced to simpler form. We obtain

$$y = \ln 57 + 3 \ln x + 2 \ln (x^2 + 4) + 4 \ln (4x - 1)$$

$$\frac{dy}{dx} = \frac{3}{x} + \frac{2 \cdot 2x}{x^2 + 4} + \frac{4 \cdot 4}{4x - 1}$$

$$= \frac{44x^3 - 7x^2 + 112x - 12}{x(x^2 + 4)(4x - 1)}$$

If the student will perform the differentiation without first reducing to a sum of simpler logarithms, he will see the advantage of the method used here.

Example 3. Find $\frac{dy}{dx}$ if $y = \ln \sqrt{\frac{1 + x}{1 - x}}$. Again it is of great advantage first to reduce the right-hand expression.

$$y = \tfrac{1}{2}[\ln (1 + x) - \ln (1 - x)]$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1 + x} + \frac{1}{1 - x}\right) = \frac{1}{1 - x^2}$$

Observe that y is real only if $-1 < x < 1$.

Example 4. Find $\frac{dy}{dx}$ if $y = \log^2 (3x + 4)$. This notation is commonly used for $[\log (3x + 4)]^2$.

$$\frac{dy}{dx} = [2 \log (3x + 4)]\left(\frac{3}{3x + 4}\right) \log e = \frac{6M \log (3x + 4)}{3x + 4}$$

Example 5. Find $\frac{dy}{dx}$ if $y = \ln \ln x$. This notation is commonly used for $\ln (\ln x)$.

$$\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

50. Logarithmic Differentiation. It is frequently of considerable advantage to take the logarithm of a function before differentiating. For example, suppose we wish to calculate the derivative of a product by this method. Let $y = uv$, where u and v are differentiable functions of x. We may first take logarithms as follows:

$$\ln y = \ln u + \ln v$$

Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

EXERCISES

Find the derivative of each of the following functions (Ex. 1 to 45):

1. $y = \ln (x - 2)$
2. $y = \ln (3x + 7)$
3. $v = \ln 5w$
4. $z = \ln (15 - 8x)$
5. $y = \ln (x^3 - 9)$
6. $y = \ln (4x^2 + 11)$
7. $z = \ln u(3u - 1)$
8. $z = \ln u^2$
9. $y = \ln (5x + 1)^3$
10. $y = \ln x^2(3x + 1)$
11. $y = \ln (x^2 + 1)(x^3 - 8)$
12. $s = \ln (4t - 7)^2(3t + 2)^3$

13. $y = \ln (9 - 3x)(4 + 6x)$

14. $y = \ln \dfrac{x^2 + 1}{x^2 - 1}$

15. $s = \ln \dfrac{5 - 2t}{5 + 2t}$

16. $y = \ln \dfrac{(x + 4)(2x - 3)}{x - 4}$

17. $y = \ln \sqrt{x^2 + 16}$

18. $s = \ln (t^3 - 27)^{1/3}$

19. $z = \log (3t^2 - 5t + 1)^{3/2}$

20. $y = \log \sqrt{(x + 2)(x - 3)}$

21. $y = \log (5x^3 + 7x + 1)$

22. $y = \log \sqrt{\dfrac{1 + x}{1 - x}}$

23. $y = \ln \sin \theta$

24. $y = \ln \sin^2 \theta$

25. $z = \ln \cos 3\varphi$

26. $z = \ln \tan \left(\varphi + \dfrac{\pi}{4}\right)$

27. $y = \log \sec \frac{1}{2}\varphi$

28. $r = \log \cot \frac{3}{4}\theta$

29. $r = \log \csc^2 \theta$

30. $r = \log^2 \sin 3\theta$

31. $y = \ln^2 \tan 2x$

32. $y = \ln \ln (x + 1)$

33. $r = \ln \ln \tan \theta$

34. $z = \ln \ln \ln x$

35. $s = \sqrt{\ln t}$

36. $y = \ln \ln x^2$

37. $r = \ln \ln 3 \cot \theta$

38. $y = x^2 \ln x$

39. $y = \sec x \ln \sec x$

40. $z = \dfrac{\ln \cos y}{\cos y}$

41. $v = \dfrac{\log u}{u}$

42. $w = \ln \sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}}$

43. $r = \ln \sqrt{\dfrac{1 + \tan \theta}{1 - \tan \theta}}$

44. $r = \ln \sqrt{1 + \cos \theta}$

45. $y = \log \tan^2 \frac{1}{2}x$

Find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$ (Ex. 46 to 50),

46. $y = \ln x$

47. $y = \ln \sin x$

48. $y = \ln \sqrt{x}$

49. $y = \ln x^2$

50. $y = \ln^2 x$

In Exercises 51 to 56, find $\dfrac{dy}{dx}$ by taking the logarithm of both sides before differentiation.

51. $y = (x + 3)^4(x - 1)^3$

52. $y = (x^2 + 4)^3(x^2 - 9)^2$

53. $y = (2x + 5)^{1/2}(3x - 1)^{3/4}$

54. $y = (x^2 - 6)^{3/2}(x^4 + 7)^{5/2}$

55. $y = \dfrac{(3x^3 - 5)^6}{(x^2 - 9)^3}$

56. $y = \dfrac{(x + 1)^2}{(x - 3)^2}$

57. Use logarithmic differentiation to derive the formula for $\dfrac{dy}{dx}$ if $y = uvw$ where u, v, w are differentiable functions of x.

58. Same as Exercise 57 for $y = u/v$

59. For what values, if any, is $\log_a x (a > 1)$ an increasing function? A decreasing function?

60. Find the equation of the tangent to the curve $y = \ln x$ at the point (x_1,y_1) on the curve. Find the y intercept of the tangent. Can the result be used to construct the tangent at any point of the curve? If so, how?

61. Find the equation of the tangent to $y = \ln x$ that is parallel to the line

$$2x - y + 1 = 0$$

62. Find the equation of the tangent to the curve $y = \ln x$ perpendicular to the line $3x + 2y - 7 = 0$.

63. Find the equation of the normal to the curve $y = \ln (1/x)$ having slope 2.

64. Can a tangent with slope $-\frac{3}{4}$ be drawn to the curve $y = \ln x$? Explain your answer.

65. Find $\dfrac{d^ny}{dx^n}$ if $y = \ln x$.

51. Derivative of the Exponential Function. Since the exponential and logarithmic functions are the inverses of one another, we can find the derivative of a^x as follows:

$$y = a^x \qquad (a > 0)$$

and therefore $\qquad \log_a y = x$

Therefore $\qquad \dfrac{dx}{dy} = \dfrac{1}{y} \log_a e$

Consequently $\qquad \dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dfrac{y}{\log_a e} = y \ln a = a^x \ln a$

[see Art. 48, formula (2)].

In general, if $y = a^u$ where u is a differentiable function of x,

★ $$\frac{d}{dx}(a^u) = a^u \frac{du}{dx} \ln a$$

If $a = e$, so that $y = e^u$, we have

★ $$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Example 1. If $y = e^{-x^2}$,

$$\frac{dy}{dx} = e^{-x^2}(-2x) = -2xe^{-x^2}$$

Example 2. If $y = 10^{\sin x}$,

$$\frac{dy}{dx} = 10^{\sin x} \cdot \cos x \cdot \ln 10 = (10^{\sin x}) \cdot \frac{\cos x}{M}$$

Example 3. If $x = e^{-\alpha t} \cdot \sin \omega t$,

$$\frac{dx}{dt} = \omega e^{-\alpha t} \cdot \cos \omega t - \alpha e^{-\alpha t} \cdot \sin \omega t$$
$$= e^{-\alpha t}(\omega \cos \omega t - \alpha \sin \omega t)$$

We are now in a position to prove the formula for the derivative of x^n for any given constant n, rational or irrational. To this end we

observe that

$$y = x^n = e^{\ln x^n} = e^{n \ln x}$$

Whence
$$\frac{dy}{dx} = e^{n \ln x} \cdot \frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

The reader will recall that this formula was derived in Art. 18 for n a positive integer. We have now established it for *any* power of x.*

52. Derivative of u^v. We have learned how to differentiate u^n, a variable to a constant power, and a^v, a constant to a variable power. Suppose we have a variable to a variable power, $y = u^v$, where u and v are differentiable functions of x.

We express our function in the following way:

$$y = u^v = e^{\ln u^v} = e^{v \ln u}$$

Differentiating,
$$\frac{dy}{dx} = e^{v \ln u} \cdot \frac{d}{dx}(v \ln u)$$
$$= u^v \left(\frac{v}{u}\frac{du}{dx} + \frac{dv}{dx}\ln u\right)$$

or

★
$$\frac{d}{dx}(u^v) = v \cdot u^{v-1} \cdot \frac{du}{dx} + u^v \frac{dv}{dx}\ln u$$

This formula is easy to remember if we observe the appearance of the two terms: only the first term would be obtained if v were a constant; only the second term would be obtained if u were a constant. Note further that the derivatives of u^n and a^v are included in this formula as special cases.

Example. If $y = x^{x^2}$, find $\frac{dy}{dx}$. Here $u = x$, $v = x^2$. Hence

$$\frac{dy}{dx} = x^2(x^{x^2-1}) \cdot 1 + x^{x^2}(\ln x) \cdot 2x$$
$$= x^{x^2+1} + 2x^{x^2+1}(\ln x)$$
$$= x^{x^2+1}(1 + 2\ln x)$$

EXERCISES

Find the derivative of each of the following functions (Ex. 1 to 37):

1. $y = e^{3x}$

2. $y = e^{-x}$

3. $z = e^{2x+1}$

4. $s = e^{-\frac{t^2}{2}}$

* Strictly speaking, we have proved this only for x positive (since $\ln x$ is real only for positive x). Somewhat more detailed consideration is necessary to establish the formula for negative x.

5. $s = e^{t^2+4}$

6. $r = e^{\sin\theta}$

7. $r = e^{\tan\theta}$

8. $y = xe^x$

9. $y = x^2e^{-x}$

10. $y = 10^{-x^2}$

11. $y = 2^{3x}$

12. $y = 3^{x^2}$

13. $s = e^{\cos^2 t}$

14. $s = e^{\sec t}$

15. $v = e^{\ln u}$

16. $v = e^u \sin u$

17. $w = e^{2u}\cos 3u$

18. $y = x \ln e^x$

19. $r = \theta e^{\tan\theta}$

20. $y = \sin e^x$

21. $y = \cos^2 e^{3x}$

22. $y = \cot 10^x$

23. $y = \arcsin e^{2x}$

24. $y = \arctan e^{x^2}$

25. $s = \operatorname{arcsec} e^t$

26. $s = e^{\frac{1}{2}t}/t^2$

27. $z = e^{-x^2}/x$

28. $y = x^x$

29. $y = x^{3x}$

30. $y = (\sin x)^x$

31. $y = (\ln x)^x$

32. $y = x^{x^n}$

33. $y = x^{x^x}$

34. $y = x^{\ln x}$

35. $y = x^{\frac{1}{\ln x}}$

36. Find $\dfrac{d^ny}{dx^n}$ if $y = e^x$.

37. Find $\dfrac{d^ny}{dx^n}$ if $y = xe^x$.

38. For what values of x, if any, is $y = e^x$ increasing? Decreasing? Concave upward? Concave downward?

39. Find the equation of the tangent line to the curve $y = e^x$ at the point where $x = 0$.

40. Find any maximum, minimum, and inflection points, and sketch the curve $y = e^{-\frac{x^2}{2}}$. This curve is important in the study of probability and statistics. The "normal law of error" is represented by a curve essentially of this form.

41. A rectangle has one side on the x axis and the ends of the opposite side on the curve $y = e^{-x^2}$. Find the vertices if the area of the rectangle is a maximum.

42. (a) Find any maximum, minimum, and inflection points, and sketch the curve $y = \dfrac{4}{1 + e^{-x}}$.

(b) Find the equation of the tangent at the point of inflection.

43. Same as Exercise 40 for $y = \dfrac{k}{1 + e^{ax+b}}$. This curve is important in the statistical study of population growth.

44. If $\dfrac{dy}{dx}$ is assumed to exist where $y = x^n$, use logarithmic differentiation to find the formula $\dfrac{dy}{dx} = nx^{n-1}$. *Note:* The method of logarithmic differentiation serves as a means of calculating this derivative, but it does not establish its *existence* as does the method of Art. 51.

45. If $\dfrac{dy}{dx}$ is assumed to exist where $y = u^v$ and u and v are differentiable functions of x, use logarithmic differentiation to find the formula of Art. 52. *Note:* This method does not establish the existence of $\dfrac{dy}{dx}$ (as does the method of Art. 52), but it is a convenient method for calculating the derivative in special cases as well as in the general case.

46. Use logarithmic differentiation instead of the formula of Art. 52 to solve Exercises 28 to 35.

47. Find $\frac{dy}{dx}$ for Exercises 13, 19, 21, 24, and 25 on page 120.

53. Hyperbolic Functions. Certain combinations of exponential functions occur frequently, and it is convenient to designate them by means of special names and notation. For instance, e^x often occurs in the combination $\frac{1}{2}(e^x - e^{-x})$. This particular function is called the *hyperbolic sine* of x and is denoted by the symbol $\sinh x$ or by the shorter $\text{sh}\, x$. The curve whose equation is $y = \frac{1}{2}(e^x + e^{-x})$ is called a *catenary* curve (a perfectly flexible string, if allowed to hang of its own weight from two supports, will assume the shape of a catenary). The combination of exponential functions on the right-hand side of this equation is called the *hyperbolic cosine* of x and is denoted by $\cosh x$ or $\text{ch}\, x$. The hyperbolic sine and cosine are combined in various ways to define the hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant. Thus we have the following definitions of the hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\text{csch}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

These functions are related to an equilateral hyperbola in somewhat the same way as the trigonometric or circular functions are related to a circle, although in the above definitions x is *not* to be regarded as the measure of an angle. The discussion of this relationship will be deferred to Chap. 19. Furthermore, their relations with one another are analogous to the familiar identities of trigonometry. The investigation of many of their properties will be left as exercises for the student. Tables of $\sinh x$, $\cosh x$, and $\tanh x$ are included in any good set of mathematical tables.

Derivatives of these functions are easily found by use of their definitions: If $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$, then $\frac{dy}{dx} = \frac{1}{2}(e^x + e^{-x}) = \cosh x$. In general, therefore,

$$\star \qquad \frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

Let the student verify that

★ $$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Example 1. Show that $\sinh x$ is an *odd* function. We have

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\left(\frac{e^{x} - e^{-x}}{2}\right) = -\sinh x$$

Example 2. Show that $\sinh x > 0$ for all $x > 0$.
We have, for all $x > 0$,

$$e^{x} > 1 > \frac{1}{e^{x}} = e^{-x}$$

and therefore
$$\sinh x = \frac{e^{x} - e^{-x}}{2} > 0$$

Example 3. Trace the curve $y = \sinh x$. Since

$$\sinh(-x) = -\sinh x,$$

the curve is symmetrical to the origin. Also, $\sinh 0 = 0$, so the curve passes through the origin. When $x \to +\infty$, $\sinh x \to +\infty$. Furthermore, $\frac{dy}{dx} = \cosh x$ is positive for all x; hence, the curve always rises. The second derivative $\frac{d^2y}{dx^2} = \sinh x$ is negative for $x < 0$, and the curve is concave downward; it is positive for $x > 0$, and the curve is concave upward; it is zero for $x = 0$, and the curve has a point of inflection at the origin, at which the slope is $\cosh 0 = 1$. Since $\frac{d^2y}{dx^2} > 0$ for $x > 0$, the slope of the curve increases as x increases from zero. The curve is shown in Fig. 59.

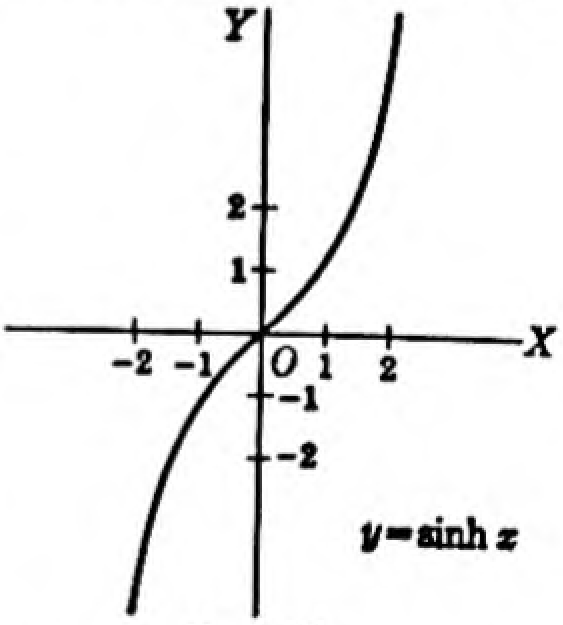


FIG. 59.

Example 4. Show that $\cosh^2 x - \sinh^2 x = 1$. From the definitions, we have

$$\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2 = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1$$

EXERCISES

1. Determine which of the hyperbolic functions are *odd* and which *even* functions.
2. Establish the formulas for the derivatives given on this page.

3. Investigate the value of each of the six hyperbolic functions for $x = 0$.

4. Find the values of $\sinh x$, $\cosh x$, and $\tanh x$ for $x = \ln 2$.

Establish the following formulas (Ex. 5 to 16):

5. $1 - \tanh^2 x = \operatorname{sech}^2 x$

6. $\coth^2 x - 1 = \operatorname{csch}^2 x$

7. $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

8. $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

9. $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

10. $\coth (x \pm y) = \dfrac{1 \pm \coth x \coth y}{\coth x \pm \coth y}$

11. $\sinh 2x = 2 \sinh x \cosh x$

12. $\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1$

13. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

14. $\coth 2x = \dfrac{1 + \coth^2 x}{2 \coth x}$

15. $\sinh (x/2) = \pm \sqrt{\dfrac{\cosh x - 1}{2}}$

16. $\cosh (x/2) = \sqrt{\dfrac{\cosh x + 1}{2}}$

Find the first derivative of each of the following functions (Ex. 17 to 42):

17. $y = \sinh (x/2)$

18. $s = \sinh (2t - 1)$

19. $s = \cosh (4 - 3t)$

20. $w = \cosh u^2$

21. $z = \tanh \dfrac{16y - 1}{5}$

22. $u = \tanh (1 - v^3)$

23. $y = \coth (1/x)$

24. $y = \coth (x^2/2)$

25. $z = \operatorname{sech} \sqrt{t}$

26. $x = \operatorname{sech} \left(\dfrac{t^2}{2} + 1\right)$

27. $y = \operatorname{csch} (16 - t^2)$

28. $y = \operatorname{csch} \sqrt{1 - t^2}$

29. $y = \tanh^2 x$

30. $z = \sqrt{\sinh 2x}$

31. $y = \cosh^4 (x^2/8)$

32. $w = \coth^{3/2} u^2$

33. $z = y \sinh y$

34. $z = e^y \cosh y$

35. $w = e^u \tanh u$

36. $v = e^{\sinh u}$

37. $y = \cosh e^x$

38. $y = \ln \sinh x$

39. $y = \ln \cosh x$

40. $y = \ln \operatorname{sech} x$

41. $y = \ln \tanh (x/2)$

42. $y = \ln \coth (x/2)$

Find maximum, minimum, and inflection points, and sketch each curve (Ex. 43 to 50):

43. $y = \cosh x$

44. $y = \tanh x$

45. $y = \coth x$

46. $y = \operatorname{sech} x$

47. $y = \operatorname{csch} x$

48. $y = \sinh 2x$

49. $y = a \cosh (x/a)(a > 0)$

50. $y = \sinh^2 x$

54. Inverse Hyperbolic Functions. The student is already familiar with the inverse trigonometric functions. We define the inverse hyperbolic functions in a similar way. If $\sinh y = x$, we write $y = \operatorname{argsinh} x$, read "$y$ is the number whose hyperbolic sine is x," and similarly for the other inverse hyperbolic functions. We use the notation $y = \operatorname{argsinh} x$ rather than $y = \operatorname{arcsinh} x$ in order to emphasize the fact that y is simply a number, or "argument," of the function $x = \sinh y$. It does not ordi-

narily represent the measure of an angle or of an arc as does y when $y = \arcsin x$. The symbol $\sinh^{-1} x$ is frequently used to represent the inverse hyperbolic sine, but we shall avoid it (following the convention adopted in Art. 44). Since hyperbolic functions are defined in terms of exponential functions, we may expect to find a connection between the inverse hyperbolic functions and the inverse of exponential functions, that is, logarithms. Thus, if

$$y = \operatorname{argsinh} x$$

then
$$x = \sinh y = \frac{e^y - e^{-y}}{2} \qquad \text{and} \qquad 2x = e^y - e^{-y}$$

Multiplying both sides by e^y and rearranging terms, we have

$$e^{2y} - 2xe^y - 1 = 0$$

which is a quadratic equation in e^y. Hence

$$e^y = x \pm \sqrt{x^2 + 1}$$

However, for real values of y, e^y is positive; and since $x < \sqrt{x^2 + 1}$ for all real values of x, we must reject the minus sign with the square root. Therefore

$$e^y = x + \sqrt{x^2 + 1}$$

and we have

$$\operatorname{argsinh} x = y = \ln (x + \sqrt{x^2 + 1}) \qquad \text{for all real } x \tag{3}$$

It will be left to the student to verify that

$$\operatorname{argcosh} x = \ln (x \pm \sqrt{x^2 - 1}) \qquad \text{for } x \geqq 1 \tag{4}$$

$$\operatorname{argtanh} x = \frac{1}{2} \ln \frac{1 + x}{1 - x} \qquad \text{for } -1 < x < 1 \tag{5}$$

$$\operatorname{argcoth} x = \frac{1}{2} \ln \frac{x + 1}{x - 1} \qquad \text{for } x < -1 \text{ and } x > 1 \tag{6}$$

$$\operatorname{argsech} x = \ln \frac{1 \pm \sqrt{1 - x^2}}{x} \qquad \text{for } 0 < x \leqq 1 \tag{7}$$

$$\left.\begin{aligned} \operatorname{argcsch} x &= \ln \frac{1 + \sqrt{x^2 + 1}}{x} \qquad \text{for } x > 0 \\ \operatorname{argcsch} x &= \ln \frac{1 - \sqrt{x^2 + 1}}{x} \qquad \text{for } x < 0 \end{aligned}\right. \tag{8}$$

The derivatives of the inverse hyperbolic functions are readily found. If $y = \operatorname{argsinh} x$, then $\sinh y = x$, and therefore

$$\frac{dx}{dy} = \cosh y \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{\cosh y}$$

By Example 4 of the preceding section,

$$\cosh y = \sqrt{\sinh^2 y + 1} = \sqrt{x^2 + 1}$$

where the plus sign must be taken for the square root since $\cosh y$ is always positive. Therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

and, in general, if u is a differentiable function of x,

$$\frac{d}{dx}(\operatorname{argsinh} u) = \frac{\frac{du}{dx}}{\sqrt{u^2 + 1}} \qquad \text{for all real } u \tag{9}$$

By use of the identities established in the exercises on page 130, the student can easily verify that

$$\frac{d}{dx}(\operatorname{argcosh} u) = \frac{\pm \frac{du}{dx}}{\sqrt{u^2 - 1}} \qquad \text{for } u > 1 \tag{10}$$

$$\frac{d}{dx}(\operatorname{argtanh} u) = \frac{\frac{du}{dx}}{1 - u^2} \qquad \text{for } -1 < u < 1 \tag{11}$$

$$\frac{d}{dx}(\operatorname{argcoth} u) = \frac{\frac{du}{dx}}{1 - u^2} \qquad \text{for } u < -1 \text{ or } u > 1 \tag{12}$$

$$\frac{d}{dx}(\operatorname{argsech} u) = \frac{\pm \frac{du}{dx}}{u\sqrt{1 - u^2}} \qquad \text{for } 0 < u < 1 \tag{13}$$

$$\left.\begin{aligned} \frac{d}{dx}(\operatorname{argcsch} u) &= -\frac{\frac{du}{dx}}{u\sqrt{1 + u^2}} \qquad \text{for } u > 0 \\ \frac{d}{dx}(\operatorname{argcsch} u) &= \frac{\frac{du}{dx}}{u\sqrt{1 + u^2}} \qquad \text{for } u < 0 \end{aligned}\right\} \tag{14}$$

Example 1. Show that argsinh x is an odd function. Let $y = \operatorname{argsinh}(-x)$; then $\sinh y = -x$. Let $z = \operatorname{argsinh} x$; then $\sinh z = x$. Therefore,

$$\sinh y = -\sinh z = \sinh(-z)$$

Hence

$$y = -z$$

and argsinh x is an odd function.

Example 2. Trace the curve $y = \operatorname{argsinh} x$. If $x = 0$, then $y = 0$, and the curve goes through the origin. If $x \to +\infty$, then $y \to +\infty$ (see Example 3 of the preceding section). Since argsinh x is an odd function, its graph is symmetrical to the origin. We have

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}} > 0$$

and y is an increasing function for all x. The second derivative is $\dfrac{d^2y}{dx^2} = -\dfrac{x}{(x^2 + 1)^{3/2}}$

which is negative for $x > 0$ so that the curve is concave downward; $\frac{d^2y}{dx^2}$ is positive for $x < 0$, so that the curve is concave upward; $\frac{d^2y}{dx^2}$ is zero for $x = 0$ and changes sign; the origin is, therefore, a point of inflection at which the slope of the curve is seen to be 1. Since $\frac{d^2y}{dx^2}$ is negative for $x > 0$, the slope is a decreasing function, and the

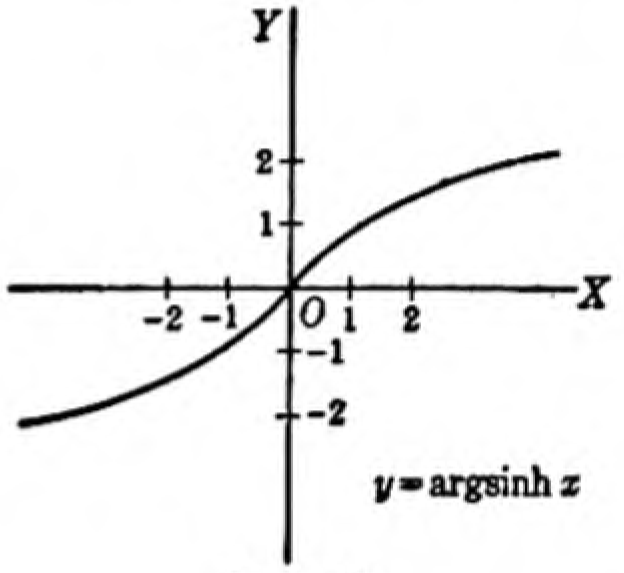


FIG. 60.

curve flattens out as x increases from zero. The graph is shown in Fig. 60. Note that this curve is the *reflection* of the curve $y = \sinh x$ in the line $y = x$.

EXERCISES

1. Verify the formulas on page 131 giving the inverse hyperbolic functions in terms of logarithms.

2. Verify the formulas on page 132 for the derivatives of the inverse hyperbolic functions.

3. Determine which of the following are *odd* and which *even* functions: argcosh x, argtanh x, argcoth x, argsech x, argcsch x.

Establish the following formulas (Ex. 4 to 15):

4. $\text{argsinh } x = \text{argcsch } (1/x)$

5. $\text{argcoth } x = \text{argtanh } (1/x)$

6. $\text{argsinh } x = \text{argcosh } \sqrt{1 + x^2}$

7. $\text{argtanh } (x/\sqrt{1 + x^2}) = \text{argcsch } (1/x)$

8. $\text{argtanh } x = \text{argsinh } (x/\sqrt{1 - x^2})$

9. $\text{argcoth } x = \text{argsinh } (1/\sqrt{x^2 - 1})$

10. $\sinh (2 \text{ argsinh } x) = 2x \sqrt{1 + x^2}$

11. $\cosh (2 \text{ argsinh } x) = 1 + 2x^2$

12. $\sinh (2 \text{ argtanh } x) = \frac{2x}{1 - x^2}$

13. $\cosh (2 \text{ argtanh } x) = \frac{1 + x^2}{1 - x^2}$

14. $\tanh (2 \text{ argsinh } x) = \frac{2x \sqrt{1 + x^2}}{1 + 2x^2}$

15. $\tanh (2 \text{ argtanh } x) = \frac{2x}{1 + x^2}$

Find $\frac{dy}{dx}$ for each of the following functions, and specify the values of x for which your result holds (Ex. 16 to 30):

16. $y = \text{argsinh } 2x$

17. $y = \text{argsinh } x^3$

18. $y = \text{argcosh } (x/3)$

19. $y = \text{argcosh } (1 - x)$

20. $y = \text{argtanh } 4x$

21. $y = \text{argtanh } x^3$

22. $y = \text{argcoth } (2x - 1)$

23. $y = \text{argcoth } (x/2)$

24. $y = \text{argsech } 5x$

25. $y = \text{argcsch } (x^2/2)$

26. $y = x \text{ argsinh } x$

27. $y = x^2 \text{ argtanh } x^2$

28. $y = (\text{argsinh } 3x)^2$

29. $y = \left(\text{argtanh } \frac{x}{2}\right)^2$

30. $y = (\text{argsech } x)^2$

Find any maximum, minimum, and inflection points, and sketch the curve (Ex. 31 to 35):

31. $y = \text{argcosh } x$

32. $y = \text{argtanh } x$

33. $y = \text{argcoth } x$

34. $y = \text{argsech } x$

35. $y = \text{argcsch } x$

55. Summary of the Rules for Differentiation. It will be helpful to repeat the most important of our formulas for differentiation. The student should know all of them as a result of his practice in *using* them, not because he has made a task of memorizing them. Above all, he must remember that these rules simply provide short-cut methods for writing down the *limit* $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ without the necessity of evaluating it in each problem that may arise. It will be noted that not all the formulas given in the preceding chapters have been repeated here, but references to earlier sections are given for those omitted.

★ $$\frac{dc}{dx} = 0 \qquad \text{(derivative of a } \textit{constant}\text{)}$$

★ $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{(derivative of a } \textit{sum}\text{)}$$

★ $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(derivative of a } \textit{product}\text{)}$$

★ $$\frac{d}{dx}(cu) = c\frac{du}{dx} \qquad \text{(derivative of a } \textit{constant times a function}\text{)}$$

★ $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{(derivative of a } \textit{quotient}\text{)}$$

★ $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(derivative of a } \textit{function of a function}\text{)}$$

★ $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{(derivative of the } \textit{inverse of a function}\text{)}$$

★ $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{(derivative in \textit{parametric representation})}$$

★ $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \text{(derivative of a \textit{power of a function})}$$

It is helpful to notice that "co" in the name of a trigonometric function means a minus sign in the derivative; also, that tangent appears with secant and cotangent with cosecant.

★ $$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\arcsin u) = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}} \qquad \text{where } -\frac{\pi}{2} \leqq \arcsin u \leqq \frac{\pi}{2}$$

★ $$\frac{d}{dx}(\arccos u) = \frac{-\frac{du}{dx}}{\sqrt{1 - u^2}} \qquad \text{where } 0 \leqq \arccos u \leqq \pi$$

★ $$\frac{d}{dx}(\arctan u) = \frac{\frac{du}{dx}}{1 + u^2} \qquad \text{where } -\frac{\pi}{2} < \arctan u < \frac{\pi}{2}$$

Derivatives of the other inverse trigonometric (circular) functions can be found on pages 114–115.

★ $$\frac{d}{dx}(\log_a u) = \frac{1}{u}\frac{du}{dx}\log_a e$$

★ $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

★ $$\frac{d}{dx}(a^u) = a^u \frac{du}{dx}\ln a$$

★ $$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

★ $$\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + u^v \frac{dv}{dx}\ln u$$

★ $$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

★ $$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

Derivatives of the other hyperbolic functions can be found on page 129.

★ $$\frac{d}{dx}(\text{argsinh } u) = \frac{\frac{du}{dx}}{\sqrt{1 + u^2}} \qquad \text{for all } u$$

Derivatives of the other inverse hyperbolic functions can be found on page 132.

It is instructive to note that many of these rules were derived, or could have been derived, from comparatively few fundamental rules. For example, if we evaluate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ for y a constant, for y the sum of a finite number of functions of x, for y a function of a function of x, for $y = \log_a x$, for $y = \sin x$ and if we establish the relation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

all the other rules for differentiation can be obtained from these.

MISCELLANEOUS EXERCISES

Find the first derivative of each of the following functions (Ex. 1 to 50):

1. $y = x^{3/5} + 3x^2 + \dfrac{2}{\sqrt{x}}$
2. $y = \sqrt{\dfrac{2}{x}} + x^5 - \dfrac{1}{x^4}$
3. $y = (x^2 + 1)^3(2x - 1)^2$
4. $z = (3y - 2)^2(5y + 4)^4$
5. $z = \dfrac{6y + 2}{y^2 + 1}$
6. $w = \dfrac{u}{\sqrt{a^2 - u^2}}$
7. $w = \log \sqrt{\dfrac{u + 1}{u - 1}}$
8. $u = \log (v^2 - 8)^4(3v + 1)^2$
9. $v = \dfrac{1 + \ln u}{u}$
10. $z = e^y \ln y$
11. $z = y \ln \ln y$
12. $y = \ln \ln e^x$
13. $y = x^2e^{2x+1}$
14. $s = e^{-2t} \cos (4t - 1)$
15. $s = \arctan \dfrac{t + 1}{2}$
16. $r = \sin^3 4\theta$
17. $r = \csc^2 \frac{3}{2}\theta$
18. $r = e^{\cos 2\theta}$
19. $y = x^{\sin x}$
20. $y = e^{x^2}$
21. $y = x^{e^x}$
22. $s = \arctan\ t + \ln \sqrt{\dfrac{t - 1}{t + 1}}$

23. $s = \arcsin t + \ln \sqrt{1 - t^2}$

24. $y = \arcsin \dfrac{x}{\sqrt{x^2 + 4}}$

25. $y = \cosh (x^2/4)$

26. $z = \tanh x + \coth x$

27. $y = \sinh^3 (x/3)$

28. $y = e^x \tanh x$

29. $y = e^{\cosh x}$

30. $z = \ln \sinh y$

31. $z = \ln \ln \cosh y$

32. $v = \ln \operatorname{argsinh} u$

33. $v = \operatorname{argsinh} \log u$

34. $\varphi = \arctan \sinh x$

35. $x = \operatorname{argsinh} \tan \varphi$

36. $y = \arctan 10^x$

37. $y = \arcsin a^x$

38. $y = \operatorname{argsinh} a^x$

39. $y = \sqrt{\tan 3x}$

40. $y = \dfrac{(a^2 + x^2)^{3/2}}{x}$

41. $y = e^{\frac{1}{x}}(x^2 - 1)$

42. $z = \cot^2 (\theta/2) + \ln \csc^2 (\theta/2)$

43. $r = \ln (\csc \theta - \cot \theta)$

44. $r = \ln \tan \left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right)$

45. $y = \frac{1}{5}e^{2x}(2 \sin x - \cos x)$

46. $y = e^x(x^2 - 2x + 2)$

47. $z = \frac{1}{2}[y \sqrt{y^2 + a^2} + a^2 \ln (y + \sqrt{y^2 + a^2})]$

48. $r = \frac{1}{2}[\sec \theta \tan \theta + \ln (\sec \theta + \tan \theta)]$

49. $z = -y \cos y + \sin y$

50. $z = \frac{1}{2}e^y(\sin y - \cos y)$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the following cases (Ex. 51 to 60):

51. $x^2 + 4y^2 = 16$

52. $x^2 + y^4 = 1$

53. $xy^2 = 8$

54. $x^3 - 8y^3 = 1$

55. $x = e^{-t}$
 $y = e^{-t} \sin t$

56. $x = e^t$
 $y = t$

57. $x = \ln t$
 $y = e^t$

58. $x = \cosh u$
 $y = \sinh u$

59. $x = a \cosh^3 u$
 $y = a \sinh^3 u$

60. $x = \operatorname{sech} 2u$
 $y = \tanh 2u$

61. Let $f(x)$ be defined as follows:

$$f(x) = x^4\left(1 + \sin \frac{1}{x}\right) + e^{\frac{-1}{x^2}} \qquad \text{for } x \neq 0 \text{ and } f(0) = 0$$

Assume as known the fact that

$$\lim_{h \to 0} \frac{1}{h^n} e^{\frac{-1}{h^2}} = 0 \qquad \text{for } n > 0$$

(a) Show that $f(x)$ is continuous at $x = 0$.

(b) Show that $f'(0) = 0$. $\left[\textit{Hint:} \text{ Find the limit as } h \to 0 \text{ of } \dfrac{f(0 + h) - f(0)}{h}.\right]$

(c) Show, further, that $f'(x)$ is continuous at $x = 0$.

62. (a) Show that the function $f(x)$ of Exercise 61 has a *minimum* at $x = 0$ by showing that $f(x) > 0$ for all $x \neq 0$.

(b) Show that $f'(x)$ has no properly defined sign in any interval that includes $x = 0$, so that $f'(x)$ *does not "change sign"* as x passes through 0, even though $f'(x)$ is continuous at $x = 0$ (see Art. 32).

CHAPTER 8

THE DIFFERENTIAL

56. The Differential. Let us recall the definition of the derivative of a function, say of $y = f(x)$. We have

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

provided the indicated limit exists.

We remember that $\frac{dy}{dx}$ was regarded as a *single* symbol for the derivative. There is some advantage in changing this point of view. It is clear, first of all, that

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon$$

where $\epsilon \to 0$ when $\Delta x \to 0$. Consequently

$$\Delta y = \underset{\text{principal part}}{f'(x)\,\Delta x} + \underset{\text{negligible part}}{\epsilon\,\Delta x} \tag{1}$$

As indicated in (1), we call $f'(x)\,\Delta x$ the *principal part of* Δy. In Fig. 61, suppose the graph of $y = f(x)$ is drawn with equal scales on OX and OY.

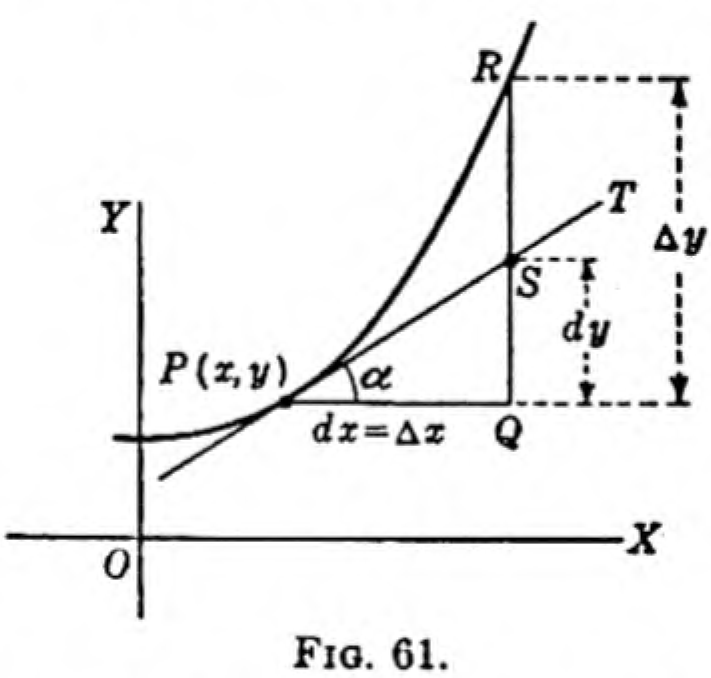


FIG. 61.

Then $f'(x) = \tan\alpha$. Let $PQ = \Delta x$. It is then clear that $QR = \Delta y$, and $QS = f'(x)\,\Delta x$ = "principal part of Δy." The length

$$SR = \epsilon\,\Delta x = \text{"negligible part of } \Delta y\text{"}$$

In general, disregarding those rare cases where $\epsilon = 0$, we see from (1) that $\Delta y \neq f'(x)\,\Delta x$. In other words, the equation

$$\Delta y = f'(x)\,\Delta x \tag{2}$$

is not true. Yet this equation, although not true, comes closer and closer to the truth as Δx becomes smaller and smaller. Figure 61 illustrates this fact. Following Leibnitz, we imagine quantities dy and dx, instead of Δy and Δx, for which the equation (2) is *exactly true:*

$$dy = f'(x)\,dx \tag{3}$$

We call dy and dx *differentials*, whereas Δy and Δx are *differences* (that is, "increments"). Purely logically, dy and dx are variable quantities which are required to satisfy the relation (3), nothing else. Intuitively, the ratio of dy to dx is seen to be the same as the ratio of the *principal part of* Δy to Δx. A geometrical interpretation is suggested in Fig. 61. Dividing both sides of (3) by dx, we obtain our old equation $\dfrac{dy}{dx} = f'(x)$, but we may now, if we so desire, regard the left-hand side as the quotient of the differentials dy and dx.

The advantage of the differential notation, especially for computational purposes, is the increased freedom in algebraic manipulation that results from its use. This will appear in Art. 58.

We may illustrate the foregoing discussion with the function

$$y = f(x) = x^3$$

This gives $f'(x) = 3x^2$, and, therefore, $dy = 3x^2\,dx$ is the differential of x^3. Further, we observe that

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^3 = x^3 + 3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3 \\ \Delta y &= 3x^2\,\Delta x + (3x\,\Delta x + \Delta x^2)\,\Delta x \\ &= f'(x)\,\Delta x + \epsilon\,\Delta x \end{aligned}$$

We see that, for any fixed x, $\epsilon = 3x\,\Delta x + \Delta x^2$ approaches 0 as Δx approaches 0, and that $3x^2\,\Delta x$ is the principal part of Δy. The reader will find it instructive to sketch the curve $y = x^3$, take some point P, say (1,1), draw the tangent line, and compare with Fig. 61.

A concrete idea of the approximation obtained by using the principal part of Δy instead of Δy itself in the example of the preceding paragraph results from considering a cube of edge x whose volume is $y = x^3$. Increase the length of the edge by Δx. The increase in volume Δy is

$$\Delta y = 3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3$$

The comparative importance of the three terms in this expression is illustrated in Fig. 62. There are three square slabs, each of which has base x^2 and thickness Δx, and hence volume $x^2\,\Delta x$. These are heavily out-

lined in the figure and constitute the principal part of Δy. There are, in addition, three rectangular prisms which fit along three edges of the cube. Each has length x, cross-sectional area Δx^2, and volume $x\,\Delta x^2$. These, together with the single cube of edge Δx and volume Δx^3 that fits into the space at the upper corner, account for the terms $3x\,\Delta x^2 + \Delta x^3$ that are neglected in the approximation $3x^2\,\Delta x$.

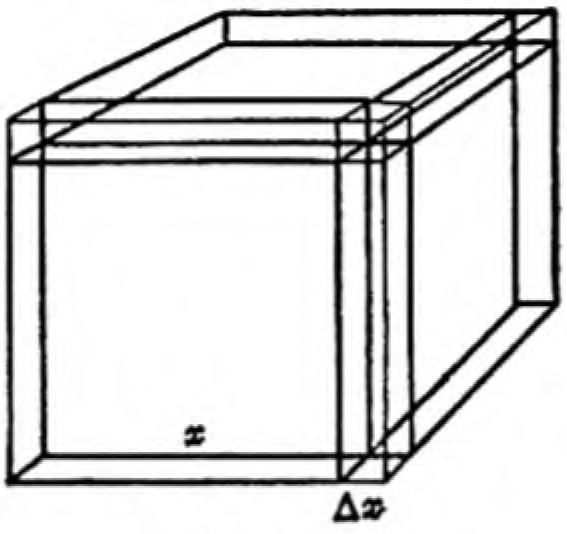


FIG. 62.

In our discussion we have used x as an independent variable. We may well ask if formula (3) still holds if x is not independent but is some function of an independent variable t. To answer this question, we note that if $y = f(x)$ where $x = g(t)$, then actually y is a function of t:

$$y = f[g(t)] = F(t)$$

Consequently, by (3), $dy = F'(t)\,dt$. We calculate $F'(t)$ by the usual method for finding the derivative of a function of a function (Art. 23):

$$F'(t) = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = f'(x)g'(t)$$

where $\frac{dy}{dx}$ and $\frac{dx}{dt}$ represent derivatives as indicated. Hence

$$dy = f'(x)g'(t)\,dt$$

But t is an independent variable; consequently, by (3), $g'(t)\,dt = dx$. Therefore

$$dy = f'(x)\,dx$$

and formula (3) is valid whether or not x is an independent variable.

Hereafter, when we speak of "differentiating" a function, we shall mean either finding its derivative or finding its differential.

57. Formulas for Differentials of Functions. It is clear that our formulas for derivatives are easily modified to become formulas for differentials. For example, the differential of the product uv where u and v are functions of x is easily obtained. For, if $y = uv$, $dy = y'\,dx$. Since $y' = uv' + u'v$, where the primes indicate differentiation with respect to x,

$$dy = (uv' + u'v)\,dx = u(v'\,dx) + v(u'\,dx) = u\,dv + v\,du$$

Similarly

$$d(c) = 0$$
$$d(u + v) = du + dv$$
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$
$$d(x^n) = nx^{n-1}\,dx$$
$$d(\sin x) = \cos x\,dx$$
$$d(\ln x) = \frac{dx}{x}$$
$$d(e^x) = e^x\,dx$$

The student can easily fill in other formulas.

EXERCISES

Find the differential of each of the following functions (Ex. 1 to 40):

1. $y = x^4 - 7x^2 + 6$
2. $y = 3 + 5x - 7x^2 - 11x^3$
3. $y = \sqrt{x^2 + 4}$
4. $y = (3x^2 - 8)^{3/2}$
5. $z = (x^2 - 9)^4$
6. $z = (27 - x^3)^5$
7. $w = (t^2 - 1)^2(3t + 4)$
8. $w = (t^2 + 4)^3(t^2 - 9)^2$
9. $s = \dfrac{r^2 - 1}{r^2 + 1}$
10. $s = r/\sqrt{r^2 - 4}$
11. $r = \sin 4\theta$
12. $r = \tan \frac{1}{2}\theta$
13. $r = \theta - \tan \theta$
14. $y = 3 \sin x - \sin^3 x$
15. $y = \sqrt{\cot x}$
16. $y = x + \cot x - \frac{1}{3} \cot^3 x$
17. $z = 2x + \sin 2x$
18. $z = \sec^2 2x$
19. $z = \arcsin \frac{1}{2}x$
20. $y = \arctan x^2$
21. $y = \dfrac{1}{a} \arctan \dfrac{x}{a}$
22. $y = \sqrt{x} - \arctan \sqrt{x}$
23. $y = \operatorname{arccsc} x$
24. $y = -\dfrac{1}{a} \arcsin \dfrac{a}{x}$
25. $y = \sqrt{x^2 - a^2} + a \operatorname{arccsc} (x/a)$
26. $s = \ln (t^2 - 1)$
27. $s = \log t$
28. $v = \ln \sin u$
29. $v = \dfrac{1}{2a} \ln \dfrac{a - u}{a + u}$
30. $z = \ln (x + \sqrt{x^2 + a^2})$
31. $z = \ln \tan (x/2)$
32. $v = \operatorname{sech} u$
33. $v = \sinh u$
34. $y = \ln \tanh (x/2)$
35. $y = \ln \cosh x$
36. $y = e^{-\frac{x^2}{2}}$
37. $z = e^{y^2-1}$
38. $z = e^{\sin y}$
39. $z = e^{\ln \cos y}$
40. $z = u^v$

Find the following differentials (Ex. 41 to 45):

41. $d(xy^2)$
42. $d(x^2y^2)$
43. $d(x^2/y)$
44. $d(y/x^2)$
45. $d(x^2 \sin y)$

Using differentials, find $\dfrac{dy}{dx}$ in the following cases (Ex. 46 to 49):

46. $x^2 + 4y^2 = 16$
47. $x^2y^3 - 3xy^4 = a^5$
48. $x^4y^4 + 2x^2y^6 = a^8$
49. $x^{2/3} + y^{2/3} = a^{2/3}$

Using the fact that $y' = dy \div dx$ and that $y'' = dy' \div dx$, find y' and y'' in the following cases (Ex. 50 to 55):

50. $x = t^3$
$y = 2t^4 + 1$

51. $x = a \cos \theta$
$y = b \sin \theta$

52. $x = a(\theta - \sin \theta)$
$y = a(1 - \cos \theta)$

53. $x = a \cosh u$
$y = b \sinh u$

54. $x = e^t$
$y = \ln t$

55. $x = \arcsin u$
$y = \arctan u$

58. Use of Differentials in Approximate Computation. We have seen in the discussion of Art. 56 that, when Δx is small, $f'(x)\,\Delta x$ is a fairly close approximation to Δy. We frequently wish to calculate the increment in a function $y = f(x)$, but the computation of the actual increment may be laborious. Often an approximation sufficiently close for the purpose at hand results from computing the value of $f'(x)\,\Delta x$, that is, of the principal part of Δy. To do this, we find $dy = f'(x)\,dx$, then replace dx by Δx. We gain a considerable advantage from the fact that dy is usually much more easily computed than Δy. This will be made clear in the following examples.

Example 1. What is the approximate change per minute of angle in the sine of θ in the neighborhood of $\theta = 45°$? We have $y = \sin \theta$. We wish to find the approximate value of Δy for a change in angle $\Delta\theta$ of $1'$. If we write the differential dy for an approximate value of Δy, we obtain

$$y = \sin \theta \qquad dy = \cos \theta \, d\theta$$

where we take $\theta = 45°$ and put $d\theta = 1' = \frac{1}{60} \cdot \pi/180 = 0.00029$ radian. Hence $dy = (0.70711)(0.00029) = 0.00021$. Let the student obtain Δy directly by use of a five-place table of sines.

Example 2. What is the approximate volume of a thin circular cylindrical shell with fixed height h? Let the inside radius of the shell be r and let the thickness be Δr. The volume of the shell is then exactly equal to the volume of a cylinder of radius $r + \Delta r$ minus the volume of a cylinder of radius r. This is equivalent to saying that the volume of the shell is the *change in volume* of a cylinder of radius r produced by increasing the radius by Δr. That is, if $V = \pi r^2 h$ is the volume of the cylinder, then

$$\Delta V = \pi(r + \Delta r)^2 h - \pi r^2 h = 2\pi r h \,\Delta r + \pi h \,\Delta r^2$$

is the volume of the shell. If Δr is *small compared with* r, this is very closely approximated by $2\pi r h\,\Delta r$ which becomes, for $dr = \Delta r$, $2\pi r h\, dr = dV$. That is, the volume of the shell is approximately its *circumference* times its *height* times its thickness. Note that either the inside or the outside radius will serve.

Example 3. An angle can be measured with an error not exceeding $1'$. It is required to obtain the sine of such an angle with an error not exceeding 0.00010. For what range of acute angles can this be done? If we call $\Delta\theta$ the error in measuring an angle θ and call Δy the error in $y = \sin \theta$, we have given $|\Delta\theta| \leq 1' = 0.00029$ radian (to five places), and we require $|\Delta y| \leq 0.00010$. Now

$$\Delta y = \sin (\theta + \Delta\theta) - \sin \theta$$

which is very awkward to work with. However

$$dy = \cos \theta \, d\theta$$

Hence $\cos\theta\,\Delta\theta$ is approximately equal to Δy and, furthermore, it is easy to work with. We have

$$|\cos\theta\,\Delta\theta| = |\cos\theta|\cdot|\Delta\theta| < 0.00010$$

that is

$$|\cos\theta|\cdot(0.00029) < 0.00010$$

$$|\cos\theta| < \frac{0.00010}{0.00029} = \frac{10}{29} = 0.34483$$

The acute angles for which $\cos\theta < 0.34483$ are found by use of a table to be all angles from 69°50′ to 90°. Note that $\cos 69°50' = 0.34475$ and that

$$\cos 69°49' = 0.34503$$

but since the measure of the angle may be in error by as much as 1′, it would be foolish to interpolate and find that θ must be greater than 69°49.7′. We are certainly safe to take $\theta > 69°50'$.

Example 4. Find approximately $\sqrt[4]{627}$. We observe that 627 is close to 625 of which the fourth root is 5. Suppose we set $y = \sqrt[4]{x} = x^{1/4}$. Then, if $x = 625$ so that $y = 5$ and we set $\Delta x = 2$, we have

$$y + \Delta y = \sqrt[4]{x + \Delta x} = \sqrt[4]{627}$$

It is easy to calculate $dy = \frac{1}{4}x^{-3/4}\,dx$. Replacing dx by $\Delta x = 2$, we get

$$\tfrac{1}{4}(625)^{-3/4}\cdot(2) = 0.004$$

Writing $y + dy = 5 + 0.004 = 5.004$ and recalling that dy is written for the approximate value of Δy, we have $\sqrt[4]{627} = 5.004$.

Example 5. For what values of x can $\sqrt[4]{x}$ be used for $\sqrt[4]{x+1}$ if we require the result to be in error by less than 0.001? If we set $y = \sqrt[4]{x}$ and call Δy the error in y, we have

$$\Delta y = \sqrt[4]{x + \Delta x} - \sqrt[4]{x} = \sqrt[4]{x+1} - \sqrt[4]{x}$$

Since it is inconvenient to work with Δy, we shall find dy. We have $dy = \frac{1}{4}x^{-3/4}\,dx$. Replacing dx by $\Delta x = 1$, we obtain

$$dy = \tfrac{1}{4}x^{-3/4}\cdot 1$$

We require that this be less than 0.001, that is

$$\frac{1}{4x^{3/4}} < 0.001$$

This is equivalent to

$$4x^{3/4} > \frac{1}{0.001} = 1000 \qquad x^{3/4} > 250$$

or

$$x > (250)^{4/3} = 250\cdot(250)^{1/3} = 250(6.29961)$$

the cube root being taken from a table of cube roots. If x is taken greater than

$$250(6.3) = 1575$$

the error in y is less than 0.001.

It is often important to find, not the actual error involved in a computation, but the *relative error*. For instance, if Δy is the error made in cal-

culating y, then $\frac{\Delta y}{y}$ is the relative error. Obviously, $100 \frac{\Delta y}{y}$ is the *percentage error*. Since we use dy for an approximate value of Δy, $\frac{dy}{y}$ may be used as an approximation to the relative error. The form of this expression indicates the convenience of finding first $\ln y$ and then calculating $d(\ln y) = \frac{dy}{y}\cdot$ The method is illustrated in Example 6.

Example 6. The radius of a sphere is found to be 10 in., with a possible error of 0.02 in. What is the relative error in the computed volume? We have $V = \frac{4}{3}\pi r^3$. We might proceed as follows:

$$dV = 4\pi r^2\,dr$$

Let $r = 10$, and replace dr by $\Delta r = \pm 0.02$. This gives, as an approximation to ΔV, $4\pi(100)(\pm 0.02) = \pm 8\pi$ cu. in. For the relative error, we have $\frac{\Delta V}{V}\cdot$ Suppose the actual error to be positive. The relative error is then approximately

$$8\pi/\tfrac{4}{3}\pi(1000) = 0.006 = 0.6 \text{ per cent}$$

However, we need not first calculate dV. Taking logarithms, we get

$$\ln V = \ln \tfrac{4}{3}\pi + 3 \ln r$$

Differentiating, we have

$$\frac{dV}{V} = 3\frac{dr}{r}$$

Replacing dr by $\Delta r = 0.02$ (supposing the error to be positive), we have for our relative error $3(0.02)/10 = 0.006 = 0.6$ per cent.

EXERCISES

In Exercises 1 to 3, find approximate formulas:

1. Area of a narrow circular ring of inside radius r and width Δr
2. Volume of a thin spherical shell of inside radius r and thickness Δr
3. Volume of the shell obtained by increasing by Δr the radius r of the base of a cone of fixed altitude h

4. A cubical box is to be made of tin 1 mm. thick. If the box measures 10 cm. along the outside edge, find approximately the volume of tin required.

5. A rectangular chest of inside dimensions 2 by 2 by 5 ft. is to be lined with felt $\frac{1}{8}$ in. thick (including the cover of the chest). Find approximately the reduction in volume.

6. The edge of a cubical box is measured with an error less than 0.01 in., and the volume computed. If the volume must be in error by less than 6 cu. in., find the largest box for which the process will suffice.

7. The side of a square is measured with an error less than 0.01 cm. The area must be computed with an error less than 2 sq. cm. How large a square can be satisfactorily measured?

8. The radius of a sphere is to be measured and the volume computed. If the volume must be accurate to 2 cu. in. and the radius can be measured to 0.01 in., how great a radius will permit use of the process?

9. In Exercise 8, find the relative error in the volume when $r = 2$.

10. The volume of a sphere is found with an error less than 1 cc. by measuring the volume of water it displaces. If the radius must be computed with an error less than 0.01 cm., how large must the sphere be?

11. The edge of a cube is measured with a possible error of 0.05 in. Find the relative error in the computed volume when the edge is found to be 7.49 in.

12. If, in Exercise 11, the relative error in the volume must not exceed 1 per cent, how small a cube can be satisfactorily measured?

13. Find approximately the change per degree in the sine of an angle for angles near 45, 0, and 90 deg. Check results by reference to a table of sines.

14. Find the relative error in $\sin\theta$ near $\theta = 30°$ if θ is in error by $30'$.

15. (a) Find the error in the reciprocal of x caused by an error Δx in x itself.

(b) Find the relative error in this reciprocal.

16. Calculate approximately the reciprocal of 997.

17. Calculate approximately the reciprocal of 102.

By use of differentials, calculate approximately the values of the following (Ex. 18 to 23):

18. $\sqrt[3]{66}$ **19.** $\sqrt[3]{122}$

20. $\sqrt[4]{252}$ **21.** $\sqrt[4]{82}$

22. 103^3 **23.** 79^2

24. For what values of x can $\sqrt{x+1}$ be replaced by $\sqrt{x}$ if the resulting error must be less than 0.01?

25. For what values of x can $\sqrt[3]{x+1}$ be replaced by $\sqrt[3]{x}$ if the resulting error must be less than 0.01?

26. In Exercise 24, what are the values of x if the relative error must be less than 1 per cent?

27. In Exercise 25, what are the values of x if the relative error must be less than 1 per cent?

28. The equatorial radius of the earth is approximately 4000 miles. Imagine a steel tire tightly wrapped around the earth at the Equator. If the tire were lengthened by 10 ft., how far out from the earth's surface would it stand if this distance were uniform around the Equator?

29. An angle is measured and its sine computed. The error in the sine must be less than 0.001. The angle measurement may be in error by as much as $10'$. For what acute angles will the process suffice?

30. An angle is measured and its cosine computed. The error in the cosine must be less than 0.001. The angle measurement may be in error by as much as $10'$. For what acute angles will the process suffice?

31. An angle is measured and its tangent computed. The extreme errors in the angle and the tangent are $20'$ and 0.01. For what acute angles is the process satisfactory?

32. Show that near 90° a small error in the angle causes a large error in the tangent.

33. An angle is to be found by measuring its tangent. If the tangent may be in error by 0.001 and the angle is required with an error less than $1'$, for what acute angles is the process satisfactory?

34. The sine of an angle is measured with an error less than 0.0001. The computed angle must be in error by less than $5'$. For what acute angles will the process suffice?

35. Find approximately the relative error in $\sin \theta$ due to an error $\Delta\theta$ in the angle.

36. An angle θ is found to be 45°, with a possible error of 10′. What is the relative error in $\sin \theta$?

37. An angle is measured with an error not greater than 30′. What is the relative error in the tangent if the angle is found to be (*a*) 30°? (*b*) 45°? (*c*) 60°? (*d*) 89°?

38. An angle is measured with an error not greater than 10′. If the relative error in the cosine must be less than 1 per cent, for what acute angles can the process be used?

39. Find approximately the error in $\ln x$ due to a small error Δx in x if x is large. Find the relative error in $\ln x$.

40. Same as Exercise 39 for $\log x$.

41. Find the following common logarithms by use of differentials. Check the results by reference to a table.

(*a*) log 997 (*b*) log 10.2
(*c*) log 10004 (*d*) log 99.98

42. For what values of x can $\ln x$ be used for $\ln (x + 1)$ if the result must be in error by less than 0.001?

43. Solve Exercise 42 for $\log x$.

44. Find approximately the error in $\log \sin \theta$ due to an error $\Delta\theta$ in θ.

45. If θ can be measured with an error not exceeding 10′, find approximately the resulting error in $\log \cos \theta$ when θ is found to be 60°.

46. If θ can be measured with an error not exceeding 15′, find approximately the resulting error in $\log \tan \theta$ when θ is found to be (*a*) 23°; (*b*) 87°.

47. In finding the number N from its natural logarithm, what error results from a small error in the logarithm? (*Hint:* Let $y = \ln N$, and find dN in terms of dy.)

48. Solve Exercise 47 if $y = \log N$.

49. If $\log N = 1.88047$ with a possible error of ± 0.00002, find N.

50. If $\log N = 2.01536$ with a possible error of ± 0.00002, find N.

51. Find approximately the error in θ if $y = \log \tan \theta$ may be in error by an amount Δy.

52. Find approximately the error in θ if $y = \log \sin \theta$ may be in error by an amount Δy.

53. Find θ if $\log \tan \theta = 0.36204 \pm 0.00001$.

54. Find θ if $\log \sin \theta = 9.82941 - 10 \pm 0.00001$.

CHAPTER 9

FURTHER APPLICATIONS OF DERIVATIVES AND DIFFERENTIALS

59. Ratio of Arc to Chord. The student is probably familiar with the determination of the circumference of a circle by finding the limit of the sum of the sides of an inscribed polygon when the number of sides is indefinitely increased and the length of each side decreases toward zero. In fact, the *length* of the circumference of a circle is defined as *the limit of such a sum of straight-line segments.* A similar device will be used to determine the lengths of arcs of various curves, thus: Let arc $\widehat{AB}$ be

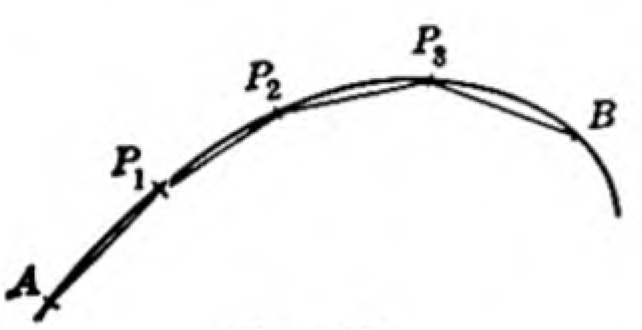


Fig. 63.

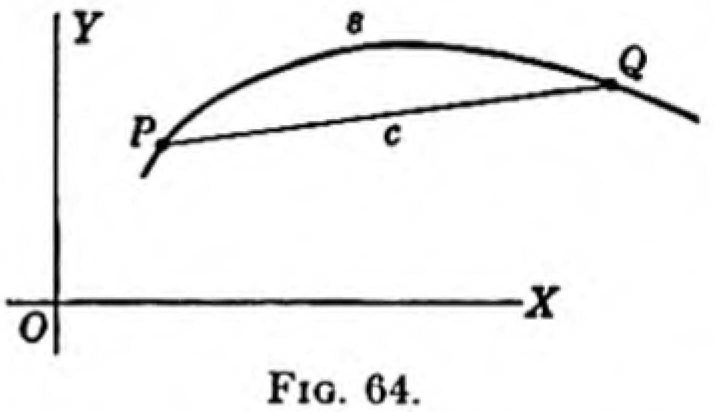


Fig. 64.

divided into several arcs by the points $P_1, P_2, P_3, \ldots, P_{n-1}$ (Fig. 63). Next, form the sum of chords $\overline{AP_1} + \overline{P_1P_2} + \cdots + \overline{P_{n-1}B}$. Then increase indefinitely the number of points of division; at the same time, let consecutive points approach coincidence along the curve. If the sum of chords approaches a limit, this limit is defined as *the length of the arc* $\widehat{AB}$. The evaluation of such limits is one of the problems of the integral calculus and will be discussed later. For the present, we shall suppose that we have an intuitive idea of what is meant by the length of an arc.

It is important for our present work to compare the length of an arc $\widehat{PQ} = s$ (Fig. 64) with the length of its corresponding chord $\overline{PQ} = c$. It seems natural to suppose that s will be very nearly equal to c if the arc is very short. In particular, it appears that under such circumstances, the ratio s/c should be very nearly unity. This definition of the length of arc involves the fact that

$$\lim_{s\to 0} \frac{s}{c} = 1$$

For the present we shall assume this without proof. It will be verified in Art. 114.

60. Differential of Arc Length. Suppose that the function $y = f(x)$ has a graph as illustrated in Fig. 65. Let A be some *fixed* point upon the curve, and let $P(x,y)$ be *any* point on the curve. Let s be the distance *along the curve* from A to P. Then a given x determines a value of y and also a value of s. Hence, s is a function of x, and let us suppose (as indicated in the figure) that s increases as x increases. We may calculate the derivative of s with respect to x by the usual method.

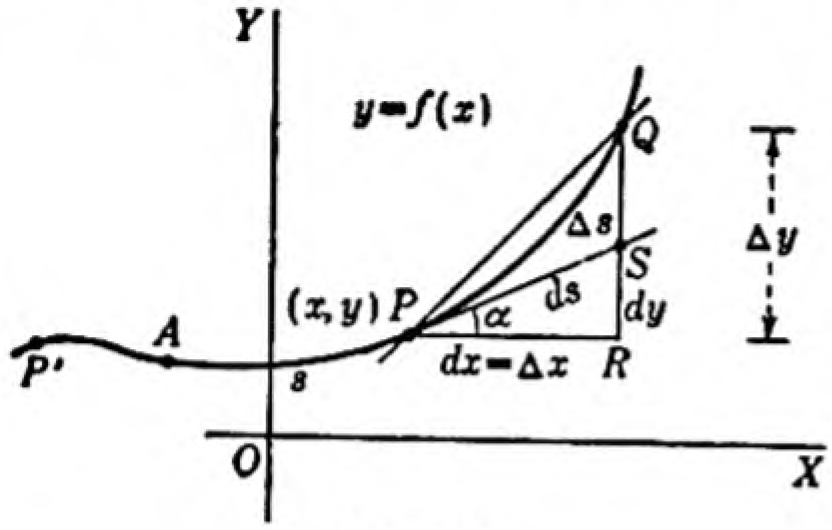


FIG. 65.

If x increases by an amount Δx, y will change by an amount Δy, and s will increase by an amount Δs. If Q is the point with coordinates $x + \Delta x$, $y + \Delta y$, then arc $\widehat{PQ} = \Delta s$.

We also have chord

$$\overline{PQ} = \sqrt{\Delta x^2 + \Delta y^2} \tag{1}$$

We wish to find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta s}{\Delta x}$. In order to have an expression whose limit can be conveniently evaluated, we multiply numerator and denominator of $\dfrac{\Delta s}{\Delta x}$ by the length of the chord $\overline{PQ}$. Thus

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{\overline{PQ}}\,\frac{\overline{PQ}}{\Delta x} = \frac{\Delta s}{\overline{PQ}}\,\frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} = \frac{\Delta s}{\overline{PQ}}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Therefore
$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta s}{\overline{PQ}} \cdot \lim_{\Delta x \to 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

From the preceding section, the first factor on the right-hand side is 1; since the square root is a continuous function, the second factor is $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$; hence

$$\star \qquad \frac{ds}{dx} = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2}$$

In case the point P is chosen on the other side of A (for instance, at P'), then s will *decrease* when x increases, and consequently $\dfrac{ds}{dx}$ should be negative. This is accomplished by choosing the negative sign for the square root in (1). The result is $\dfrac{ds}{dx} = -\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

We may find the *differential of arc length*, ds, from (2). Since the differential is the derivative $\frac{ds}{dx}$ multiplied by dx, this gives

★ $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Recalling that derivatives can be regarded as quotients of differentials, we then have

★ Simple way $$ds^2 = dx^2 + dy^2 \qquad (3)$$ from fig

as a relation among the differentials. If, following Art. 56, we take $dx = PR$, then $dy = RS$, and $ds = PS$, as indicated in Fig. 65.

We might have regarded s as a function of y, and in that case the derivative $\frac{ds}{dy}$ could have been found. Again regarding derivatives as quotients of differentials, we can find this derivative from (3).

★ $$\frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \qquad \text{and} \qquad ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

From Fig. 65, it is clear that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 \alpha} = \sec \alpha$$

where α is the angle which the tangent at P makes with the horizontal. Since

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

we have

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \cot^2 \alpha} = \csc \alpha$$

Or we may regard x and y as functions of s and write

★ $$\frac{dx}{ds} = \cos \alpha \qquad \text{and} \qquad \frac{dy}{ds} = \sin \alpha$$

Example. Find approximately the distance along the curve $y = x^2$ from the point (3,9) to the point (3.01,9.0601). Following Art. 56, we calculate an approximate value of Δs by replacing dx by Δx in the formula

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Thus we have

$$\frac{dy}{dx} = 2x, \qquad \left.\frac{dy}{dx}\right]_{x=3} = 6, \qquad \text{and} \qquad \Delta x = 0.01$$

Hence

$$ds = (\sqrt{1 + 36})(0.01) = 0.01\sqrt{37} = 0.01(6.08276)$$
$$= 0.0608$$

61. Curvature. Consider two circles such as (a) and (b) of Fig. 66, with radii $r_1 > r_2$. Imagine a point P to move along the circumference of the circle (a), and think of the line tangent to the circle at P. Suppose that P traverses an arc $\widehat{P_1Q_1}$ of some particular length, for instance, 1 cm. The tangent line will turn through an angle $\Delta\alpha_1$ radians. Now suppose a point traverses the arc $\widehat{P_2Q_2}$ on the circle (b) of the same length as the arc $\widehat{P_1Q_1}$ (for instance, in this case, 1 cm.); the tangent will turn through an angle $\Delta\alpha_2$ radians. It is intuitively evident that, since circle (b) has

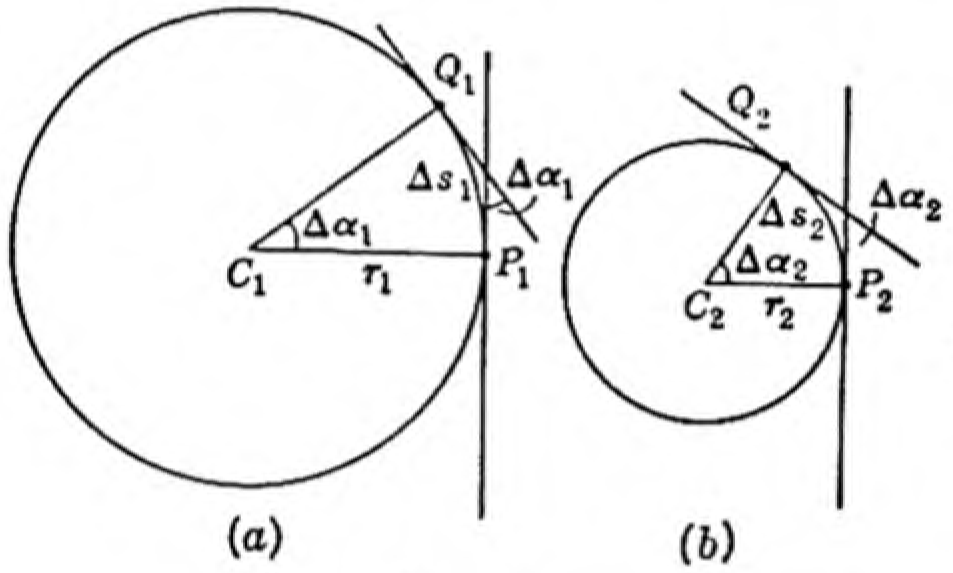


FIG. 66.

a smaller radius than circle (a), the angle $\Delta\alpha_2$ will be greater than the angle $\Delta\alpha_1$. It is natural to say that in moving along circle (b) the point P changes its direction of motion more per unit arc length than it does in moving along circle (a). The measure of this rate of change of direction along the circle is called its *curvature*. The curvature is obtained by comparing the angle through which the tangent turns with the length of arc traversed by P. Thus, if $P_1Q_1 = \Delta s_1$, then the average curvature from P_1 to Q_1 is $\dfrac{\Delta\alpha_1}{\Delta s_1}$ units of angle per unit of arc length. This is easily calculated in the case of the circle, thus:

We have (Fig. 66)

$$\angle P_1C_1Q_1 = \Delta\alpha_1 \qquad \text{(radians)}$$
$$\Delta s_1 = r_1\,\Delta\alpha_1$$

Hence
$$\frac{\Delta\alpha_1}{\Delta s_1} = \frac{\Delta\alpha_1}{r_1\,\Delta\alpha_1} = \frac{1}{r_1}$$

We might now hold P_1 fixed, let $\Delta s_1 \to 0$, and take the limit of the ratio

$$\lim_{\Delta s_1 \to 0} \frac{\Delta\alpha_1}{\Delta s_1} = \frac{d\alpha_1}{ds_1} = \frac{1}{r_1}$$

and call this the *curvature* of the circle *at* the point P_1. Note that the circle has the same curvature at all points, namely, $1/r_1$. In the same way, the curvature of the circle (*b*) is $\dfrac{d\alpha_2}{ds_2} = \dfrac{1}{r_2}$ at all points of that circle. We observe, as already remarked, that since $r_1 > r_2$, $1/r_1 < 1/r_2$ so that the curvature of (*a*) is less than the curvature of (*b*).

Now suppose that $y = f(x)$ has a graph, as shown in Fig. 67. Let $P(x,y)$ be any point on this curve. Draw the tangent at P, and let it make an angle α (measured in radians) with the horizontal. Now, as we

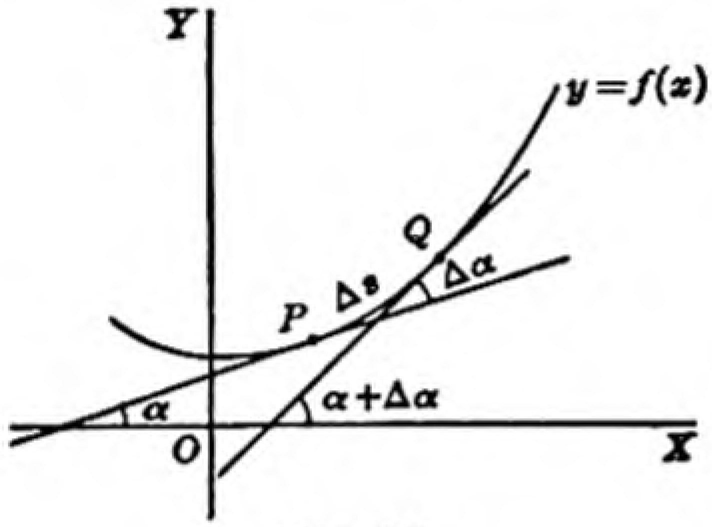


FIG. 67.

move along the curve to some nearby point Q, the tangent will turn through an angle $\Delta\alpha$. As in the case of the circle, we call $\dfrac{\Delta\alpha}{\Delta s}$ the *average curvature* of the arc $\widehat{PQ}$. Unlike the circle, where $\dfrac{\Delta\alpha}{\Delta s}$ was the same for all positions of P and Q, this curve will give different values to $\dfrac{\Delta\alpha}{\Delta s}$ depending upon the positions of P and Q. We therefore define *curvature at the point* P to be

★ $$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}$$

To evaluate $\dfrac{d\alpha}{ds}$ in terms of the coordinates of P, we shall use the fact that a derivative is the quotient of differentials. First note that the slope of the tangent line at P is

$$\tan\alpha = f'(x) = y' \qquad \text{or} \qquad \alpha = \arctan y'$$ $\alpha = \tan^{-1} y'$

Therefore $$\frac{d\alpha}{dx} = \frac{1}{1 + y'^2}\cdot\frac{dy'}{dx} = \frac{1}{1 + y'^2}\cdot\frac{d^2y}{dx^2} = \frac{y''}{1 + y'^2}$$ $\frac{d\alpha}{dx} = \frac{1}{1+y'^2}\frac{dy'}{dx}$

and $$d\alpha = \frac{y''\,dx}{1 + y'^2}$$ $= y''$

In the preceding section we found that

$$ds = \sqrt{1 + y'^2}\,dx$$

Dividing, we have

$$\star \qquad \kappa = \frac{d\alpha}{ds} = \frac{y''}{(1 + y'^2)^{3/2}} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

Observe that, if s increases with x so that

$$ds = + \sqrt{1 + y'^2}\, dx$$

the expression κ will have the same sign as the second derivative. Consequently, if κ turns out to be positive, the curve is concave upward; if negative, concave downward. However, it is quite common to regard the *curvature as given by the positive value of* κ.

Hence, when speaking of the curvature at point P, we shall always use $|\kappa|$. Thus the point of maximum curvature is the point at which the *numerical value* of the curvature is a maximum. When $\dfrac{d^2y}{dx^2} = 0$, we have $\kappa = 0$, or "zero curvature."

Note further that the definition of curvature does not require that α be the angle made by the tangent line with the horizontal. It could well be the angle made by the tangent with any fixed line in the plane. In fact, the definition of curvature is quite independent of the coordinate system in which the equation of the curve may be given. However, if the equation of the curve is given in rectangular coordinates, the above choice of α is convenient, for it enables us to express the curvature readily in terms of the coordinates of P. Let the student show that if α is taken to be the angle between the tangent line and the y axis, then

$$|\kappa| = \frac{\left|\dfrac{d^2x}{dy^2}\right|}{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}$$

It is instructive to note that since α is measured in radians the unit of curvature is *radians per unit of length*, for example, radians per centimeter.

62. Radius of Curvature. We saw in the preceding section that the curvature at any given point of a circle is equal to $1/r$. In other words, the radius of the circle is the reciprocal of its curvature. We shall define the *radius of curvature* ρ *at any point* P of a curve to be the reciprocal of the numerical value of the curvature at that point. Thus, if $y = f(x)$ is the equation of the curve, its radius of curvature at $P(x,y)$ is

$$\star \qquad \rho = \frac{1}{|\kappa|} = \frac{(1 + y'^2)^{3/2}}{|y''|} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\left|\dfrac{d^2y}{dx^2}\right|} = \left|\frac{ds}{d\alpha}\right|$$

Example 1. Find the curvature and radius of curvature of the curve $y = \frac{2}{3}x^{3/2}$ at the point $(8, \frac{32}{3}\sqrt{2})$. We have

$$\frac{dy}{dx} = x^{1/2} \quad \text{and} \quad \left.\frac{dy}{dx}\right]_{x=8} = \sqrt{8}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right]_{x=8} = \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}$$

Hence

$$\kappa = \frac{1/4\sqrt{2}}{(1+8)^{3/2}} = \frac{1/4\sqrt{2}}{27} = \frac{1}{108\sqrt{2}}$$

and

$$\rho = 108\sqrt{2}$$

Example 2. Find the curvature of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point (x,y). We have, differentiating with respect to x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{b^2}{a^2}\cdot\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\cdot\frac{y - x\dfrac{dy}{dx}}{y^2} = -\frac{b^2}{a^2}\cdot\frac{y + \dfrac{b^2}{a^2}\cdot\dfrac{x^2}{y}}{y^2} = -\frac{b^2}{a^2}\cdot\frac{a^2y^2 + b^2x^2}{a^2y^3}$$

The original equation is equivalent to $b^2x^2 + a^2y^2 = a^2b^2$, and therefore

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\cdot\frac{a^2b^2}{a^2y^3} = -\frac{b^4}{a^2y^3}$$

Also

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^4x^2}{a^4y^2} = \frac{a^4y^2 + b^4x^2}{a^4y^2}$$

Consequently

$$\kappa = \frac{-b^4/a^2y^3}{\left[\dfrac{a^4y^2 + b^4x^2}{a^4y^2}\right]^{3/2}} = \frac{-b^4/a^2y^3}{\dfrac{(a^4y^2 + b^4x^2)^{3/2}}{a^6y^3}}$$

of which the numerical value is

$$|\kappa| = \frac{a^4b^4}{(a^4y^2 + b^4x^2)^{3/2}}$$

Example 3. Find the points of maximum curvature on the curve $y = \sin x$. We must first find the curvature as a function of x, then the value of x that makes $|\kappa|$ a maximum. We have

$$\frac{dy}{dx} = \cos x \quad \text{and} \quad \frac{d^2y}{dx^2} = -\sin x$$

Therefore

$$\kappa = \frac{-\sin x}{(1 + \cos^2 x)^{3/2}}$$

Since this is to be a numerical maximum, we first calculate

$$\frac{d\kappa}{dx} = -\frac{(1+\cos^2 x)^{3/2}\cos x - \sin x\cdot\frac{3}{2}(1+\cos^2 x)^{1/2}(2\cos x)(-\sin x)}{(1+\cos^2 x)^3}$$

$$= -\frac{\cos x(1 + \cos^2 x + 3\sin^2 x)}{(1+\cos^2 x)^{5/2}}$$

This fraction can be zero only if $\cos x = 0$, and it never becomes infinite. Hence, $\cos x = 0$ and $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ give critical values of κ. Note that at each such point $\frac{d\kappa}{dx}$ changes sign, for its sign is opposite to that of $\cos x$ since

$$(1 + \cos^2 x + 3\sin^2 x)$$

and $(1 + \cos^2 x)^{5/2}$ are positive for all x. Now when $x = \pi/2$, $\kappa = -1$; but when $x = -\pi/2$, $\kappa = +1$. Hence it would appear that $x = \pi/2$ gives a minimum whereas $x = -\pi/2$ gives a maximum value of κ. But we wish the points where the *numerical value of* κ is a maximum, and consequently both points suffice. Similarly for the other points mentioned.

Actually, we might have observed that $|\kappa|$ is a maximum for values of x that make $\cos x = 0$, for such values of x make $\sin x = \pm 1$; hence these give κ the smallest denominator and numerically largest numerator. Ordinarily, in order to find maximum or minimum values of a function (such as the curvature), it is necessary to find the derivative of the function. But if any maximum or minimum values can be found readily by inspection, it may be possible to avoid calculation of the derivative.

Note that the points of numerically smallest curvature are those for which $\kappa = 0$, namely, $x = 0, \pm\pi, \pm 2\pi, \ldots$. These results are intuitively evident from the graph of $y = \sin x$. The student should inspect this curve carefully.

EXERCISES

Find the curvature and radius of curvature of each of the following curves at the points indicated (Ex. 1 to 14):

1. $y = x^2$
   (a) at (0,0)
   (b) at (2,4)
2. $y = 16x - x^2$ at (8,64)
3. $y = x^3 - 4x^2 + 2x - 1$ at $(2, -5)$
4. $y = 8 + 3x - x^3$ at $(3, -10)$
5. $y = \sin x$ at $x = \pi/6$
6. $y = \cos x$ at $x = \pi/4$
7. $y = \tan x$
   (a) at $x = \pi/4$
   (b) at $x = 0$
8. $y = \ln x$ at $x = 1$
9. $y = \log x$ at $x = 1$
10. $y = e^x$ at $x = 0$
11. $y = 10^x$ at $x = 0$
12. $y = \ln \sin x$ at $x = \pi/2$
13. $y = \cosh x$ at $x = 0$
14. $y = \sinh x$ at $x = 0$

Find the radius of curvature of each of the following curves at any point (x,y) of the curve (Ex. 15 to 27):

15. $y = \sin x$
16. $y = \ln x$
17. $y = e^x$
18. $y = \cosh x$
19. $y = \arcsin x$
20. $y = \arctan x$
21. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (hyperbola)
22. $2xy = a^2$ (hyperbola)
23. $x^2 = 4ay$ (parabola)
24. $y^2 = 4ax$ (parabola)
25. $x^{2/3} + y^{2/3} = a^{2/3}$ (hypocycloid)
26. $x^{1/2} + y^{1/2} = a^{1/2}$ (arc of parabola)
27. $x^3 + y^3 = a^3$

Find points of maximum and minumum (numerical) curvature on the following curves (Ex. 28 to 36):

28. $y = x^2$
29. $y = x^3$
30. $y = \cos x$
31. $y = \ln x$
32. $y = \ln \sin x$
33. $y = e^x$
34. $y = \cosh x$
35. $y = \sinh x$
36. A parabola

63. Curvature and Radius of Curvature in Parametric Representation. Suppose a curve is given in parametric form, for example, $x = t^2$, $y = 2t + 1$. To find the curvature at the point $t = \sqrt{3}$, we have

$$y' = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\dfrac{dy'}{dt}}{\dfrac{dx}{dt}} = \frac{-\dfrac{1}{t^2}}{2t} = -\frac{1}{2t^3}$$

and therefore, for any $t \neq 0$

$$\kappa = \frac{-1/2t^3}{(1 + 1/t^2)^{3/2}} = \frac{-1/2t^3}{(t^2 + 1)^{3/2}/t^3} = -\frac{1}{2(t^2 + 1)^{3/2}}$$

or

$$|\kappa| = \frac{1}{2(t^2 + 1)^{3/2}}$$

At the point where $t = \sqrt{3}$, $|\kappa| = 1/2(4)^{3/2} = \frac{1}{16}$. Thus, in general, if a curve is given by parametric equations, we can find its curvature at any point by first calculating $\dfrac{dy}{dx}$ by the methods of Art. 28 and then employing the formula for κ. However, we may express κ in terms of the derivatives of x and y with respect to the parameter if we wish. Thus, if $x = f_1(t)$ and $y = f_2(t)$, then, from Art. 28, we have

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}$$

Therefore

$$\kappa = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3} \frac{1}{\left[1 + \left(\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right)^2\right]^{3/2}} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]^{3/2}}$$

Applying the formula to our example, we have

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2 \qquad \frac{d^2x}{dt^2} = 2 \qquad \frac{d^2y}{dt^2} = 0$$

and
$$\kappa = \frac{2t \cdot 0 - 2 \cdot 2}{(4t^2 + 4)^{3/2}} = -\frac{4}{8(t^2 + 1)^{3/2}} = -\frac{1}{2(t^2 + 1)^{3/2}}$$

or
$$|\kappa| = \frac{1}{2(t^2 + 1)^{3/2}}$$

as already obtained in the previous calculation. The radius of curvature is given at once by $\rho = 1/|\kappa|$.

EXERCISES

Find the curvature and radius of curvature of the given curve at the points indicated (Ex. 1 to 4).

1. $\begin{cases} x = t - 2 \\ y = 3t^2 \end{cases}$ at $t = 1$

2. $\begin{cases} x = t^2 + 2t \\ y = 1/t \end{cases}$ at $t = 2$

3. $\begin{cases} x = \ln t \\ y = 1/t \end{cases}$ at $t = 1$

4. $\begin{cases} x = e^t \\ y = e^{-t} \end{cases}$ at $t = 0$

Find the curvature of the given curve at any point of the curve (assume a and b positive) (Ex. 5 to 10).

5. $\begin{cases} x = a \sin \alpha \\ y = a \cos \alpha \end{cases}$

6. $\begin{cases} x = a \sec \varphi \\ y = a \tan \varphi \end{cases}$

7. $\begin{cases} x = a \cos^3 \varphi \\ y = a \sin^3 \varphi \end{cases}$ (hypocycloid)

8. $\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases}$ (ellipse)

9. $\begin{cases} x = a \cosh u \\ y = b \sinh u \end{cases}$ (hyperbola)

10. $\begin{cases} x = a(\theta - \sin \theta) \\ y = a(1 - \cos \theta) \end{cases}$ (cycloid)

Find points of maximum and minimum curvature for:

11. An ellipse (Exercise 8)
12. A hyperbola (Exercise 9)
13. A cycloid (Exercise 10)
14. A hypocycloid (Exercise 7)

64. Circle of Curvature; Center of Curvature. Consider the curve $y = f(x)$, and upon it choose any point $P(x,y)$ (Fig. 68). Suppose the tangent and normal at P to be drawn. On the normal lay off, toward the concave side of the curve, a segment PC equal in length to the radius of curvature ρ of the curve at P. With C as a center and ρ as radius, draw a circle. This circle has curvature $|\kappa| = 1/\rho$, which is the same as the curvature of the given curve at the point P. Furthermore, the circle and the curve have a common tangent line at P. The circle is called the *circle of curvature or osculating circle*, and C the *center of curvature* of the given curve at the point P. It can be shown that this particular circle

fits the curve "more closely" in the neighborhood of P than does any other circle. In fact, let P_0 and P_1 be any two points on the given curve and near to P (Fig. 69); they may be on the same or opposite sides of P, and in general they do not lie in a straight line. Let C' be the center of

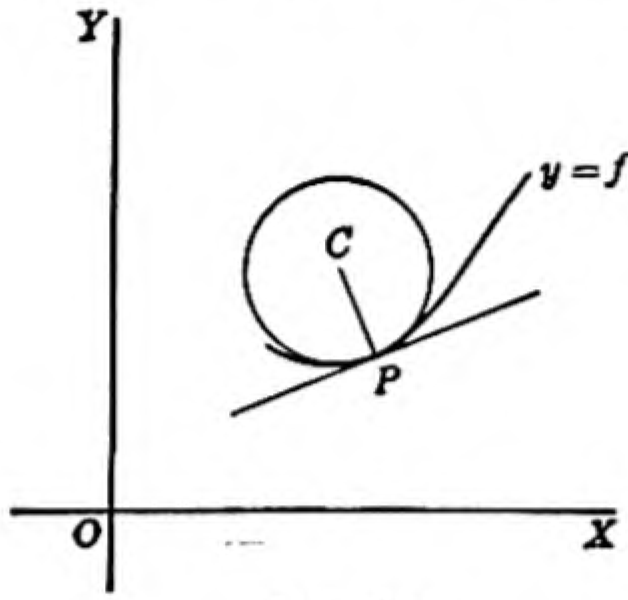


Fig. 68.

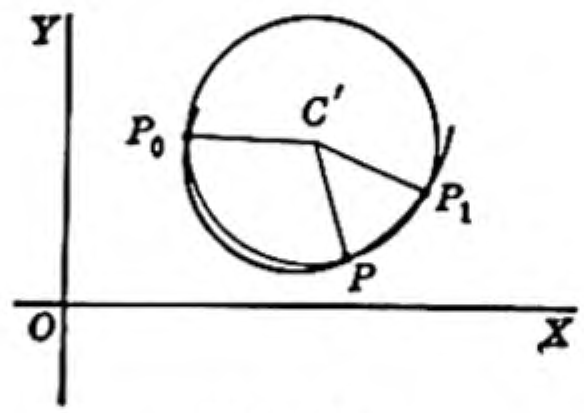


Fig. 69.

the circle through these points. Now let P_0 and P_1 both approach P along the curve. It can be shown that the circle approaches the circle of curvature as a limiting form, that C' approaches the center of curvature as a limiting position, and that $C'P$ approaches ρ as a limiting value.

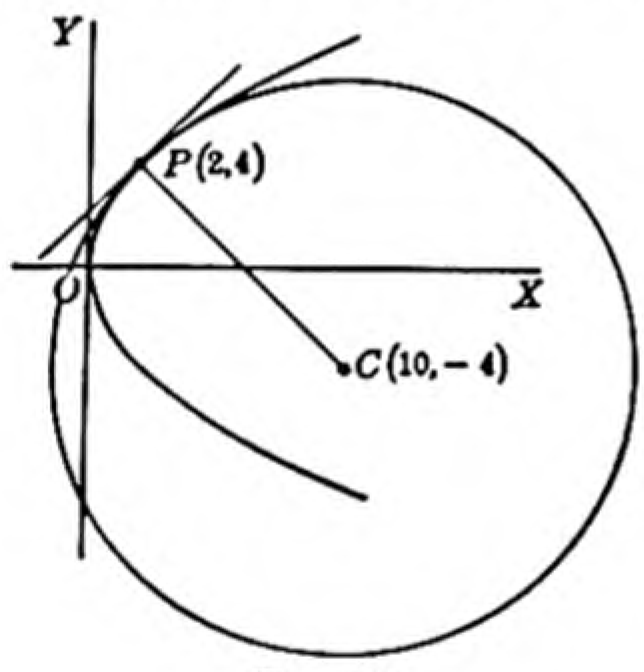


Fig. 70.

Example. Find the center of curvature of the parabola (Fig. 70) $y^2 = 8x$ at the point $P(2,4)$. We first find the radius of curvature at P, thus

$$2y\frac{dy}{dx} = 8, \qquad \frac{dy}{dx} = \frac{4}{y}, \qquad \text{and} \qquad \left.\frac{dy}{dx}\right]_{y=4} = 1$$

$$\frac{d^2y}{dx^2} = -\frac{4}{y^2}\frac{dy}{dx} = -\frac{16}{y^3} \qquad \text{and} \qquad \left.\frac{d^2y}{dx^2}\right]_{y=4} = -\frac{16}{64} = -\frac{1}{4}$$

Therefore
$$\rho = \frac{(1+1)^{3/2}}{\frac{1}{4}} = 4(2^{3/2}) = 8\sqrt{2}$$

To find the coordinates X, Y of C, we must find a point on the normal at P which is $8\sqrt{2}$ units from P. The slope of the normal is -1, and the equation is

$$y - 4 = -(x - 2)$$

or $x + y - 6 = 0$. Since C is at a distance $8\sqrt{2}$ from P, its coordinates (X,Y) must satisfy the condition

$$(X - 2)^2 + (Y - 4)^2 = (8\sqrt{2})^2 = 128 \tag{4}$$

Since C also lies on the normal,

$$X + Y - 6 = 0$$

Thus $Y = 6 - X$; and, substituting into (4),

$$\begin{aligned} (X - 2)^2 + (2 - X)^2 &= 128 \\ 2(X - 2)^2 &= 128 \\ (X - 2)^2 &= 64 \\ X - 2 &= \pm 8 \end{aligned}$$

and

$$X = 10 \text{ or } -6$$

If $X = 10$, $Y = -4$; if $X = -6$, $Y = 12$.

Since it was specified that C was to be on the concave side of the curve and since the curve is concave downward at $P(2,4)$, our required point is $C(10,-4)$. The circle is drawn in Fig. 70.

Suppose the center of curvature to have been found for every point of a given curve. Then these centers of curvature will form another curve called the *evolute* of the given curve.

EXERCISES

Find the center of curvature of each of the following curves corresponding to the points indicated. Sketch the curve and the circle of curvature in each case (Ex. 1 to 5).

1. The parabola $x^2 = 8y$ corresponding to the vertex
2. The parabola $x^2 = 8y$ corresponding to the point $(4,2)$
3. The hyperbola $xy = 16$ corresponding to the point $(4,4)$
4. The hyperbola $x^2 - y^2 = 16$ corresponding to the point $(5,3)$
5. The curve $y = x^3$ at $(2,8)$

6. Using the method indicated in the example of Art. 64, show that the coordinates X, Y of the center of curvature corresponding to the point (x_1,y_1) of the curve $y = f(x)$ are

$$X = x_1 - \frac{\{1 + [f'(x_1)]^2\}f'(x_1)}{f''(x_1)}$$

$$Y = f(x_1) + \frac{1 + [f'(x_1)]^2}{f''(x_1)} \qquad \text{where } f''(x_1) \neq 0$$

Use the formulas of Exercise 6 to find the center of curvature of each of the following curves corresponding to the points indicated (Ex. 7 to 10):

7. The parabola $x^2 = 4ay$ corresponding to the vertex
8. The rectangular hyperbola $xy = a^2$ corresponding to the vertex (a,a)
9. The parabola $y^2 = 4ax$ corresponding to (x_1,y_1) on the parabola
10. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at any point (x_1,y_1) on the ellipse

11. Since the *evolute* of a curve is the locus of centers of curvature, the formulas of Exercise 6 are the parametric equations (in which x_1 is the parameter) of the evolute of $y = f(x)$. Elimination of x_1 gives the equation $F(X,Y) = 0$ of the evolute. Find the evolute of the parabola $x^2 = 4y$, and sketch the two curves.

12. If the coordinates X, Y of the center of curvature are expressed by the formulas of Exercise 6 in terms of x_1 and y_1, the equation of the evolute may be found by eliminating x_1 and y_1 from these equations and the original equation of the curve. Find the evolute of the parabola $y^2 = 4ax$ (Exercise 9) by this method, and draw the two curves.

13. Use the method of Exercise 12 to find the evolute of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Exercise 10). Sketch the two curves.

14. If the parametric equations of a curve are given, the parametric equations (in terms of the same parameter) of the evolute are found by use of the formulas of Exercise 6. Find the evolute of the curve whose parametric equations are

$$x = 2t \qquad y = t^2 - 1$$

15. Find the evolute of the cycloid, and sketch both curves:

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$$

16. Find the evolute of the ellipse

$$x = a\cos\varphi \qquad y = b\sin\varphi$$

Compare with Exercise 13.

17. Find the evolute of the rectangular hyperbola

$$x = \pm a\cosh u \qquad y = a\sinh u$$

18. Find the evolute of the hyperbola

$$x = \pm a\cosh u \qquad y = b\sinh u$$

19. In Exercise 18, express the equation of the evolute in cartesian form.

65. Rectilinear Motion. Suppose that a point P is moving along a straight line l. Let us choose as x axis the line of motion and take any desired point as origin. Let t represent time, measured in some convenient unit, before or after a certain instant called the *time origin.* Thus, a negative value of t means simply so many units of time *before* the time origin, whereas a positive value of t means so many units *after* the time origin. For example, if twelve o'clock noon is chosen as time origin and if t is measured in hours, then $t = -2$ means ten o'clock in the morning, whereas $t = 4$ means four o'clock in the afternoon. Now suppose that the abscissa of P is a function of t, say $x = f(t)$. We are able to calculate the position of P at any time.

We recall (Art. 13) that the time rate of change of distance x at a given instant is called the *velocity* of the moving point at that instant. Thus $v = \dfrac{dx}{dt} = f'(t)$ units of distance per unit of time (miles per hour, feet per second, etc.) is the velocity at any time. Note that, when v is positive, x

is an increasing function of t, and P is moving toward the right. If v is negative, x is a decreasing function of t, and P is moving toward the left. If v is a constant, we call the motion *uniform.* The *numerical value* of the velocity is called the *speed.* The time rate of change of the velocity is called the *acceleration* of P, which, therefore, is given by the derivative of v with respect to t. Thus

$$j = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t)$$

is the acceleration of P at any time. Note that this is measured in units of velocity per unit of time, for example, in *feet per second per second* which may be written ft./sec.^{2}. If the acceleration is a constant, the motion is said to be *uniformly accelerated.* The "dot" notations $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$ are often used for derivatives with regard to time.

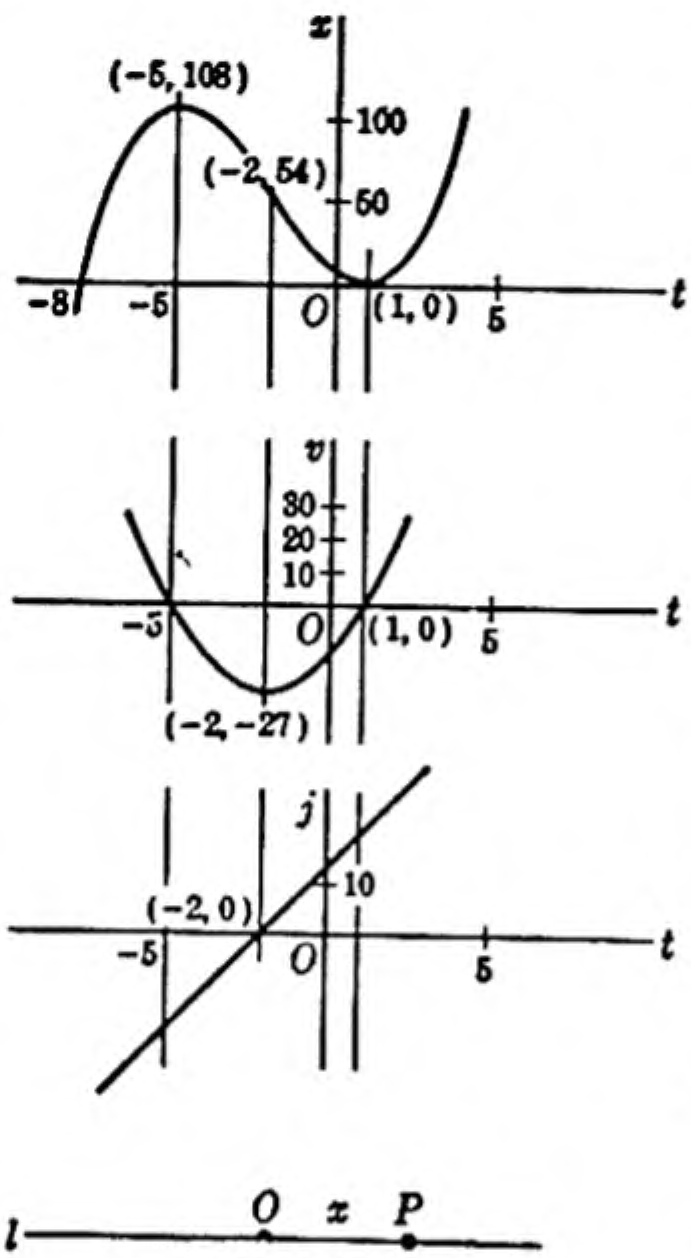


FIG. 71.

Example. A particle moves in a straight line with the law of motion

$$x = t^3 + 6t^2 - 15t + 8 = (t-1)^2(t+8)$$

Describe the motion. We have, at once,

$$\begin{aligned} v &= 3t^2 + 12t - 15 = 3(t^2 + 4t - 5) = 3(t+5)(t-1) \\ j &= 6t + 12 = 6(t+2) \end{aligned}$$

If we draw a graph of the function $x = t^3 + 6t^2 - 15t + 8$, then we may represent v and j by graphs that are the first and second derived curves of the function (see Art. 36). This is done in Fig. 71. Notice that the same scale is used on the horizontal (the t) axis but that the vertical scales are different for x, v, and j.

The graphs mentioned above will enable us to describe the motion, but first let us list various facts for reference.

x	v	j
If $t < -8$, $x < 0$	If $t < -5$, $v > 0$	If $t < -2$, $j < 0$
If $t = -8$, $x = 0$	If $t = -5$, $v = 0$	If $t = -2$, $j = 0$
If $-8 < t < 1$, $x > 0$	If $-5 < t < 1$, $v < 0$	If $t > -2$, $j > 0$
If $t = 1$, $x = 0$	If $t = 1$, $v = 0$	
If $t > 1$, $x > 0$	If $t > 1$, $v > 0$	

The reader should keep clearly in mind the fact that the moving point P does *not* traverse any of these curves. It remains on the straight line l, and its *entire motion*

takes place upon that line. The curves are to help us discover the way in which the abscissa of P varies. First, suppose t to be numerically large but negative. The first curve shows us that $x < 0$, the second that $v > 0$, and the third that $j < 0$. Hence, the point P is on the line l, far to the left of 0, and moving toward the *right* with decreasing velocity and speed. When $t = -8$, P goes through 0. It continues to move toward the right until $t = -5$. Then $v = 0$, and $x = 108$.

Since $v = 0$, P stops moving momentarily. As t increases beyond -5, v becomes negative, and so P starts back toward the left. Note that, for $-5 < t < -2$, $v < 0$ and also $j < 0$. Hence, v is a decreasing function; and since it is negative, its *numerical value* is *increasing*. Therefore, P speeds up until $t = -2$. Here $j = 0$; and when $t > -2$, $j > 0$, and v becomes an increasing function. However, since v is still negative, it now decreases in numerical value; hence, P, although still moving toward the left, slows down. When $t = 1$, $v = 0$ and P stops moving toward the left. Since $x = 0$ at $t = 1$, P is now at 0. As soon as $t > 1$, $v > 0$ and P starts moving toward the right. Since j is still positive, $v > 0$ is an increasing function that must therefore increase numerically, and hence P moves to the right with increasing speed. As t becomes numerically large and positive, P continues to move to the right with increasing speed. Observe that the *speed* is increasing whenever v and j have the *same* sign and decreasing when v and j have *opposite* signs. The student should convince himself that this is true in general.

The procedure to be followed in discussing the behavior of a particle moving in a straight line according to a given law of motion, $x = f(t)$, may be summarized as follows:

1. Find expressions for

$$v = \frac{dx}{dt} \qquad \text{and} \qquad j = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

2. Sketch roughly the graph of x as a function of t, and then exhibit v and j as first and second derived curves.

3. Let t increase from numerically large negative values to large positive values, and list in order values of t that produce a change in sign or a zero value of x, v, j.

4. Write a brief description of the motion, starting with negative t and permitting t to increase to large positive values.

EXERCISES

A point moves in a straight line according to the law of motion given. Discuss fully the motion in each case (Ex. 1 to 30).

1. $x = 96t - 16t^2$
2. $x = 9 - t^2$
3. $x = t^3 - t^2$
4. $x = t^3 - t$
5. $x = 4t - t^2$
6. $x = t^2 - 16$
7. $x = t(2 - t)^2$
8. $x = 3t^2 - t^3$
9. $x = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 1$
10. $x = \frac{1}{3}t^3 + \frac{5}{2}t^2 + 6t$
11. $x = 4 - 12t + 6t^2 - t^3$
12. $x = t^4 - 5t^2 + 4$
13. $x = 5t^4 - 20t^5$
14. $x = \sin t$
15. $x = \cos 3t$
16. $x = -4 \sin 2t$
17. $x = -5 \cos \frac{1}{3}t$
18. $x = e^{-t}$

19. $x = \ln(1 + t)$
20. $x = te^{-t}$
21. $x = t - \sin t$
22. $x = t - \cos t$
23. $x = t \sin t$
24. $x = \cosh t$
25. $x = \sinh t$
26. $x = e^{-t} \sin t$
27. $x = e^{-3t} \cos 3t$
28. $x = a \cos kt$
29. $x = a \sin kt$
30. $x = a \cos\left(kt + \frac{\pi}{4}\right)$

31. Show that a particle moving in a straight line reaches a maximum distance from O when v changes sign and that it attains a maximum or minimum speed when j changes sign.

32. If $x = a \cosh kt$, show that the acceleration is k^2x.

33. If $x = ae^t + be^{-t}$, show that the acceleration is equal in magnitude to x.

34. If $x = \sqrt{t}$, show that the acceleration is $-2v^3$.

35. If $x = r \cos(kt + \varphi)$, show that the acceleration is $-k^2x$. This rectilinear motion is known as *simple harmonic motion.*

36. In Exercise 35, show that (*a*) the moving point P oscillates with maximum distance from the origin (*center of motion*) $|OP| = r$ (*amplitude*); (*b*) the time required for one complete oscillation from extreme right to extreme left and back again is $2\pi/k$ (*period*); (*c*) the numerically smallest value of t for which P is at the extreme right-hand position is $-\varphi/k$ (*phase*). (*d*) Find the times at which P is at the center of motion.

37. Show that $x = r \sin(kt + \varphi)$ represents a simple harmonic motion with amplitude r, period $2\pi/k$, and phase $-\varphi/k$ where phase is defined to be the numerically smallest value of t for which P is at the center of motion.

38. Show that $x = A \cos kt + B \sin kt$ represents a simple harmonic motion with amplitude $\sqrt{A^2 + B^2}$, period $2\pi/k$, and phase $\frac{1}{k} \arctan \frac{B}{A}$. [*Hint:* reduce to the form $x = r \cos(kt + \varphi)$.]

39. Verify directly that if $x = A \cos kt + B \sin kt$, then $j = -k^2x$.

40. A flywheel 3 ft. in diameter turns at the uniform rate of 1 r.p.s. Find the law describing the motion of the projection upon the ground of a point on the rim of the wheel.

41. A flywheel 4 ft. in diameter revolves at the uniform rate of 80 r.p.m. Find the law describing the motion of the projection upon the ground of a point on the rim of the wheel (t is measured in seconds).

42. A particle moves in a straight line with simple harmonic motion such that, when $v = 5$ ft. per second, $x = 5$ ft., and when $v = 13$ ft. per second, $x = 3$ ft. Find the equation of motion if the phase is zero.

43. A particle moves in a straight line with simple harmonic motion such that, when $v = 4$ ft. per second, $x = 1$ ft. and, when $v = 2$ ft. per second, $x = 5$ ft. Find the equation of motion if the phase is zero.

44. Show that the equation $x = 2 - 2\sin^2 4t$ describes a simple harmonic motion with center at $x = 1$. Find the amplitude and period.

45. Show that the equation $x = 16\cos^2 \frac{2}{3}t - 11$ describes a simple harmonic motion with center at $x = -3$. Find the amplitude and period.

66. Vectors. A quantity that has *direction* as well as magnitude is called a *vector quantity.* Such a quantity may be represented by a line segment called a *vector*, whose length is proportional to the magnitude and whose direction is the direction of the given quantity. Thus (Fig.

72), vectors **a** and **b** have the same direction but different magnitudes; **c** has a direction opposite to that of **a**; **d** has a direction different from the other three. The magnitude of the vector **a** is denoted by a and is called a *scalar* quantity. Since velocity has magnitude and direction, it is a vector quantity and may be represented by a vector. Similarly, acceleration may be represented by a vector. If a particle receives an acceleration, we say that the acceleration is imparted by a *force*, which is defined to be equal to the product of the acceleration by the mass of the particle. Since the mass is a quantity possessing only magnitude, this product is a vector quantity.

Vectors may be combined in various ways. We pay attention only to the *addition* of vectors. To add vectors **a** and **b** (Fig. 73), lay them off starting from a common point O. Through A, draw a line parallel to the direction of **b** and through B a line parallel to the direction of **a**. Let these lines intersect at C. Draw OC.

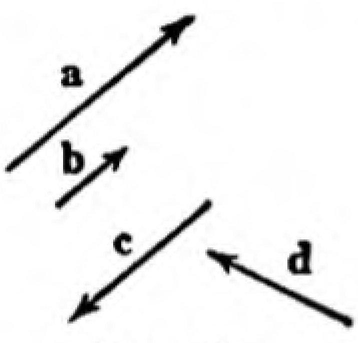


FIG. 72.

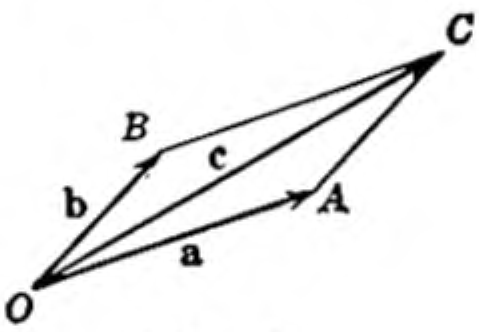


FIG. 73.

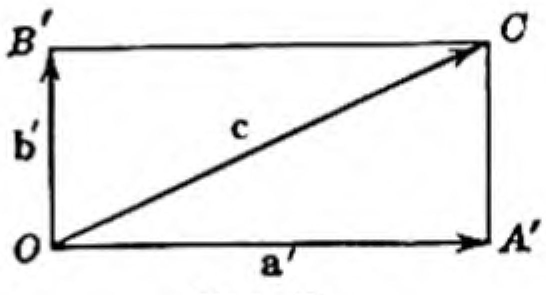


FIG. 74.

The vector **c** of length OC is called the *sum*, or the *resultant*, of vectors **a** and **b**. Vectors **a** and **b** are called the *components* of **c** in the directions OA and OB. We may *resolve* **c** into components in any two distinct directions. For example, let us resolve **c** into components, one of which is horizontal and the other vertical. We need only draw horizontal lines through O and C and find their intersections A' and B' with vertical lines drawn through O and C, as indicated in Fig. 74. It is clear that **c** is the sum of **a**′ and **b**′.

Let us refer to Exercises 35 to 39, page 162, on simple harmonic motion. We saw that if $x = r \cos (kt + \varphi)$, then $j = -k^2x$. Therefore, if a particle moves according to this law of motion, the force acting on the particle must be a multiple of j and therefore proportional to x. Furthermore, notice that, when $x > 0$, $j < 0$; when $x < 0$, $j > 0$. Hence the vector that represents the force acting at any particular instant will be horizontal and directed toward the center of the motion.

67. Curvilinear Motion. Suppose that a point moves along a plane curve and that the coordinates of its position at time t are given by

$$x = f_1(t) \qquad \text{and} \qquad y = f_2(t)$$

These are the parametric equations of the path of motion (Fig. 75). The time rates of change of the x and y coordinates are called the *components of velocity* in the directions, respectively, of the x and y axes. Since

velocities have directions as well as magnitudes, they may be represented by vectors. Thus, let these components of velocity be represented by vectors as in Fig. 75. Their magnitudes are

★ $$v_x = \frac{dx}{dt} = f_1'(t) \qquad \text{and} \qquad v_y = \frac{dy}{dt} = f_2'(t)$$

and their directions are parallel to the coordinate axes. The direction of each (left or right, up or down) will depend upon the algebraic signs

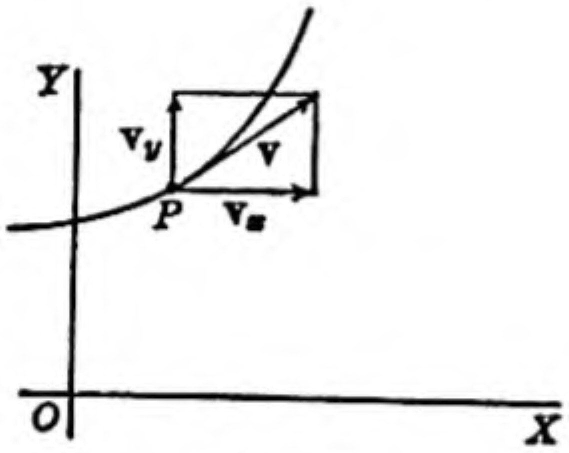


FIG. 75.

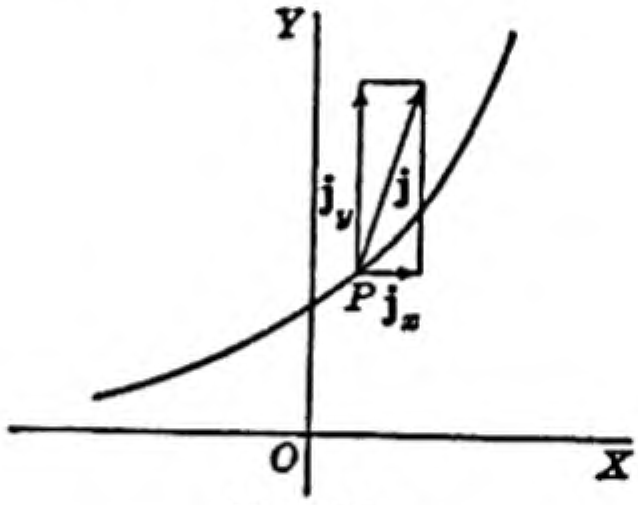


FIG. 76.

of $f_1'(t)$ and $f_2'(t)$, respectively. The resultant of these two vectors is called the *velocity* of the point *in its path*, and its magnitude is clearly

$$v = \sqrt{v_x^2 + v_y^2}$$

Its direction can be specified by giving the angle α that the vector $\mathbf{v}$ makes with the horizontal,

★ $$\tan \alpha = \frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Hence, the vector $\mathbf{v}$ has the *direction of the tangent line.* Note further that

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left|\frac{ds}{dt}\right|$$

The time rates of change of v_x and v_y are called the *components of acceleration* in the direction of the x and y axes. Thus,

★ $$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = f_1''(t)$$

★ $$j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = f_2''(t)$$

These components of acceleration may be represented by vectors (Fig. 76) with magnitudes j_x and j_y and directions parallel to the coordinate axes. Their resultant is called the *acceleration* of P. This vector has magnitude

★ $$j = \sqrt{j_x^2 + j_y^2} \tag{5}$$

and if φ is the angle that it makes with the horizontal, then

★ $$\tan \varphi = \frac{j_y}{j_x} = \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$$

Observe that the magnitude j is *not* the derivative of the magnitude of v. Neither is the vector **j** directed, in general, along the tangent line.

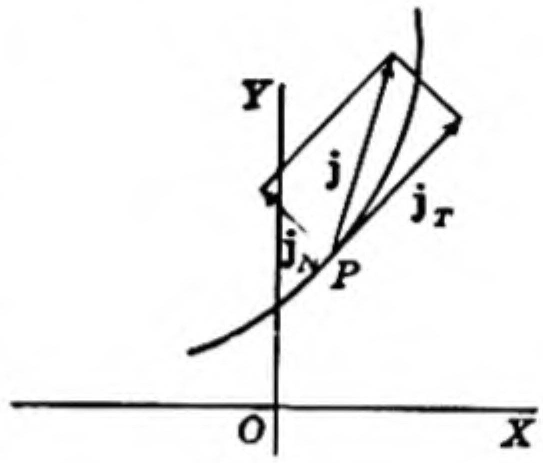


FIG. 77.

We may resolve the vector **j** into components in any two desired directions. In particular, let us resolve it into components one of which is in the direction of the tangent line and the other in the direction of the normal (Fig. 77). These components are called the *tangential acceleration* and *normal acceleration* and are denoted by $\mathbf{j}_T$ and $\mathbf{j}_N$. Since $\mathbf{j}_T$ is the acceleration in the direction of motion, its magnitude must be simply the time rate of change of the velocity v. Thus

$$j_T = \left|\frac{dv}{dt}\right|$$

But since $v^2 = v_x^2 + v_y^2$, we have

$$2v\frac{dv}{dt} = 2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt} = 2v_xj_x + 2v_yj_y$$

or

★ $$j_T = \left|\frac{dv}{dt}\right| = \frac{|v_xj_x + v_yj_y|}{v} \tag{6}$$

To find the magnitude of $\mathbf{j}_N$, note that (Fig. 77)

$$j^2 = j_T^2 + j_N^2$$

Hence $j_N = \sqrt{j^2 - j_T^2}$. Substituting values of j and j_T from (5) and (6), we have

$$j_N^2 = j_x^2 + j_y^2 - \frac{v_x^2j_x^2 + 2v_xv_yj_xj_y + v_y^2j_y^2}{v^2}$$

But since $v^2 = v_x^2 + v_y^2$,

$$j_N^2 = \frac{(j_x^2 + j_y^2)(v_x^2 + v_y^2) - (v_x^2j_x^2 + 2v_xv_yj_xj_y + v_y^2j_y^2)}{v^2} = \frac{(v_xj_y - v_yj_x)^2}{v^2}$$

or

★ $$j_N = \frac{|v_xj_y - v_yj_x|}{v}$$

Example 1. Show that $j_N = v^2|\kappa| = v^2/\rho$, where $|\kappa| = 1/\rho$ is the curvature of the path of motion at point P. We recall (Art. 63) that

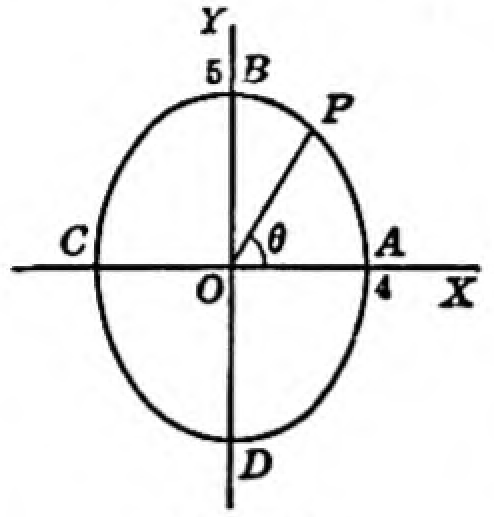


FIG. 78.

$$\kappa = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]^{3/2}} = \frac{v_xj_y - v_yj_x}{v^3}$$

Therefore $j_N = v^2|\kappa|$; that is, the normal acceleration is proportional to the curvature. Thus, if a particle of mass m moves along a path (for instance, a smooth wire), it exerts a force proportional to the curvature of the path and the square of the velocity and acting at right angles to the path.

Example 2. A point moves with equations of motion

$$x = 4 \cos 3t \qquad y = 5 \sin 3t$$

Find the equation of the path, v, j, j_T, and j_N at any time t. To eliminate t from the two equations, we have $\dfrac{x^2}{16} = \cos^2 3t$, $\dfrac{y^2}{25} = \sin^2 3t$; therefore $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ is the equation of the path (Fig. 78).

$$v_x = -12 \sin 3t \qquad v_y = 15 \cos 3t$$

and
$$\begin{aligned}
v &= \sqrt{144 \sin^2 3t + 225 \cos^2 3t} = 3\sqrt{16 \sin^2 3t + 25 \cos^2 3t} \\
j_x &= -36 \cos 3t \\
j_y &= -45 \sin 3t \\
j &= \sqrt{36^2 \cos^2 3t + 45^2 \sin^2 3t} = 9\sqrt{16 \cos^2 3t + 25 \sin^2 3t} \\
&= 9\sqrt{x^2 + y^2} \\
j_T &= \frac{|v_xj_x + v_yj_y|}{v} = \frac{|(-12 \sin 3t)(-36 \cos 3t) + (15 \cos 3t)(-45 \sin 3t)|}{3\sqrt{16 \sin^2 3t + 25 \cos^2 3t}} \\
&= \frac{81|\sin 3t \cos 3t|}{\sqrt{16 \sin^2 3t + 25 \cos^2 3t}} \\
j_N &= \frac{|v_xj_y - v_yj_x|}{v} = \frac{|(-12 \sin 3t)(-45 \sin 3t) - (15 \cos 3t)(-36 \cos 3t)|}{3\sqrt{16 \sin^2 3t + 25 \cos^2 3t}} \\
&= \frac{180(\sin^2 3t + \cos^2 3t)}{\sqrt{16 \sin^2 3t + 25 \cos^2 3t}} = \frac{180}{\sqrt{16 \sin^2 3t + 25 \cos^2 3t}} \\
&= \frac{540}{v}
\end{aligned}$$

Let the student show that v^2 is a maximum at points A and C (Fig. 78) and that, therefore, these are points at which the speed is a maximum. Similarly, show that v^2, and hence the speed, is a minimum at points B and D. Find the speed at points A, B, C, D.

EXERCISES

1. Show that vectors **a** and **b** can be added as follows: Place the initial point of one vector upon the terminal point of the other, being careful to preserve directions. The vector joining the free initial and terminal points is the resultant.

2. Show that three vectors **a**, **b**, **c** may be added as follows: Place the initial point of **b** upon the terminal point of **a**, and then place the initial point of **c** upon the terminal point of **b**, being careful in each case to preserve directions. The vector joining the initial point of **a** to the terminal point of **c** is the resultant of the three vectors.

3. In Exercise 2, show that the same resultant is obtained if the operations are carried out in any order. For example, place the initial point of **a** upon the terminal point of **b** and the initial point of **c** upon the terminal point of **a**; then join the initial point of **b** to the terminal point of **c**.

4. Generalize Exercises 2 and 3 to the case of n vectors.

5. Given vectors **a** and **b**. Show how to construct a vector **c** such that $\mathbf{b} + \mathbf{c} = \mathbf{a}$. The vector **c** is defined to be the result of subtracting **b** from **a**.

A point moves in a plane curve according to the law of motion given. Find the cartesian equation of the path, v, j, j_T, j_N at the point specified, and discuss the motion in Exercises 6 to 16.

6. $\begin{cases} x = t/2 \\ y = t^2 \end{cases}$ at $t = 1$ **7.** $\begin{cases} x = t + 1 \\ y = t - 1 \end{cases}$ at any point

8. $\begin{cases} x = t^2 \\ y = t^4 \end{cases}$ at $t = 1$ **9.** $\begin{cases} x = t \\ y = \sin t \end{cases}$ at $t = \pi/2$

10. $\begin{cases} x = 2 \sin t \\ y = 2 \cos t \end{cases}$ at any point **11.** $\begin{cases} x = 3 \cos 2t \\ y = 4 \sin 2t \end{cases}$ at $t = 0$

12. $\begin{cases} x = \cosh t \\ y = \sinh t \end{cases}$ at any point **13.** $\begin{cases} x = 3 \cosh 2t \\ y = 4 \sinh 2t \end{cases}$ at $t = 0$

14. $\begin{cases} x = e^t \\ y = e^{-t} \end{cases}$ at any point

15. $\begin{cases} x = 1 + 2 \cos t \\ y = -3 + 2 \sin t \end{cases}$ at any point

16. $\begin{cases} x = t^2 \\ y = \cos t \end{cases}$ at $t = \pi/2$

17. Show that when a point moves along a curve with a constant velocity in the path ($v = k$) the acceleration is always directed along the normal to the path.

18. In Exercise 11, find points of maximum and minimum speed.

19. In Exercise 11, verify that $j_N = v^2|\kappa|$ (κ = curvature) at any point.

20. A point moves along the parabola $y^2 = 4x$ with constant velocity $v = 10$, and $v_x > 0$. Find v_x, v_y, j_x, j_y, j at (9,6).

21. A point moves along the curve $y = x^3$ with constant velocity $v = 6$, and $v_x > 0$. Find v_x, v_y, j_x, j_y, j at (2,8).

22. If a point moves on the hyperbola $x^2 - y^2 = a^2$, can v_x and v_y ever be equal?

23. A point transverses the parabola $y = x^2 - 4x$ with constant horizontal component of velocity $v_x = 2$. Find $v_y, v, j_x, j_y, j, j_T, j_N$ at (5,5).

24. A point traverses the circle $x^2 + y^2 = 25$ with constant velocity $v = 10$ and $v_x < 0$. Find $v_x, v_y, j_x, j_y, j_T, j_N$ at (3,4), and verify that $j_N = v^2|\kappa|$.

25. A point moves along the curve $y = \sin x$. Find a point at which $v_x = v_y$; at which $v_x = 2v_y$.

26. A point moves along the curve $y = \sin x$ with constant velocity $v = 5$ and $v_x > 0$. Find $v_x, v_y, j_x, j_y, j, j_T, j_N$ at $x = \pi/3$, and verify that $j_N = v^2|\kappa|$ where κ is the curvature at this point.

In the following (Ex. 27 to 30), two points move in a plane. The first two equations describe the motion of the first point, the second two describe the motion of the second point. In each case, find where, if anywhere, the points collide.

27. $x = 2t$ and $x = t + 2$
$y = t^2 - 1$ $\qquad y = 2t^2 - 5$

28. $x = t^2 + 2$, $y = t - 2$ and $x = 3t^2$, $y = 4t - 5t^2$

29. $x = 2 \sin t$, $y = 2 \cos t$ and $x = \tan t$, $y = \cot t$

30. $x = \sin t$, $y = \tan t$ and $x = \sin 2t$, $y = -3 \cot 2t$

68. Angular Velocity. If a point P moves along a curve (Fig. 79) with equations of motion $x = f_1(t)$ and $y = f_2(t)$, the angle θ between the x axis and the radius vector OP is a function of the position of P and therefore of the time t. The time rate of change of θ is called the *angular velocity* of OP and is denoted by ω. Thus

★ $$\omega = \frac{d\theta}{dt}$$

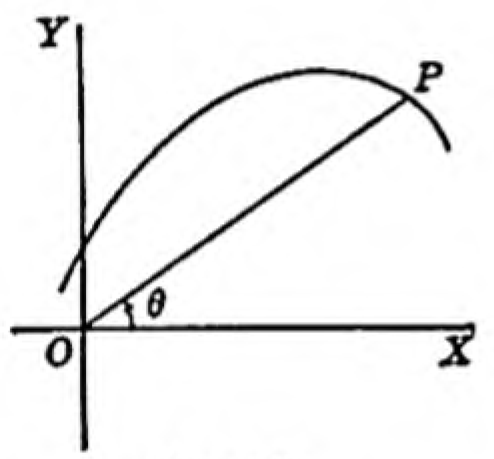


FIG. 79.

Since $\theta = \arctan(y/x)$, we have

$$\frac{d\theta}{dt} = \frac{1}{1 + (y/x)^2} \cdot \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$$

and therefore

★ $$\omega = \frac{xv_y - yv_x}{x^2 + y^2} \tag{7}$$

This is measured in radians per unit of time. If desired, it may be expressed in revolutions per unit of time. Note that ω may be found entirely in terms of t through the given equations of motion.

The time rate of change of the angular velocity is called the *angular acceleration*, which we denote by α. Thus

★ $$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Example 1. In Example 2 of the preceding section, we had point P moving on the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ (see Fig. 78) with equations of motion

$$x = 4 \cos 3t \qquad y = 5 \sin 3t$$
$$v_x = -12 \sin 3t \qquad v_y = 15 \cos 3t$$

Let t be measured in seconds. Let us find the angular velocity of OP when P is at the point $B(0,5)$. This occurs when $3t = \pi/2$. Hence $v_x = -12$ and $v_y = 0$. Therefore, by formula (7)

$$\omega = \frac{0 - 5(-12)}{0 + 25} = \frac{12}{5} \qquad \text{(radians/sec.)}$$

To change to revolutions per second, note that one revolution is 2π radians. Hence $\omega = \frac{12}{5} \div 2\pi = 6/5\pi$ r.p.s.

We may find ω at any time from formula (7), thus

$$\omega = \frac{(4\cos 3t)(15\cos 3t) - (5\sin 3t)(-12\sin 3t)}{16\cos^2 3t + 25\sin^2 3t}$$
$$= \frac{60}{16\cos^2 3t + 25\sin^2 3t} \qquad \text{(radians/sec.)}$$

Note that this is always positive; hence, θ is always increasing with time, and P moves around the ellipse in a counterclockwise direction.

Example 2. A point P moves with constant velocity $v = 4$ ft. per second along the line $3x - 2y + 12 = 0$ (Fig. 80). Find the angular velocity of OP at any time. We must calculate v_x and v_y in order to use formula (7).

Differentiating

$$3x - 2y + 12 = 0$$

with respect to t, we have $3v_x - 2v_y = 0$, and therefore $v_y = \frac{3}{2}v_x$. We also have

$$v = 4 = \sqrt{v_x^2 + v_y^2}$$

and therefore $\quad v_x^2 + v_y^2 = 16$

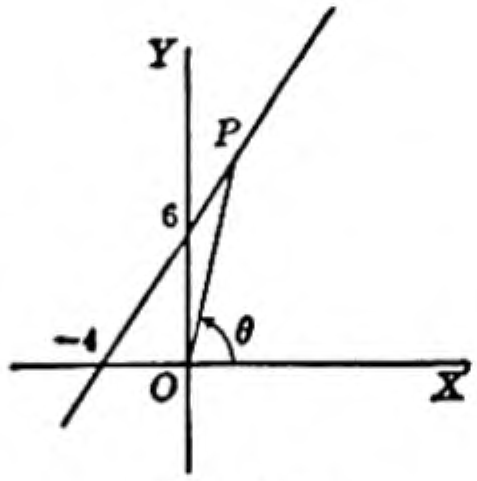


FIG. 80.

Substituting for v_y, we have

$$v_x^2 + \tfrac{9}{4}v_x^2 = 16$$
$$13v_x^2 = 64$$
$$v_x = \pm\frac{8}{\sqrt{13}}$$

Hence $$v_y = \pm\frac{3}{2}\cdot\frac{8}{\sqrt{13}} = \pm\frac{12}{\sqrt{13}}$$

This gives, by formula (7),

$$\omega = \pm\left(\frac{x\cdot\dfrac{12}{\sqrt{13}} - y\cdot\dfrac{8}{\sqrt{13}}}{x^2 + y^2}\right) = \pm\frac{4}{\sqrt{13}}\left(\frac{3x - 2y}{x^2 + y^2}\right)$$
$$= \mp\frac{48}{\sqrt{13}\,(x^2 + y^2)} \qquad \text{(radians/sec.)}$$

since $3x - 2y = -12$. The plus sign is to be used if P moves "down" the line and θ therefore increases with t; the minus sign is to be used if P moves "up" the line. Note that, if P is at a remote distance from O, x and y are large, and ω is small. Evidently, as P moves indefinitely far from 0, x and y become infinite, and ω approaches zero.

EXERCISES

1. Find the equation of the path, the angular velocity ω, and the velocity in the path, v, for the motion

$$x = a\cos t \qquad y = a\sin t$$

2. Find the equation of the path, the angular velocity, and the angular acceleration at any point for the motion

$$x = t + 1 \qquad y = 3t - 2$$

3. A wheel of radius 10 in. is rotating with constant angular velocity ω. A point on the rim 6 in. above the level of the center is found to have $v_y = 72$ in. per second. Find ω. (*Hint:* Take the origin at the center of the wheel.)

4. If a point P moves around a circle of radius r ft. with velocity in the path of motion of $\dfrac{ds}{dt} = v$ ft. per second, show that $|\omega| = \dfrac{v}{r}$.

5. A flywheel 10 ft. in diameter makes 40 r.p.m. Find v_x, v_y, and v for a point on the rim 4 ft. above and to the right of the center (let $\omega > 0$).

6. A point P moves along the line $3x - 2y + 6 = 0$ with $v = \sqrt{13}$ ft. per second. Find the angular velocity of the line OP (joining P to the origin) at any time.

7. A point P moves along the line $3x + 5y - 15 = 0$ with $v = 4\sqrt{34}$ ft. per second. Find the angular velocity of OP when P is at $(-5,6)$.

8. Find the angular acceleration in Exercises 6 and 7.

9. A point A on the rim of a wheel 2 ft. in diameter follows the cycloid

$$x = (\theta - \sin\theta) \qquad y = 1 - \cos\theta$$

If the wheel makes 30 r.p.m., find v_x, v_y, and v for the point A.

10. In Exercise 9, find j, j_T, j_N.

11. A wheel 3 ft. in diameter rolls along a level roadway at 40 r.p.m. Find v_x, v_y, v, j, j_T, j_N for a point A on the rim.

12. Evaluate v_x, v_y, v, j, j_T, j_N in Exercise 11 for A at points for which $y = 0$, $y = \frac{3}{2}$ ft., $y = 3$ ft.

13. A point P traverses the parabola $y^2 = 4x$ with $v_x = 2$ ft. per second. Find the angular velocity of the line OP for P at any point of the parabola.

14. A point moves with the law of motion $x = 4\cos t$, $y = 3\sin t$. Find the angular velocity of the line joining the point to the origin, and sketch the path of motion.

15. A point P moves with law of motion $x = 6t - 2$, $y = 8t + 1$. Find the angular velocity of OP, and sketch the path.

16. A point P moves with law of motion $x = 4t^2$, $y = t^4$. Sketch the path and find the angular velocity of OP when P is at the point (16 16).

17. Find the angular velocity of OP for each of the motions and for the indicated positions of P in Exercises 6 to 16, page 167.

MISCELLANEOUS EXERCISES

Find the radius of curvature at the indicated point (Ex. 1 to 12).

1. $y = x^{3/2}$ at $(4,8)$ **2.** $y = x^{2/3}$ at $(8,4)$

3. $4x^2 + 9y^2 = 36$ at any point

4. $3x^2 + 2y^2 = 11$ at $(-1,2)$

5. $y^2 + 2y + 3x - 6 = 0$ at $(2,-2)$

6. $y = \arcsin x$ at $(\frac{1}{2}, \pi/6)$ **7.** $y = \arctan x$ at $(1, \pi/4)$

8. $y = \ln\cosh x$ at $(0,0)$ **9.** $y = \ln\operatorname{sech} x$ at $(0,0)$

10. $y = \operatorname{argcosh} x$ at $(1,0)$ **11.** $\begin{cases} x = a\csc\alpha \\ y = a\cot\alpha \end{cases}$ at any point

12. $\begin{cases} x = a\cosh^3 t \\ y = a\sinh^3 t \end{cases}$ at any point. (Also, find the cartesian equation, and sketch the curve.)

13. Find points of maximum curvature, and sketch the hyperbola

$$x = a\sec\varphi \qquad y = a\tan\varphi$$

14. Let $P_1(x_1,y_1)$ be any point on the curve $y = a\cosh(x/a)$. Let N_1 be the x intercept of the normal at P_1. Show that the radius of curvature at P_1 is equal in length to the distance P_1N_1.

A point moves in a straight line according to the law of motion given. Discuss fully the motion in each case (Ex. 15 to 21).

15. $x = (t-2)^3$

16. $x = t^3 - 3t^2 + 4$

17. $x = e^t - 2t$

18. $x = e^{-t}\cos t$

19. $x = \sin t - \cos t$

20. $x = \sin 2t + \cos 3t$

21. $x = e^{-2t}\sin 3t$

22. The law of motion of a point moving in a straight line is given by the following table of data. Sketch the first and second derived curves, and then discuss the motion.

t	0	1	2	3	5	7	9	10	11	12	15	20
x	0	2	3.4	4.3	5	3.7	2.5	2.1	2	2.3	3.5	11

23. Same as Exercise 22 for the following data:

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x	5	3	1.5	0.4	−0.7	−1	−0.7	0	1.5	3.4	4.6	5.6	6	5.6	4.9	3.6	2	0	−2

24. Show that $x = 10\cos^2 \frac{4}{5}t - 7$ describes a simple harmonic motion with center at $x = -2$. Find the amplitude and period.

25. Show that $x = 2 - 14\sin^2 6t$ describes a simple harmonic motion with center at $x = -5$. Find the amplitude and period.

A point moves in a plane curve according to the law of motion given. Find the cartesian equation of the path, v, j, j_T, j_N at the point specified, and discuss the motion (Ex. 26 to 31).

26. $\begin{cases} x = 3t + 1 \\ y = t^2 - 4 \end{cases}$ at $t = 2$

27. $\begin{cases} x = t^2 \\ y = 1/t \end{cases}$ at $t = 1$

28. $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ at any point

29. $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$ at $t = \pi/4$

30. $\begin{cases} x = \ln t \\ y = t^2 \end{cases}$ at $t = 2$

31. $\begin{cases} x = \tan t \\ y = \sec t \end{cases}$ at $t = \pi/4$

32. A point traverses the parabola $y^2 + 2y - 3x - 3 = 0$ with constant horizontal component of velocity $v_x = 2$. Find v_y, v, j_x, j_y, j, j_T, j_N at $(4,3)$.

33. A point traverses the hyperbola $x^2 - y^2 = 16$ with constant vertical component of velocity $v_y = 1$. Find v and j at $(5,3)$.

34. A point traverses the curve $y = \cos x$ with constant velocity $v = 10$ and $v_x > 0$. Find v_x, v_y, j_x, j_y, j at $x = \pi/6$.

35. In constructing a railroad track a "transition curve" is used to pass from the straight-line track to a circular curve. Suppose that this transition curve has for its equation $y = ax^3$ and that a car runs along the track with constant velocity v. Using the fact that $j_N = v^2|\kappa|$, show that the normal component of acceleration (and hence the force normal to the curve of the track) at any point of the transition curve approaches zero as x approaches zero and also as x increases indefinitely.

36. Work Exercise 35 if the transition curve has for its equation $y = ax^4$.

37. Find a formula for the angular acceleration of the radius vector OP in terms of x, y, v_x, v_y, j_x, j_y.

CHAPTER 10

POLAR COORDINATES

69. Polar Coordinates. So far we have dealt with functions whose graphical representation has involved only a rectangular coordinate system. Polar coordinates are useful in writing the equations of certain curves as well as in expressing the conditions in various physical problems, and it is important to see how some of the properties of such curves can be found by use of the calculus. We recall the relations between the rectangular and polar coordinates of a point. Let the *pole* be taken at the origin of the rectangular system, and let the *initial line* (or *polar axis*) be taken on the positive half of the x axis (Fig. 81). Let P be any point in the plane with rectangular coordinates x, y. The *radius vector* r* and the *vectorial angle* θ are connected with x and y by the equations

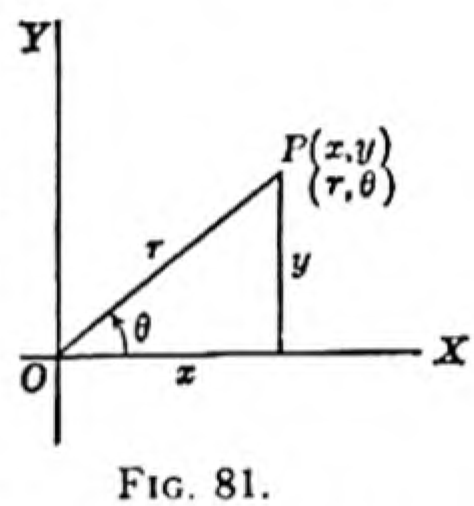


FIG. 81.

$$x = r \cos \theta \qquad y = r \sin \theta \tag{1}$$

or

$$r^2 = x^2 + y^2 \qquad \theta = \arctan \frac{y}{x} \tag{2}$$

70. Angle between the Tangent and Radius Vector. Suppose that $r = f(\theta)$ is a continuous curve with continuously turning tangent line. Let $P(r,\theta)$ be a point on this curve. We may determine the direction of the tangent at P by considering the angle ψ measured counterclockwise from the radius vector OP to the tangent line (Fig. 82).

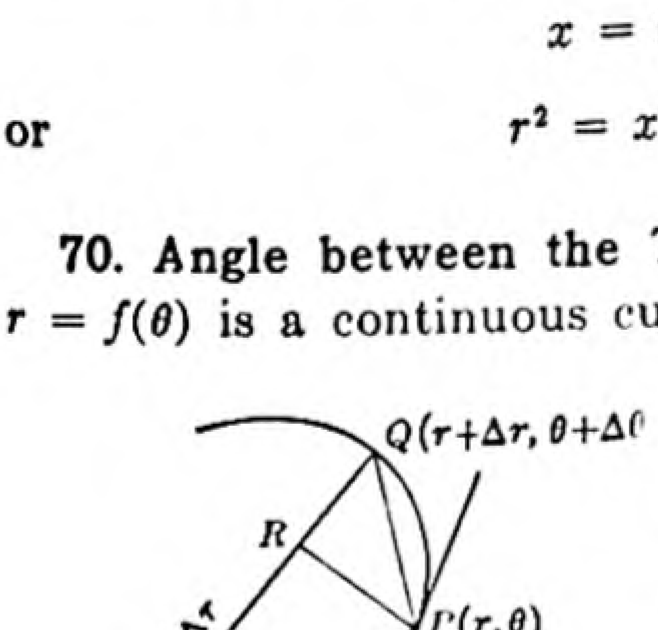


FIG. 82.

Let Q be a point on the curve near to P, and let the coordinates of Q be $r + \Delta r$, $\theta + \Delta\theta$, where θ and $\Delta\theta$ are measured in radians.

Draw the chord PQ and the perpendicular PR upon OQ. If Q is allowed to approach P along the curve, the secant line PQ will approach the position of the

* There are two systems of polar coordinates in general use, one in which r is restricted to positive values and another in which r may have both positive and negative values. We shall understand that the latter system is meant in this book.

tangent line PT, and the angle RQP will approach ψ. We shall, therefore, express $\tan \angle RQP$ in terms of r, θ, Δr, $\Delta\theta$ and then take the limit as $Q \to P$.

To this end, we note that, in the right triangle PRQ,

$$\tan \angle RQP = \frac{PR}{RQ}$$

Now $PR = r \sin \Delta\theta$. Also

$$\begin{aligned} RQ &= OQ - OR = r + \Delta r - r \cos \Delta\theta \\ &= \Delta r + r(1 - \cos \Delta\theta) = \Delta r + 2r \sin^2 \frac{\Delta\theta}{2} \end{aligned}$$

Therefore
$$\tan \angle RQP = \frac{r \sin \Delta\theta}{\Delta r + 2r \sin^2 \dfrac{\Delta\theta}{2}}$$

Divide numerator and denominator by $\Delta\theta$, obtaining

$$\tan \angle RQP = \frac{r \cdot \dfrac{\sin \Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r \sin^2 \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}}}$$

Now let Q approach P along the curve. Then $\Delta\theta$ and Δr approach zero, and we have

$$\tan \psi = \lim_{Q \to P} \tan \angle RQP = \frac{r}{\dfrac{dr}{d\theta}}$$

since
$$\lim_{\Delta\theta \to 0} \frac{\sin^2 \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}} = \lim_{\Delta\theta \to 0} \frac{\sin \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}} \cdot \lim_{\Delta\theta \to 0} \sin \frac{\Delta\theta}{2} = 1 \cdot 0 = 0$$

We have

★
$$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}} = r \frac{d\theta}{dr}$$

In Fig. 82 we may find $\tan \alpha$, the slope of the tangent line PT, as follows:

$$\alpha = \theta + \psi$$

$$\tan \alpha = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\tan \theta + r \dfrac{d\theta}{dr}}{1 - r \dfrac{d\theta}{dr} \tan \theta} \tag{3}$$

By drawing figures illustrating various positions for P, OP, TP, the student will see that in any case α differs from $\theta + \psi$ by some multiple of

π; hence, $\tan \alpha = \tan (\theta + \psi)$, and (3) holds. It is customary to choose $0 \leq \alpha < \pi$.

This formula might have been obtained from equations (1) by use of differentials, as follows:

$$\begin{aligned} dy &= r \cos \theta \, d\theta + \sin \theta \, dr \\ dx &= -r \sin \theta \, d\theta + \cos \theta \, dr \\ \tan \alpha &= \frac{dy}{dx} = \frac{\sin \theta \, dr + r \cos \theta \, d\theta}{\cos \theta \, dr - r \sin \theta \, d\theta} \end{aligned}$$

Now divide numerator and denominator by $\cos \theta \, dr$.

$$\tan \alpha = \frac{\tan \theta + r \dfrac{d\theta}{dr}}{1 - r \dfrac{d\theta}{dr} \tan \theta}$$

Hence $\tan \alpha = \tan (\theta + \psi)$, and we note that, as before, α either equals $\theta + \psi$ or differs from it by some multiple of π.

Example. Find ψ, $\tan \alpha$, and α for any point of the cardioid $r = a(1 + \cos \theta)$ (Fig. 83); also for the points $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$.

We have $\dfrac{dr}{d\theta} = -a \sin \theta$, and therefore

$$\tan \psi = \frac{r}{-a \sin \theta} = \frac{a(1 + \cos \theta)}{-a \sin \theta} = -\cot \frac{\theta}{2}$$

Therefore $$\psi = \frac{\pi}{2} + \frac{\theta}{2}$$

This gives $$\alpha = \theta + \psi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{3\theta}{2}$$

Therefore $$\tan \alpha = \tan \left(\frac{\pi}{2} + \frac{3\theta}{2} \right) = -\cot \frac{3\theta}{2}$$

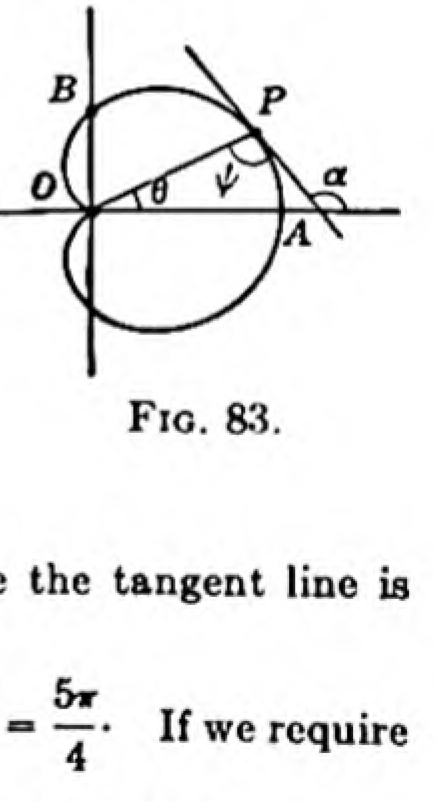


FIG. 83.

At point A we have $\theta = 0$, $\psi = \pi/2$, $\alpha = \pi/2$, and therefore the tangent line is vertical and has no slope.

At point B, we have $\theta = \dfrac{\pi}{2}$, $\psi = \dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{3\pi}{4}$, and $\alpha = \dfrac{\pi}{2} + \dfrac{3\pi}{4} = \dfrac{5\pi}{4}$. If we require α to be always between 0 and π, this gives $\dfrac{5\pi}{4} - \pi = \dfrac{\pi}{4}$ for α. The slope of the tangent line is therefore $\tan \alpha = 1$. Formula (3) cannot be used in this case since $\tan \theta$ has no value for $\theta = \pi/2$.

As point P approaches O along the curve, the angle θ approaches π. Since

$$\psi = \frac{\pi}{2} + \frac{\theta}{2}$$

angle ψ also approaches π. Thus the limiting position of the tangent line coincides with OX, and we take $\alpha = 0$.

71. Angle between Two Curves. Let two curves, C_1 with equation $r = f_1(\theta)$ and C_2 with equation $r = f_2(\theta)$, intersect in a point P (Fig. 84). Let PT_1 be the tangent to C_1 at P, and let PT_2 be the tangent to C_2 at P. Let φ be the angle from PT_1 to PT_2. Then $\varphi = \psi_2 - \psi_1$. Hence

$$\tan \varphi = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_1 \tan \psi_2}$$

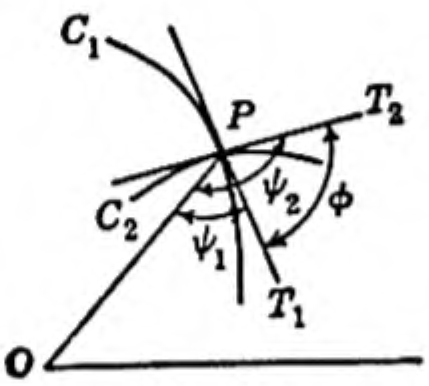


FIG. 84.

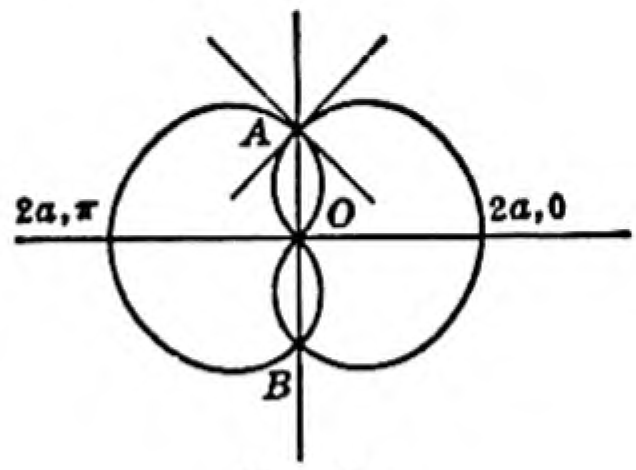


FIG. 85.

Example. Find the angle between the curves (Fig. 85)

$$r_1 = a(1 - \cos \theta) \qquad \text{and} \qquad r_2 = a(1 + \cos \theta)$$

We first find points of intersection. Solving simultaneously, we have

$$1 - \cos \theta = 1 + \cos \theta \qquad \cos \theta = 0$$

For $\theta = \pi/2$, $r_1 = r_2 = a$; this is point A. Similarly, point $B(a,3\pi/2)$ is a point of intersection. The pole is also an intersection, for both curves pass through O, although $r_1 = 0$ for $\theta = 0$, while $r_2 = 0$ for $\theta = \pi$.

Consider first point A. Here

$$\tan \psi_1 = \frac{a(1 - \cos \theta)}{a \sin \theta} = \tan \frac{\theta}{2} = 1$$

$$\tan \psi_2 = \frac{a(1 + \cos \theta)}{-a \sin \theta} = -\cot \frac{\theta}{2} = -1$$

Therefore $\psi_1 = \pi/4$, and $\psi_2 = 3\pi/4$. Hence $\varphi = \psi_2 - \psi_1 = \pi/2$. It is instructive to note that, for point A, the formula for $\tan \varphi$ has denominator $1 + \tan \psi_1 \tan \psi_2 = 0$, while the numerator is 2. Consequently, at point A, φ is an angle with no tangent; therefore, it is a right angle. Because of symmetry, at point B, $\varphi = \pi/2$.

Next consider point O. When $\theta = 0$, $\tan \psi_1 = 0$. When $\theta = \pi$, $\tan \psi_2 = 0$. In both cases the tangents coincide with OX. Hence $\varphi = 0$.

EXERCISES

Find the angle between the radius vector and the tangent, and also the slope of each of the following curves at the points indicated (Ex. 1 to 8):

1. The cardioid $r = a(1 + \sin \theta)$ at $\theta = \pi/6$

2. The circle $r = 2a \cos \theta$ at $\theta = \pi/6$

3. The parabola $r = \dfrac{a}{1 - \cos \theta}$ at $\theta = \dfrac{\pi}{2}$

4. The four-leaved rose $r = a \cos 2\theta$ at $\theta = \pi/6$
5. The lemniscate $r^2 = a^2 \cos 2\theta$ at $\theta = \pi/6$
6. The ellipse $r = \dfrac{1}{2 - \cos \theta}$ at $\theta = \dfrac{\pi}{3}$
7. The hyperbola $r = \dfrac{1}{1 + 3 \cos \theta}$ at $\theta = \dfrac{\pi}{3}$
8. The three-leaved rose $r = a \sin 3\theta$ at $\theta = \pi/6$

9. Show that ψ is constant for the logarithmic, or "equiangular," spiral $r = e^{a\theta}$. Find ψ for $a = 1$ and for $a = 2$.
10. Show that $\tan \psi = \theta$ for the spiral of Archimedes $r = a\theta$. Find ψ when $\theta = 45°$, $\theta = 2\pi$, $\theta = 10\pi$, $\theta = 100\pi$.
11. Find the slope of $r = a\theta$ at $\theta = 45°$ (Exercise 10).
12. Find the slope of $r = e^\theta$ at $\theta = \pi/2$ (Exercise 9).
13. Find the slope of $r\theta = a$ ("hyperbolic spiral") at $\theta = \pi$
14. Find $\tan \psi$ for the curve $r = a/\sqrt{\theta}$ ("lituus") at any point.

Find the angle of intersection of the following curves (Ex. 15 to 22):

15. The circles $r = \frac{1}{2}$ and $r = \cos \theta$
16. The cardioid $r = a(1 - \cos \theta)$ and the circle $r = a \cos \theta$
17. The spirals $r = a\theta$ and $r = a/\theta$
18. The circle $r = \sin \theta$ and the four-leaved rose $r = \cos 2\theta$
19. The two four-leaved-rose curves $r = a \sin 2\theta$ and $r = a \cos 2\theta$
20. The two parabolas $r = \sec^2 (\theta/2)$ and $r = \csc^2 (\theta/2)$
21. The circle $r = 10 \sin \theta$ and the line $r = 4 \sec \theta$
22. The curves $r = a \tan \theta$ and $r = a \cot \theta$

23. Show that the cardioids $r = a(1 - \cos \theta)$ and $r = b(1 + \cos \theta)$ intersect at right angles for all positive values of a and b. Assign various values to a and b, and sketch the curves.
24. If $r = f(\theta)$ is the equation of some curve, find the tangent of the angle ψ between the radius vector and the tangent line at the point (r,θ) by the following method:

$$r = f(\theta) \qquad x = r \cos \theta \qquad y = r \sin \theta$$

$$\text{Slope of tangent} = \tan \alpha = \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

$$\tan \psi = \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta}$$

Compare with Art. 70.

72. Polar Subtangent and Polar Subnormal. In addition to the subtangent and subnormal already defined (Art. 30), namely, the projections of the lengths of tangent and normal upon OX, we shall consider the projections of tangent and normal at P upon a line through O perpendicular to the radius vector at P. Let $r = f(\theta)$ have a tangent and a normal at a point P as shown in Figs. 86a and 86b. Draw OP, the radius vector at P, and through O draw a line perpendicular to OP. Let the tangent at P intersect this line at T, and let the normal intersect it at N. The seg-

ment PT is called the *length of the polar tangent*, and its projection OT is called the *polar subtangent;* the segment PN is called the *length of the polar normal*, and its projection ON is called the *polar subnormal.* We consider two cases.

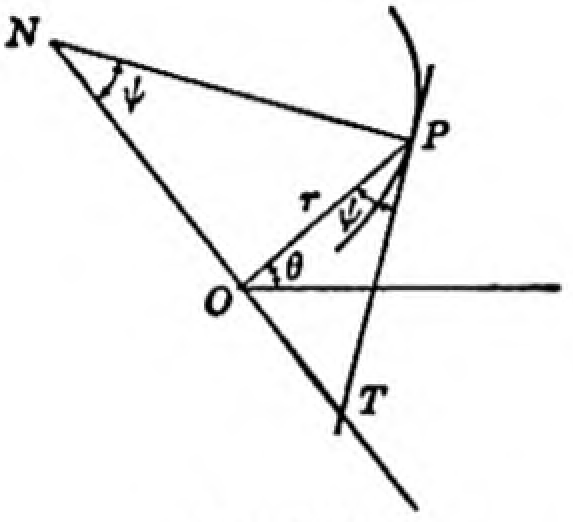


FIG. 86a.

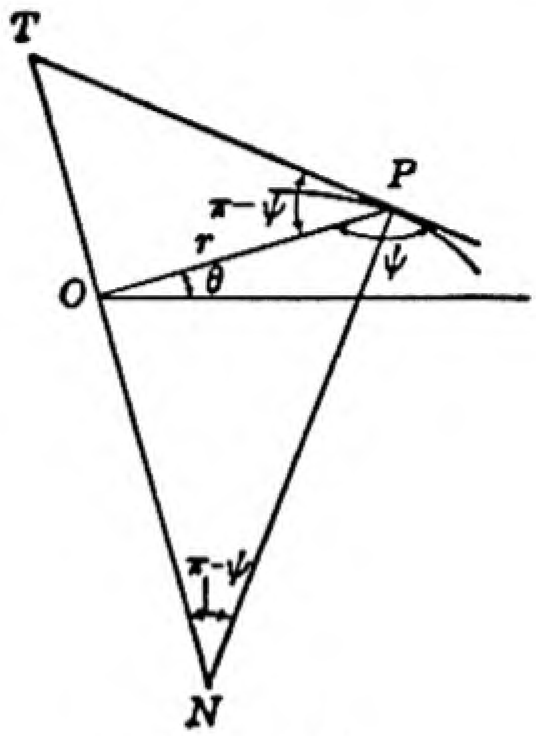


FIG. 86b.

First suppose that r increases with θ as in Fig. 86a. Then $\dfrac{dr}{d\theta}$ is positive, and

$$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}} = r\frac{d\theta}{dr}$$

is positive. Hence ψ is an acute angle as shown. Since $\tan \psi = OT/r$

$$\star \qquad OT = r \tan \psi = r^2 \frac{d\theta}{dr} \qquad \text{(the polar subtangent)} \tag{4}$$

We also have $\tan \psi = r/ON$, whence

$$\star \qquad ON = \frac{r}{\tan \psi} = \frac{dr}{d\theta} \qquad \text{(the polar subnormal)} \tag{5}$$

We may find PT and PN from the right triangles, thus

$$PT = \sqrt{\overline{OP}^2 + \overline{OT}^2} = \sqrt{r^2 + r^4\left(\frac{d\theta}{dr}\right)^2} = r\sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$$

$$PN = \sqrt{\overline{OP}^2 + \overline{ON}^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Note that when $\dfrac{dr}{d\theta}$ is positive the polar subtangent is measured to the right of an observer standing at O and looking along OP.

Next suppose that r decreases with increasing θ as shown in Fig. 86b.

Then $\frac{dr}{d\theta}$ is negative, and $\tan \psi = r \frac{d\theta}{dr}$ is negative. Hence ψ is an obtuse angle as shown. Formulas (4) and (5) now give negative values for the subtangent and subnormal. The reader will notice, of course, that, if we use triangle OTP of Fig. 86*b*, we obtain

$$OT = r \tan (\pi - \psi) = -r \tan \psi = -r^2 \frac{d\theta}{dr}$$

which is positive since $\frac{d\theta}{dr}$ is negative. Note that, when $\frac{dr}{d\theta}$ is negative, the polar subtangent is measured to the left of the observer standing at O and looking along OP.

Example. If $r = a(1 + \cos \theta)$, find the polar subtangent, polar subnormal, and the lengths of the polar tangent and polar normal at the point where $\theta = \arctan \frac{3}{4}$

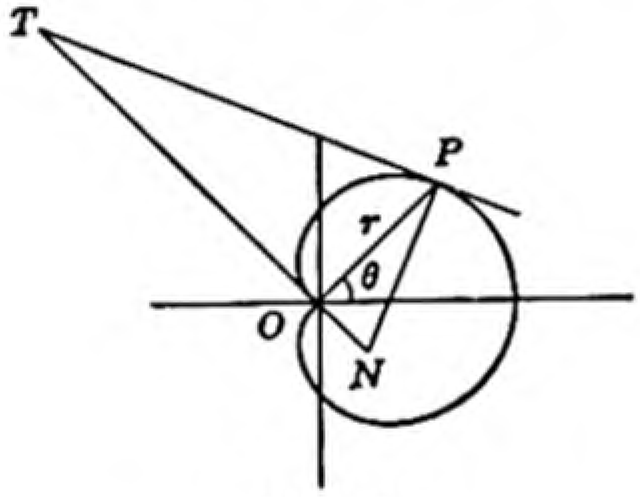


FIG. 87.

(Fig. 87). We have

$$\left.\frac{dr}{d\theta}\right]_{\theta=\arctan\frac{3}{4}} = -a \sin \theta\Big]_{\theta=\arctan\frac{3}{4}} = -\frac{3}{5}a \qquad \text{and} \qquad r = a(1 + \tfrac{4}{5}) = \tfrac{9}{5}a$$

Formulas (4) and (5) give

$$OT = \frac{81}{25}a^2\left(-\frac{5}{3a}\right) = -\frac{27}{5}a$$

$$ON = -\tfrac{3}{5}a$$

and

$$PT = \sqrt{\tfrac{81}{25}a^2 + \tfrac{729}{25}a^2} = \tfrac{9}{5}\sqrt{10}\,a$$

$$PN = \sqrt{\tfrac{81}{25}a^2 + \tfrac{9}{25}a^2} = \tfrac{3}{5}\sqrt{10}\,a$$

EXERCISES

In the following cases, find the lengths of the polar subtangent, subnormal, tangent, and normal at the points indicated. Make a rough sketch in each case.

1. The cardioid $r = a(1 + \sin \theta)$ at $\theta = \arctan \frac{4}{3}$
2. The four-leaved rose $r = a \sin 2\theta$ at $\theta = 30°$
3. The circle $r = 2a \cos \theta$ at $\theta = 45°$
4. The spiral $r = a\theta$ at $\theta = 1$ radian

5. The spiral $r = a\theta$ at any point. (*Note: The subnormal is constant.*)
6. The spiral $r = a/\theta$ at any point. (*Note: The subtangent is constant.*)
7. The curve $r = a \tan \theta$ at any point
8. The curve $r = a \cot \theta$ at any point
9. The curve $r = a/\sqrt{\theta}$ at any point
10. The spiral $r = a^\theta$ at any point. (Consider the case in which $a = e$.)

73. Derivative of Arc Length. If we measure the distance s along the curve $r = f(\theta)$ from some fixed point to the point $P(r,\theta)$, then s is a function of θ. We shall find a formula for $\frac{ds}{d\theta}$. Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$ (Fig. 88), and let Q with coordinates $r + \Delta r$, $\theta + \Delta\theta$ be a nearby point of the curve. We denote the arc PQ by Δs and chord PQ by $\overline{PQ}$. Draw PR perpendicular to OQ. In the right triangle PRQ,

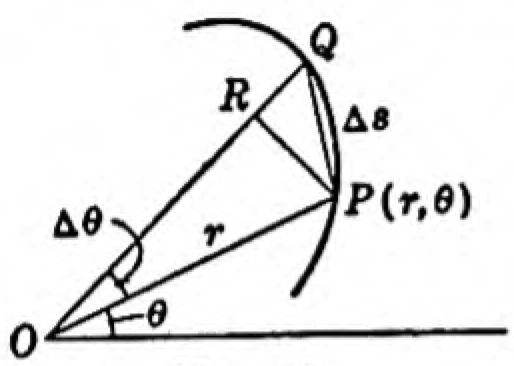


FIG. 88.

$$\overline{PQ}^2 = \overline{RP}^2 + \overline{RQ}^2$$

We also have, from triangle ORP,

$$\overline{RP} = r \sin \Delta\theta$$

Now
$$\begin{aligned}\overline{RQ} &= \overline{OQ} - \overline{OR}\\ &= r + \Delta r - r\cos\Delta\theta\\ &= \Delta r + r(1 - \cos\Delta\theta)\end{aligned}$$

We wish to find $\frac{\Delta s}{\Delta\theta}$. It is simpler to calculate $\left(\frac{\Delta s}{\Delta\theta}\right)^2$:

$$\left(\frac{\Delta s}{\Delta\theta}\right)^2 = \left(\frac{\Delta s}{\overline{PQ}}\right)^2\left(\frac{\overline{PQ}}{\Delta\theta}\right)^2$$

Now
$$\begin{aligned}\overline{PQ}^2 &= r^2\sin^2\Delta\theta + [\Delta r + r(1-\cos\Delta\theta)]^2\\ &= r^2\sin^2\Delta\theta + \left[\Delta r + 2r\sin^2\frac{\Delta\theta}{2}\right]^2\end{aligned}$$

Hence
$$\left(\frac{\Delta s}{\Delta\theta}\right)^2 = \left(\frac{\Delta s}{\overline{PQ}}\right)^2\left[r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(\frac{\Delta r}{\Delta\theta} + r\frac{\sin^2\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)^2\right]$$

Taking limits as $\Delta\theta \to 0$, we get

$$\left(\frac{ds}{d\theta}\right)^2 = 1\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]$$

or

★ $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \tag{6}$$

If s decreases with increasing θ, we should take the negative sign for this square root.

The differential of arc length is at once obtained from (6) by recalling that the differential of a function is equal to the derivative of the function multiplied by the differential of the argument. Thus

★
$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (7)$$
$$ds = \sqrt{r^2\, d\theta^2 + dr^2}$$

Let the reader show that formulas (6) and (7) can be obtained by use of differentials from the equations connecting rectangular and polar coordinates.

74. Curvature. Let P be any point on the curve $r = f(\theta)$ (Fig. 89). Let the tangent at P make angles ψ with OP and α with OX. We wish to find

$$\frac{d\alpha}{ds} = \frac{\dfrac{d\alpha}{d\theta}}{\dfrac{ds}{d\theta}}$$

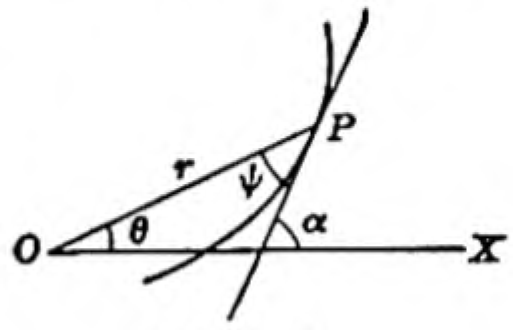


FIG. 89.

As in Art. 70, we have $\alpha = \theta + \psi$. Therefore,

$$\frac{d\alpha}{d\theta} = 1 + \frac{d\psi}{d\theta}$$

But
$$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}}$$

that is
$$\psi = \arctan \frac{r}{\dfrac{dr}{d\theta}}$$

Hence
$$\frac{d\psi}{d\theta} = \frac{1}{1 + \dfrac{r^2}{\left(\dfrac{dr}{d\theta}\right)^2}} \cdot \frac{\dfrac{dr}{d\theta}\dfrac{dr}{d\theta} - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}$$

Consequently
$$\frac{d\alpha}{d\theta} = 1 + \frac{d\psi}{d\theta} = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}$$

Since $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$, we have, finally,

★
$$\kappa = \frac{d\alpha}{ds} = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]^{3/2}}$$

The radius of curvature ρ is simply the reciprocal of $|\kappa|$. We shall, as before, regard curvature as essentially positive and so shall always use $|\kappa|$.

Example. Find the curvature and radius of curvature at any point of the cardioid $r = a(1 + \cos\theta)$; also at the point $\theta = \arctan\frac{3}{4}$. We have

$$\frac{dr}{d\theta} = -a\sin\theta \qquad \frac{d^2r}{d\theta^2} = -a\cos\theta$$

Therefore

$$\begin{aligned}\kappa &= \frac{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1+\cos\theta)(-\cos\theta)}{[a^2\sin^2\theta + a^2(1+\cos\theta)^2]^{3/2}} \\ &= \frac{a^2[1 + 3\cos\theta + 2\cos^2\theta + 2\sin^2\theta]}{a^3[\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta]^{3/2}} \\ &= \frac{3(1+\cos\theta)}{a(2+2\cos\theta)^{3/2}} = \frac{3}{2a\sqrt{2(1+\cos\theta)}} = \frac{3}{4a\cos(\theta/2)}\end{aligned}$$

Or this may be written

$$\kappa = \frac{3}{2\sqrt{2a\cdot a(1+\cos\theta)}} = \frac{3}{2\sqrt{2ar}}$$

Hence
$$\rho = \frac{4}{3}a\left|\cos\frac{\theta}{2}\right| = \frac{2}{3}\sqrt{2ar}$$

That is, the radius of curvature at any point is proportional to the square root of the magnitude of the radius vector of the point.

If $\theta = \arctan\frac{3}{4}$, then

$$\cos\theta = \tfrac{4}{5} \qquad r = a(1 + \tfrac{4}{5}) = \tfrac{9}{5}a$$

and
$$\kappa = \frac{3}{2\sqrt{2a\cdot\frac{9}{5}a}} = \frac{1}{2a\sqrt{\frac{2}{5}}} = \frac{\sqrt{5}}{2a\sqrt{2}} \qquad \rho = \frac{2a\sqrt{2}}{\sqrt{5}}$$

EXERCISES

Find the curvature and radius of curvature of the following curves at the points indicated (Ex. 1 to 11):

1. The circle $r = 2a\cos\theta$ at any point
2. The curve $r = a\tan\theta$ at $\theta = \pi/4$
3. The cardioid $r = a(1 - \sin\theta)$ at $\theta = 270°$
4. The four-leaved rose $r = a\sin 2\theta$ at $\theta = 45°$ and at $\theta = 0°$
5. The three-leaved rose $r = a\cos 3\theta$ at $\theta = 0°$ and at $\theta = 30°$
6. The rose curve $r = a\cos n\theta$ at $\theta = 0$ and at $\theta = \pi/2n$
7. The spiral $r = e^{a\theta}$ at any point
8. The spiral $r = a\theta$ at any point
9. The spiral $r = a/\theta$ at any point
10. The lemniscate of Bernoulli $r^2 = a^2\cos 2\theta$ at any point
11. The conic $r = \dfrac{ac}{1 - c\cos\theta}$ at any point (c is the eccentricity)

12. Verify in Exercise 11 that, if $c = 1$, then the equation of the parabola reduces to $r = \frac{a}{2}\csc^2\frac{\theta}{2}$. Find the curvature at any point.

13. Find the points of the cardioid $r = a(1 + \cos\theta)$ at which the radius of curvature has its maximum value.

14. Find the point at which the curvature of the parabola $r = a\sec^2(\theta/2)$ has its maximum numerical value.

15. Find the points of maximum (numerical) curvature on the hyperbola

$$r^2 = a^2 \sec 2\theta$$

75. Components of Velocity. If a point P is moving along a plane curve whose equation is given in polar coordinates, it is customary to resolve its velocity into components along OP and at right angles to OP. These are called, respectively, the *radial* and *transverse* components of velocity and are denoted by v_r and v_θ. In Fig. 90, let $\mathbf{v}$ be the vector representing the velocity of P. From Art. 67, the magnitude of this vector is $v = \left|\frac{ds}{dt}\right|$. Hence

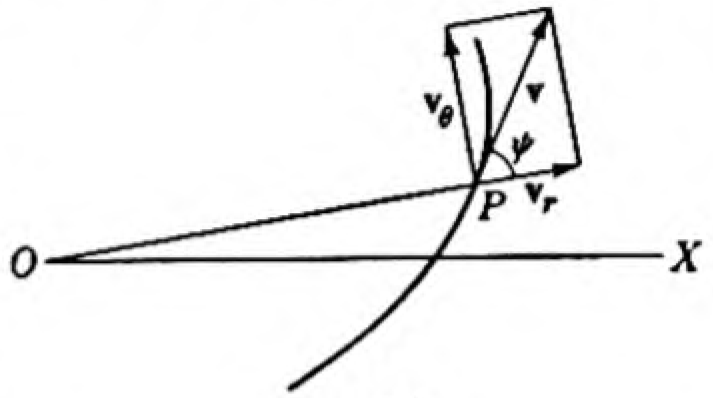


FIG. 90.

$$v = \left|\frac{ds}{dt}\right| = \left|\frac{ds}{d\theta}\frac{d\theta}{dt}\right| = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\left|\frac{d\theta}{dt}\right|$$

From Fig. 90, it is clear that $v_r = v\cos\psi$. Since

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}}$$

we have
$$\cos\psi = \frac{\frac{dr}{d\theta}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = \frac{\frac{dr}{d\theta}}{\frac{ds}{d\theta}} = \frac{dr}{ds}$$

Consequently
$$v_r = \frac{ds}{dt}\frac{dr}{ds} = \frac{dr}{dt}$$

Similarly
$$v_\theta = v\sin\psi$$

where
$$\sin\psi = \frac{r}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = \frac{r}{\frac{ds}{d\theta}} = r\frac{d\theta}{ds}$$

and therefore
$$v_\theta = \frac{ds}{dt}r\frac{d\theta}{ds} = r\frac{d\theta}{dt}$$

Observe that v_r is simply the rate at which the radius vector of P changes with time, whereas v_θ is the radius vector of P multiplied by the angular velocity of OP (see Sec. 68). Evidently

$$v^2 = v_r^2 + v_\theta^2$$

Example. A point moves around the cardioid $r = a(1 + \cos\theta)$ with the angular velocity of its radius vector equal to 10 r.p.m. (r in feet). Find v_r, v_θ, and v at any time. Find v when $\theta = \arctan \frac{3}{4}$. We have

$$v_r = \frac{dr}{dt} = -a \sin\theta \frac{d\theta}{dt}$$

Also

$$v_\theta = r\frac{d\theta}{dt} = a(1 + \cos\theta)\frac{d\theta}{dt}$$

Hence,

$$v^2 = a^2[\sin^2\theta + (1 + 2\cos\theta + \cos^2\theta)]\left(\frac{d\theta}{dt}\right)^2$$

or

$$v = a\sqrt{2 + 2\cos\theta}\,\frac{d\theta}{dt} = \sqrt{2ar}\,\frac{d\theta}{dt}$$

Since $\dfrac{d\theta}{dt} = 10 \text{ r.p.m.} = \dfrac{10(2\pi)}{60} = \dfrac{\pi}{3}$ radians/sec.

$$v = \frac{\pi}{3}\sqrt{2ar} \text{ ft./sec.}$$

When $\theta = \arctan \frac{3}{4}$,

$$r = a(1 + \tfrac{4}{5}) = \tfrac{9}{5}a$$

and

$$v = \frac{\pi}{3}\sqrt{2a \cdot \frac{9}{5}a} = \pi a\sqrt{\frac{2}{5}}$$
$$= 1.98a \text{ ft./sec. approximately}$$

MISCELLANEOUS EXERCISES

A point moves on the indicated curve with angular velocity of the radius vector as given (r in feet). Find v_r, v_θ, v at the point designated in Exercises 1 to 4.

1. The circle $r = 2a\cos\theta$; $\omega = 40$ r.p.m.; at any point
2. The four-leaved rose $r = \sin 2\theta$; $\omega = 20$ r.p.m.; at $\theta = 30°$
3. The lemniscate $r^2 = a^2\cos 2\theta$; $\omega = 15$ r.p.m.; at $\theta = 30°$
4. The spiral $r = a\theta$; $\omega = k$ radians/sec.; at any point

5. A point moves out from the pole along the first quadrant branch of the curve $r = \tan\theta$ with constant velocity in the path of $3\sqrt{5}$ in. per second. Find v_r, v_θ, and the angular velocity of the radius vector at the point (1,45°).

6. A point moves out from the pole along the spiral $r = 4\theta$ with constant velocity in the path of $5\sqrt{10}$ in. per second. Find v_r, v_θ, and the angular velocity of the radius vector when $\theta = 3$ radians.

7. Knowing that $v_r = \dfrac{dr}{dt}$ and $v_\theta = r\dfrac{d\theta}{dt}$, derive the formula for $\tan\psi$.

Find the angle between the radius vector and the tangent, and also the slope of each of the following curves at the point specified (Ex. 8 to 15):

8. The circle $r = 2a\sin\theta$	at $\theta = 30°$
9. The four-leaved rose $r = a\sin 2\theta$	at $\theta = 60°$
10. The eight-leaved rose $r = a\cos 4\theta$	at $\theta = 30°$
11. The lemniscate $r^2 = a^2\sin 2\theta$	at $\theta = 60°$
12. The curve $r = a\sin(\theta/2)$	at $\theta = 60°$
13. The curve $r = \cot\theta$	at $\theta = 45°$

14. The curve $r = a \sin^2 \theta$ at $\theta = 45°$

15. The curve $r = a \cos^2 2\theta$ at $\theta = 120°$

Find the angle of intersection of the following curves (Ex. 16 to 20):

16. The cardioids $r = a(1 + \sin \theta)$ and $r = a(1 - \sin \theta)$

17. The cardioids $r = a(1 + \cos \theta)$ and $r = a(1 + \sin \theta)$ (other than at the pole)

18. The two circles $r = 2a \cos \theta$ and $r = 2a \sin \theta$

19. The line $r = 4 \sec \theta$ and the circle $r = 10 \sin \theta$

20. The circle $r = a$ and the four-leaved rose $r = 2a \sin 2\theta$

21. Find the points at which the tangents are horizontal on the four-leaved rose $r = a \cos 2\theta$.

22. In Exercise 21, find the points at which the tangent line is vertical.

23. Find the points on the lemniscate $r^2 = a^2 \sin 2\theta$ at which the tangent line is vertical.

24. Find the points on the cardioid $r = a(1 + \cos \theta)$ at which the tangent line is horizontal; those at which the tangent line is vertical.

25. Find the length of the polar subtangent, subnormal, tangent, and normal of the curve $r = a \sin^2 \theta$ at any point.

Find the curvature and radius of curvature for the following at the points indicated (Ex. 26 to 30):

26. The curve $r = a \tan \theta$ at $\theta = 45°$

27. The lemniscate $r^2 = a^2 \sin 2\theta$ at any point

28. The curve $r = a \sin^2 \theta$ at $\theta = 90°$

29. In Exercise 28, find the center of the circle of curvature. (Sketch the curve and the circle.)

30. The curve $r = a \cos (\theta/2)$ at $\theta = 0$

In the case of each of the following curves, find approximate values for the increments in the radius vector and the arc length corresponding to a small increment $\Delta\theta$ in the angle θ as indicated:

31. The circle $r = 4 \sin \theta$ when θ changes from 40° to 40°10′

32. The curve $r = 10 \tan \theta$ when θ changes from 67° to 68°

33. The curve $r = 6\theta$ when θ changes from 155° to 157°

CHAPTER 11

THEOREM OF THE MEAN, INDETERMINATE FORMS, AND TAYLOR'S THEOREM

76. Rolle's Theorem. Suppose we have a continuous function $y = f(x)$ as illustrated in Fig. 91. Notice that there are maximum points and a minimum point whose abscissas are between a and b and that there is a definite nonvertical tangent line at each point between A and B. This last fact might be stated differently as follows: $f(x)$ has a derivative for every value of x between a and b. Now, there are points at which the tangent line is horizontal; for instance, E is such a point. In fact, any curve that we care to draw through A and B representing a continuous (of course, single-valued) function and having a definite nonvertical tangent line at all points between A and B will have a horizontal tangent at *at least one point* whose abscissa is between a and b. Let the student draw several such curves and satisfy himself that this is the case. These facts are stated in Rolle's theorem: *If $f(x)$ is continuous in the interval $a \leqq x \leqq b$ and has a derivative $f'(x)$ for all values of x between a and b (that is, for $a < x < b$), then, if $f(a) = f(b) = 0$, there is a value ξ of x between a and b for which $f'(\xi) = 0$.*

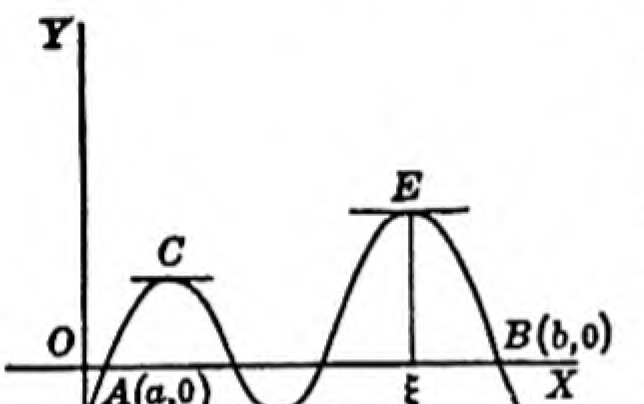


FIG. 91.

Although this theorem is geometrically evident, it is possible to give a simple analytic proof based upon the following fundamental property of a continuous function: If $f(x)$ is continuous throughout an interval $a \leqq x \leqq b$, then it has a maximum value M for at least one value of x in the interval and a minimum value m for at least one other value of x in the interval. We shall *assume* this fundamental property as known, for its proof is best deferred to a more advanced course.

We proceed to the proof of Rolle's theorem. Now, $f(x)$ has a maximum M and a minimum m, each for some value of x in the interval $a \leqq x \leqq b$. Since $f(a) = f(b) = 0$, we have, necessarily, $M \geqq 0$ and $m \leqq 0$.

Suppose $m = M = 0$. Then $f(x) = 0$ for all x in the interval. Hence $f'(\xi) = 0$ not only for one value ξ, but for *all* values of x between a and b.

If $m \neq M$, then either M is positive, or m is negative, or both. First,

suppose $M > 0$ (Fig. 92). Let $f(\xi) = M$. Then, for sufficiently small Δx, $f(\xi + \Delta x) - f(\xi)$ is negative or zero, since $f(\xi)$ is a *maximum* value of $f(x)$ in $a \leqq x \leqq b$. Consequently

$$\frac{\Delta y}{\Delta x} = \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x}$$

is positive or zero when Δx is negative and is negative or zero when Δx is positive. Hence, if Δx approaches zero from the *left*, the limit of $\dfrac{\Delta y}{\Delta x}$ is either positive or zero. Similarly, if Δx approaches zero from the right, the limit of $\dfrac{\Delta y}{\Delta x}$ is either negative or zero. But we supposed $f(x)$ to have a derivative for every value of x between $x = a$ and $x = b$, and therefore at $x = \xi$. Hence $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = f'(\xi)$ must have the same value *no matter how* Δx approaches zero. Hence $f'(\xi) = 0$.

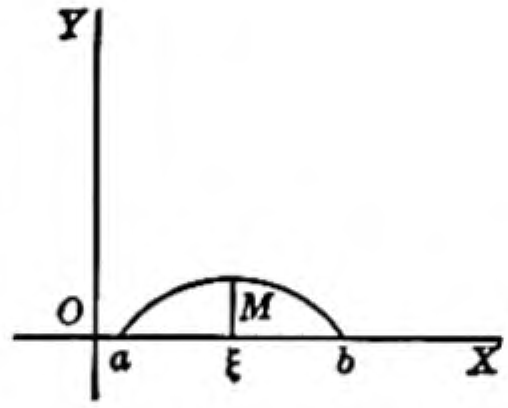


FIG. 92.

Now, suppose $m < 0$. Then, the maximum of $-f(x)$, namely, $-m$, is positive. Therefore, by the preceding paragraph, the derivative of $-f(x)$ must be zero for a suitable value of x, say $x = \xi$. That is, $-f'(\xi) = 0$, and hence $f'(\xi) = 0$, for $a < \xi < b$ (Fig. 93). This completes the proof of Rolle's theorem.

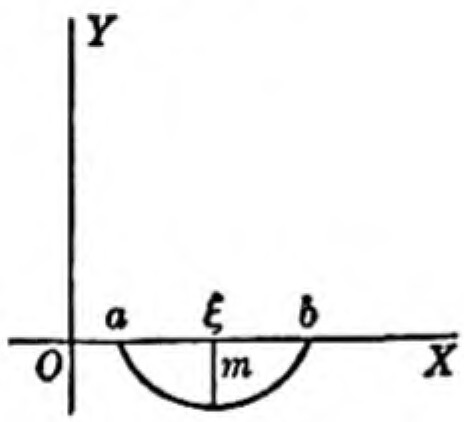


FIG. 93.

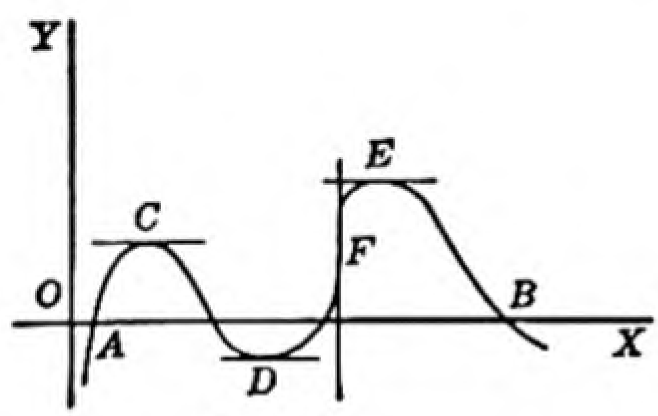


FIG. 94.

It should be noted that the existence of the derivative for every value of x between a and b is the condition upon which the proof of the theorem depends. If the derivative fails to exist for any value of x between a and b, the theorem need not hold. The conclusion will follow, however, if $f'(x)$ is infinite at some point within the interval, provided it is definitely either $+\infty$ or $-\infty$ (for example, as at point F in Fig. 94). However, in Fig. 95 the function whose graph is shown has no derivative for $x = c$; for at this point the tangent is vertical, and $\dfrac{\Delta y}{\Delta x} = \dfrac{f(c + \Delta x) - f(c)}{\Delta x}$ tends to $+\infty$ if $\Delta x \to 0^-$ but tends to $-\infty$ if $\Delta x \to 0^+$. In Fig. 96,

$$\frac{\Delta y}{\Delta x} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

tends to one limiting value for Δx approaching zero through negative values and to a different limiting value for Δx approaching zero through positive values; there is, therefore, no derivative at $x = c$. Thus, these are two cases in which Rolle's theorem does not apply.

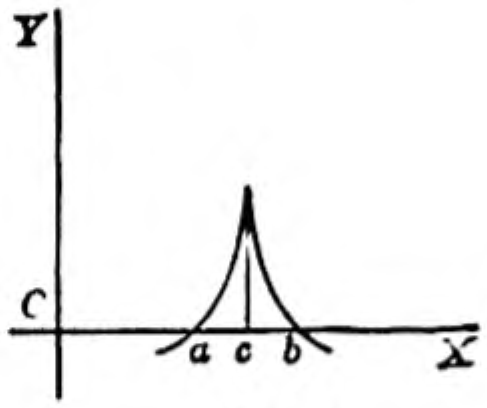


FIG. 95.

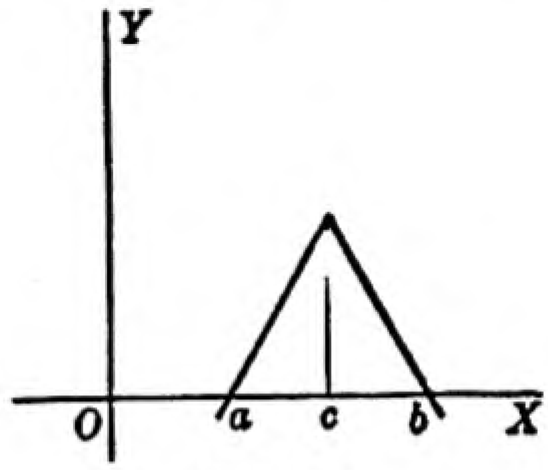


FIG. 96.

77. Theorem of the Mean. Suppose $f(x)$ is the function whose graph is shown in Fig. 97. Note that $f(x)$ is continuous with a nonvertical tangent for each value of x between $x = a$ and $x = b$. Let $AC = f(a)$ and $BD = f(b)$. Draw the chord CD, and draw CE parallel to OX. It is geometrically evident that there is some point P on the curve whose abscissa ξ is between a and b and at which the tangent to the curve is parallel to the chord CD. Since the chord has slope

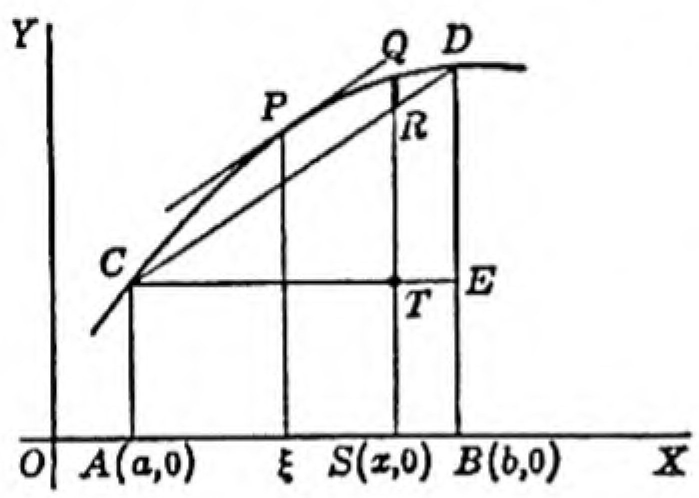


FIG. 97.

$$\frac{ED}{CE} = \frac{f(b) - f(a)}{b - a}$$

and since the tangent has slope $f'(\xi)$, we have the result

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \qquad \text{where } a < \xi < b$$

This is known as the **theorem of the mean**: *If $f(x)$ is continuous in the interval $a \leqq x \leqq b$ and has a derivative for every value of x between $x = a$ and $x = b$ (that is, for $a < x < b$), then there is a value ξ of x between a and b for which $f(b) - f(a) = (b - a)f'(\xi)$ where $a < \xi < b$.*

We need not rely upon geometrical intuition but can give a proof based upon Rolle's theorem. Let us construct the auxiliary function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

This function $F(x)$ is continuous in $a \leqq x \leqq b$ and has a derivative in $a < x < b$ since it is the sum of $f(x)$ and a polynomial of first degree. Furthermore, $F(a) = F(b) = 0$, as is evident by direct substitution. Thus, $F(x)$ fulfills the conditions of Rolle's theorem. Now

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Therefore, we must have

$$F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0$$

for some ξ between a and b. This gives at once

$$(b - a)f'(\xi) = f(b) - f(a) \qquad \text{where } a < \xi < b$$

Note (Fig. 97) the geometrical interpretation of $F(x)$:

$f(x) = SQ, f(a) = AC = ST, x - a = AS = CT, b - a = AB = CE,$ $f(b) - f(a) = ED$. Therefore

$$\begin{aligned} F(x) &= SQ - ST - \frac{ED}{CE} \cdot CT \\ &= SQ - ST - TR \\ &= RQ \end{aligned}$$

The theorem of the mean may be expressed in various ways. The following are the most frequently encountered:

$$f(b) = f(a) + (b - a)f'(\xi) \qquad a < \xi < b \tag{1}$$

This is the original statement, with $f(a)$ transferred to the right-hand side of the equation. If we put $b = a + h$, we may express the fact that ξ lies between a and b by writing $\xi = a + \theta h$ where $0 < \theta < 1$. Substituting in (1), we get

$$f(a + h) = f(a) + hf'(a + \theta h) \qquad \text{where } 0 < \theta < 1 \tag{2}$$

We may think of b as a variable, and write $b = x$ for convenience. Substituting in (1), we get

$$f(x) = f(a) + (x - a)f'(\xi) \qquad \text{where } a < \xi < x \tag{3}$$

We now prove a theorem to which we shall make reference later: *If the derivative of a function $f(x)$ is zero for all values of x in the interval $a \leqq x \leqq b$, then $f(x)$ is a constant in that interval.*

Since the function has, by hypothesis, a derivative at every point of the interval, it is continuous throughout the interval and satisfies the conditions for the theorem of the mean.

Let x be any value between a and b. Using (3), we have

$$f(x) = f(a) + (x - a)f'(\xi) \qquad \text{where } a < \xi < x$$

But we are given that $f'(\xi) = 0$. Hence $f(x) = f(a)$, a constant, and the theorem is proved.

Example 1. Verify Rolle's theorem for the function $f(x) = x^3 - 4x$. We have $f(x) = x(x + 2)(x - 2)$. Hence $f(-2) = f(0) = f(2) = 0$ (see Fig. 98). Also $f'(x) = 3x^2 - 4$ which exists for all values of x. Hence, according to Rolle's theorem, there should be a value of x, say ξ_1, between -2 and 0, and another value of x, say ξ_2, between 0 and 2, at each of which $f'(x)$ should vanish. Clearly, $f'(x) = 3x^2 - 4 = 0$ if $x^2 = \frac{4}{3}$, that is, if $x = \frac{\pm 2}{\sqrt{3}}$.

Now, $-2 < -2/\sqrt{3} < 0 < 2/\sqrt{3} < 2$. Hence, we may take $\xi_1 = -2/\sqrt{3}$, and $\xi_2 = 2/\sqrt{3}$, and therefore Rolle's theorem is verified.

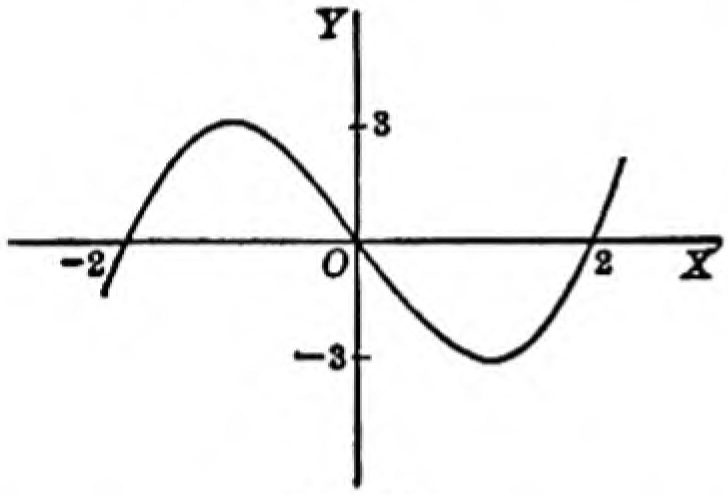


FIG. 98.

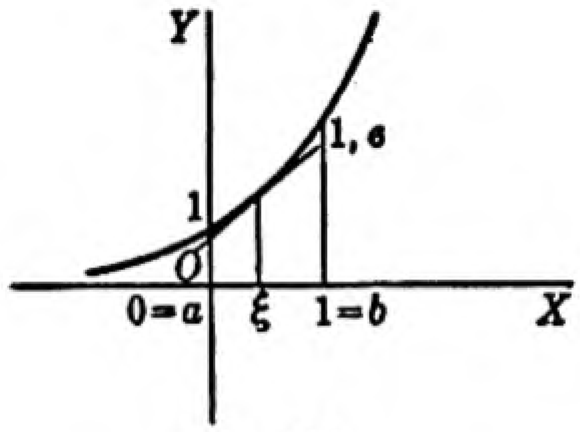


FIG. 99.

Example 2. Given $f(x) = e^x$. Verify the theorem of the mean for $a = 0$ and $b = 1$. That is, show that there is a point on the curve $y = e^x$ whose abscissa ξ is between 0 and 1 (Fig. 99) such that $f(b) = f(a) + (b - a)f'(\xi)$.

We have $f(b) = e^1 = e$, $f(a) = e^0 = 1$, $b - a = 1 - 0 = 1$; also, $f'(x) = e^x$. We therefore wish to show that there is a number ξ, between 0 and 1, such that $e = 1 + e^\xi$. This requires $e^\xi = e - 1$, or $\xi = \ln(e - 1)$. Now, since $e = 2.71828 \ldots$,

$$\ln 1.71 < \ln(e - 1) < \ln 1.72$$

that is, $0.537 < \xi < 0.542$, and consequently $0 < \xi < 1$, and the theorem is verified. Geometrically, this means that the tangent to the curve at the point (ξ, e^ξ) (approximately 0.54, 1.72) is parallel to the line joining the points (0,1) and (1,e).

EXERCISES

Verify Rolle's theorem for the following functions (Ex. 1 to 4):

1. $f(x) = 16x - x^2$ **2.** $f(x) = x^3 - x^2 - 4x + 4$
3. $f(x) = \sin x$ **4.** $f(x) = \cos(x/2)$

Rolle's theorem cannot be applied to the following functions in the interval specified. Explain why (Ex. 5 to 8).

5. $f(x) = x$ in $-1 \leqq x \leqq 1$

6. $\begin{cases} f(x) = 2x & \text{for } x \leqq 1 \\ f(x) = 4 - 2x & \text{for } x > 1 \end{cases}$ in $0 \leqq x \leqq 2$

7. $f(x) = 1 - x^{2/3}$ in $-1 \leqq x \leqq 1$

8. $f(x) = \tan x$ in $0 \leqq x \leqq \pi$

Verify the theorem of the mean for the following functions in the interval specified. Make a sketch in each case (Ex. 9 to 15).

9. $f(x) = x^2$ in $1 \leqq x \leqq 2$
10. $f(x) = x^2 - 2x + 4$ in $-1 \leqq x \leqq 2$
11. $f(x) = x^3 - 3x + 2$ in $-2 \leqq x \leqq 3$
12. $f(x) = e^x$ in $-1 \leqq x \leqq 2$
13. $f(x) = \ln x$ in $\frac{1}{2} \leqq x \leqq 2$
14. $f(x) = \sin x$ in $30° \leqq x \leqq 60°$
15. $f(x) = \cosh x$ in $-1 \leqq x \leqq 3$

78. An Alternative Form of the Theorem of the Mean. Suppose that we have two functions $f(x)$ and $g(x)$, each of which is continuous in the interval $a \leqq x \leqq b$, and each of which possesses a derivative at every point of the interval $a < x < b$. Furthermore, suppose that $g'(x)$ is not zero for any value of x *within* the interval. It then follows at once from Rolle's theorem that $g(b) \neq g(a)$. Then, there is a ξ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{for } a < \xi < b \tag{4}$$

problem

This is sometimes called the **extended law of the mean** or **extended theorem of the mean**; it is occasionally called **Cauchy's formula.*** To prove this theorem, we construct the following function:

$$\Phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}\,[g(x) - g(a)]$$

Since $\Phi(x)$ consists of a sum of constants and constant multiples of $g(x)$ and $f(x)$, it is continuous in the interval $a \leqq x \leqq b$ and possesses a derivative at every point *within* this interval. Also, $\Phi(a) = \Phi(b) = 0$, as is easily seen by direct substitution. We may therefore apply Rolle's theorem, obtaining

$$\Phi'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(\xi) = 0 \qquad \text{for } a < \xi < b$$

from which formula (4) immediately follows.

79. The Indeterminate Form 0/0. Let $f(x)$ and $g(x)$ be continuous functions, and consider the quotient $f(x)/g(x)$. It may happen that, when $x = a$, both $f(x)$ and $g(x)$ are zero, that is, $f(a) = g(a) = 0$. In this case the fraction $f(a)/g(a)$ assumes the meaningless form 0/0. The function $F(x) = f(x)/g(x)$ is, therefore, undefined for $x = a$; however, it may approach a limit as *x approaches a*. Whether it approaches such a limit or not, we say that $F(x)$ assumes the *indeterminate form* 0/0; finding the limit, or showing that it does not exist, is called *evaluating* the indeterminate form. We have already had examples of fractions that assume the form 0/0; for example, $\dfrac{\sin x}{x}$ is undefined for $x = 0$, but we have seen that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. In fact, in finding derivatives we have been forced to discover means of evaluating the limit of $\dfrac{\Delta y}{\Delta x}$ when both the numerator Δy and the denominator Δx approached zero. Again, the fraction $\dfrac{x + 2}{x^2 + 4x + 4} = F(x)$ has no meaning when $x = -2$, for both numerator and denominator are then zero.

* After the French mathematician A. L. Cauchy (1789–1857).

Further investigation shows that

$$F(x) = \frac{x+2}{x+2} \cdot \frac{1}{x+2} = \frac{1}{x+2}$$

for $x \neq -2$ becomes infinite as x approaches -2, so that the fraction does not approach a limit. Thus, a fraction that assumes the indeterminate form 0/0 when $x = a$ may or may not have a limit as x approaches a. So far we have relied upon special devices to find the limits of such fractions. We may, however, use the theorem of the mean as given in Art. 78 to obtain a convenient rule for evaluating these limits.

Suppose that $f(x)$ and $g(x)$ are continuous in the interval $a \leqq x \leqq b$, that $f(a) = g(a) = 0$, and that $f(x)$ and $g(x)$ each possess a derivative in $a < x < b$. We have, from Art. 78, letting $b = x$,

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

for some ξ such that $a < \xi < x$, that is

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{where } a < \xi < x$$

Hence $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{\xi \to a} \dfrac{f'(\xi)}{g'(\xi)}$, provided this limit exists. In case $f'(x)$ and $g'(x)$ are continuous at $x = a$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{\xi \to a} \frac{f'(\xi)}{g'(\xi)} = \frac{f'(a)}{g'(a)}$$

since ξ approaches a if x approaches a; if $g'(a) = 0$ while $f'(a) \neq 0$, then the fraction will become infinite. Although here we have supposed $x \to a^+$, the argument is easily extended to the case $x \to a^-$.

This can all be formulated in the following simple rule: To evaluate the limit of a fraction that takes the indeterminate form 0/0, differentiate the numerator and the denominator *separately*, and take the limit of the resulting fraction. *Caution:* Differentiate the numerator *and* the denominator; do *not* differentiate the whole fraction as a *quotient*.

We may let x approach a only from the *right* if we wish to find the *right-hand limit* of $f(x)/g(x)$, and our reasoning still holds; similarly for the left-hand limit.

Example 1. Evaluate

$$\lim_{x \to \frac{\pi}{4}} \frac{\tan\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$$

Both numerator and denominator have derivatives that are continuous at $x = \pi/4$, and the fraction assumes the form 0/0 at this point. Hence, applying the rule,

$$\lim_{x\to\frac{\pi}{4}} \frac{\tan\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\sec^2\left(x - \frac{\pi}{4}\right)}{1} = \frac{1}{1} = 1$$

Example 2. Evaluate $\lim\limits_{x\to 0} \dfrac{\arctan (x/2)}{x}$. Again, both numerator and denominator have derivatives that are continuous at $x = 0$, and the fraction assumes the form 0/0 at this point. Hence

$$\lim_{x\to 0} \frac{\arctan \frac{x}{2}}{x} = \lim_{x\to 0} \frac{\frac{\frac{1}{2}}{1 + (x^2/4)}}{1} = \frac{1}{2}$$

Example 3. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2}$. Here

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \infty$$

and the original fraction becomes infinite.

It may happen that $f'(x)/g'(x)$ assumes the indeterminate form 0/0. Then, if $f''(x)$ and $g''(x)$ exist in $a < x < b$ and if $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $x = b$, we need only apply our extended theorem of mean value (Art. 79) to the fraction $f'(x)/g'(x)$ and obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

If $f''(x)$ and $g''(x)$ are continuous at $x = a$, and $g''(a) \neq 0$, this is equal to $f''(a)/g''(a)$. If $f''(x)/g''(x)$ assumes the indeterminate form 0/0 for x approaching a, the method may be applied again, and so on. In evaluating indeterminate forms the student should be constantly alert for the possibility of finding the required limit without further differentiation. Frequently algebraic simplification or the use of trigonometric identities will greatly shorten the process. This is illustrated in the following example:

Example 4. $\lim\limits_{x\to 0} \dfrac{1 - 2\cos x + \cos^2 x}{x^4}$

$$\left[\frac{f'(x)}{g'(x)}\right] = \lim_{x\to 0} \frac{2\sin x - 2\sin x\cos x}{4x^3} = \lim_{x\to 0} \frac{2\sin x - \sin 2x}{4x^3}$$

$$\left[\frac{f''(x)}{g''(x)}\right] = \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{12x^2}$$

$$\left[\frac{f'''(x)}{g'''(x)}\right] = \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{24x}$$

$$\left[\frac{f^{(4)}(x)}{g^{(4)}(x)}\right] = \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{24} = \frac{-2 + 8}{24} = \frac{1}{4}$$

Note that each fraction, except the last, takes the form 0/0.

This result could have been obtained with no differentiations as follows:

$$\lim_{x\to 0} \frac{1 - 2\cos x + \cos^2 x}{x^4} = \lim_{x\to 0} \frac{(1-\cos x)^2}{x^4} = \lim_{x\to 0} \frac{\left(2\sin^2 \frac{x}{2}\right)^2}{x^4}$$

$$= \lim_{x\to 0} \frac{1}{4}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^4 = \frac{1}{4}$$

The advantage is gained by using our knowledge that $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

It can be shown, further, that if $f(x)$ takes the indeterminate form 0/0 as x becomes infinite the same method may be used. That is,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

80. The Indeterminate Form ∞/∞. If, in the fraction $f(x)/g(x)$, both $f(x)$ and $g(x)$ become infinite as x approaches a, we say that the fraction assumes the indeterminate form ∞/∞. It can be shown, although the proof will not be given here, that then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

In case $f'(x)/g'(x)$ becomes infinite, then the original fraction becomes infinite. This is still true if x approaches $+\infty$ or $-\infty$. If $f'(x)/g'(x)$ takes the form ∞/∞ and no simpler method of evaluation is evident, the method may be used again.

Example 1. Find $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$. This is clearly of the form ∞/∞. Hence

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Example 2. Find $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$. Again, this is ∞/∞, and we have

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$

Since $2x/e^x$ also takes the form ∞/∞, we apply the rule again, obtaining

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Hence
$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0$$

Example 3. $\displaystyle\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{3x^3 + 2x + 7}$

$$\left[\frac{f'(x)}{g'(x)}\right] = \lim_{x\to\infty} \frac{3x^2 - 8x + 1}{9x^2 + 2}$$

$$\left[\frac{f''(x)}{g''(x)}\right] = \lim_{x\to\infty} \frac{6x - 8}{18x}$$

$$\left[\frac{f'''(x)}{g'''(x)}\right] = \lim_{x\to\infty} \frac{6}{18} = \frac{1}{3}$$

Example 4. Find $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$. Note that, although x becomes infinite, $\sin x$ does *not* become infinite. This is, therefore, *not* the form ∞/∞, and our rule *does not apply*. The student must guard most carefully against trying to evaluate limits of fractions that do not take one of the indeterminate forms 0/0 or ∞/∞ by applying this rule. This particular limit can, however, be shown to be zero as explained in Example 2, Art. 10.

We may summarize briefly the results of this and the last section in a convenient rule as follows:

L'Hôpital's rule:* *If $f(x)/g(x)$ assumes one of the indeterminate forms 0/0 or ∞/∞ as x approaches a (a finite or infinite), and if $f(x)$ and $g(x)$ have derivatives of all orders up to and including n, then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is equal to the first of the expressions*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}, \qquad \lim_{x\to a} \frac{f''(x)}{g''(x)}, \qquad \ldots, \qquad \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$$

which is not indeterminate, provided that the indicated limit exists; and if the first such expression becomes infinite, so does $f(x)/g(x)$.

As already noted, it may be possible to evaluate an indeterminate form very readily by use of algebraic simplifications or other simple devices. Such a possibility should be explored at each step of the work before proceeding to find the quotient of the next higher ordered derivatives. It may happen that all the above quotients are indeterminate—for example, if $f(x) = e^x$, $g(x) = e^{2x}$, and $a = \infty$. Here, $\displaystyle\lim_{x\to\infty} \frac{e^x}{e^{2x}} = \lim_{x\to\infty} e^{-x} = 0$ at once, without application of L'Hôpital's rule.

EXERCISES

Evaluate the following limits:

1. $\displaystyle\lim_{x\to 2} \frac{x^3 + x^2 - 11x + 10}{x^2 - x - 2}$

2. $\displaystyle\lim_{x\to -1} \frac{2x^3 - x^2 - 2x + 1}{x^3 + x^2 + x + 1}$

3. $\displaystyle\lim_{x\to 3} \frac{x^4 - x^3 - 6x^2 - 2x + 6}{x^3 - 7x^2 + 15x - 9}$

4. $\displaystyle\lim_{x\to 1} \frac{x^4 - 2x^3 + x^2}{x^2 + 6x - 7}$

* Named for G. F. A. de L'Hôpital (1661–1704) who discussed the form 0/0.

5. $\displaystyle\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 11x - 1}{x^3 + 5x - 22}$

6. $\displaystyle\lim_{x\to\infty} \frac{17x^5 - 21x^3 + x + 2}{x^6 + 2x^3 - 1}$

7. $\displaystyle\lim_{x\to\infty} \frac{x^4 + 12x^2 + 3}{1000x^3 - 1}$

8. $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$

9. $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$

10. $\displaystyle\lim_{\theta\to 0} \frac{\cos 3\theta - 1}{\theta^2}$

11. $\displaystyle\lim_{\theta\to 0} \frac{\sec \theta - 1}{\theta}$

12. $\displaystyle\lim_{\theta\to \pi} \frac{\cot (\theta/2)}{\pi - \theta}$

13. $\displaystyle\lim_{\theta\to 0} \frac{\sin 4\theta}{\tan \theta}$

14. $\displaystyle\lim_{x\to 0} \frac{\ln (1 + x)}{x}$

15. $\displaystyle\lim_{x\to 0} \frac{\ln \sec x}{x^2}$

16. $\displaystyle\lim_{x\to 0^+} \frac{\ln \sin x}{\csc x}$

17. $\displaystyle\lim_{x\to 0} \frac{\ln (x^2 + 1)}{x^3}$

18. $\displaystyle\lim_{x\to +\infty} \frac{\ln x}{e^x}$

19. $\displaystyle\lim_{x\to -\infty} \frac{e^{-x}}{x}$

20. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

21. $\displaystyle\lim_{x\to\infty} \frac{e^x + 1}{e^x - 1}$

22. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$

23. $\displaystyle\lim_{\theta\to 0} \frac{\sin \theta - \tan \theta}{\theta^2}$

24. $\displaystyle\lim_{z\to 0} \frac{\arcsin z^2}{z^2}$

25. $\displaystyle\lim_{z\to 0} \frac{\arctan z}{\sin z}$

26. $\displaystyle\lim_{x\to 0} \frac{\arcsin (x/2)}{x^2}$

27. $\displaystyle\lim_{x\to 0} \frac{\cosh x - 1}{x^2}$

28. $\displaystyle\lim_{x\to 0} \frac{\sinh x}{x}$

29. $\displaystyle\lim_{x\to 0} \frac{\tanh x}{x}$

30. $\displaystyle\lim_{x\to\infty} \frac{\cosh x}{x}$

31. $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{x^3}$

32. $\displaystyle\lim_{x\to\infty} \frac{\cosh x}{x^n}$ (n a positive integer)

33. $\displaystyle\lim_{z\to 0} \frac{\operatorname{argsinh} z}{z}$

34. $\displaystyle\lim_{z\to 0} \frac{\operatorname{argtanh} z}{z}$

35. $\displaystyle\lim_{x\to 1^-} \frac{1 - \sqrt{x}}{\sqrt{1 - x}}$

36. $\displaystyle\lim_{x\to\infty} \frac{\tan x}{x}$

37. $\displaystyle\lim_{x\to\infty} \frac{\sin x}{e^x}$

38. $\displaystyle\lim_{\varphi\to 0} \frac{1 - \cos^4 \varphi}{\varphi^3}$

39. $\displaystyle\lim_{\theta\to 0} \frac{\theta - \sin \theta}{\theta - \tan \theta}$

40. $\displaystyle\lim_{x\to\infty} \frac{x^a}{e^x}$ (a is not necessarily an integer. Consider the cases $a < 0$, $a = 0$, $a > 0$.)

41. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^a}$ (Consider all possible values of a.) (Compare with Exercise 40.)

42. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

43. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^a}$

44. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x^n}$ $(n \geqq 0)$

81. Other Indeterminate Forms. If $f(x)$ and $g(x)$ both become infinite as $x \to a$ (or as $x \to \infty$), then $f(x) - g(x)$ assumes the *indeterminate form* $\infty - \infty$. We then attempt to transform $f(x) - g(x)$ into an expression that will take one of the forms $0/0$ or ∞/∞ and apply L'Hôpital's rule.

Example 1. The expression $\sec x - \tan x$ takes the indeterminate form $\infty - \infty$ as x approaches $\pi/2$ from the left. We have, however,

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x}$$

which takes the form $0/0$ as $x \to \pi^-/2$. Hence

$$\lim_{x\to\frac{\pi^-}{2}} (\sec x - \tan x) = \lim_{x\to\frac{\pi^-}{2}} \frac{1 - \sin x}{\cos x} = \lim_{x\to\frac{\pi^-}{2}} \frac{\cos x}{\sin x} = 0$$

If $f(x) \to 0$ while $g(x) \to \infty$ for $x \to a$, then $f(x) \cdot g(x)$ takes the *indeterminate form* $0 \cdot \infty$. Here we may write $f(x) \cdot g(x) = \dfrac{f(x)}{1/g(x)}$, or $f(x) \cdot g(x) = \dfrac{g(x)}{1/f(x)}$, obtaining either $0/0$ or ∞/∞, and L'Hôpital's rule may be applied to whichever is the more convenient.

Example 2. The product $x^2 \cdot e^{-x}$ takes the form $\infty \cdot 0$ if $x \to \infty$. However, $x^2 \cdot e^{-x} = x^2/e^x$ takes the form ∞/∞. Hence

$$\lim_{x\to\infty} x^2 \cdot e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(See Example 2 of preceding section.)

Example 3. If $y = x \ln x$, then y takes the form $0 \cdot \infty$ as $x \to 0^+$. We have $y = \dfrac{\ln x}{1/x}$ which takes the form ∞/∞. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

The expression $[f(x)]^{g(x)}$ may assume one of the *indeterminate forms* 0^0, ∞^0, 1^∞ as x approaches a (a finite or infinite). In these cases, we first find the logarithm of the given expression, and if it takes the form $0 \cdot \infty$, the limit (if it exists) may perhaps be found. The method is best made clear by examples.

Example 4. If $y = x^{\sin x}$, then y takes the form 0^0 when $x \to 0^+$. To evaluate the limit, we have $\ln y = \sin x \cdot \ln x$. Hence, $\ln y$ takes the form $0 \cdot \infty$ when $x \to 0^+$.

We write $\ln y = \dfrac{\ln x}{\csc x}$, which takes the form ∞/∞. Hence, applying the rule,

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x}$$

By trigonometric identities, this is equal to

$$\lim_{x\to 0^+} \frac{-\sin x \cdot \tan x}{x} = -\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot \tan x\right) = -1\cdot 0 = 0$$

Hence $$\lim_{x\to 0^+} \ln y = \ln \lim_{x\to 0^+} y = 0$$

and therefore $$\lim y = e^0 = 1$$

Example 5. If $y = (\cot x)^x$, then y takes the form ∞^0 as $x \to 0^+$. We have $\ln y = x \ln \cot x = \dfrac{\ln \cot x}{1/x}$, which takes the form ∞/∞. Hence, applying the rule,

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{-\dfrac{\csc^2 x}{\cot x}}{-\dfrac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2 \csc^2 x}{\cot x}$$

$$= \lim_{x\to 0^+}\left(\frac{x^2}{\sin^2 x}\cdot \tan x\right) = 1\cdot 0 = 0$$

Hence $$\ln \lim_{x\to 0^+} y = 0 \quad\text{and}\quad \lim_{x\to 0^+} y = 1$$

Example 6. If $y = x^{\frac{1}{1-x}}$, then y takes the form 1^∞ if $x \to 1$. We have

$$\ln y = \frac{\ln x}{1-x}$$

which takes the form 0/0. Hence, applying the rule,

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1$$

and $$\ln \lim_{x\to 1} y = -1$$

Therefore $$\lim_{x\to 1} y = e^{-1} = \frac{1}{e}$$

EXERCISES

Evaluate the following limits:

1. $\lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$
2. $\lim_{x\to 0^+} (\csc x - \cot x)$
3. $\lim_{\theta\to 0}\left(\dfrac{1}{1-\cos\theta} - \dfrac{2}{\sin^2\theta}\right)$
4. $\lim_{x\to 2}\left(\dfrac{4}{x^2-4} - \dfrac{1}{x-2}\right)$
5. $\lim_{z\to 0}\left[\dfrac{1}{\ln(z+1)} - \dfrac{z+1}{z}\right]$
6. $\lim_{y\to 0}\left(\dfrac{1}{y} - \dfrac{1}{e^y - 1}\right)$
7. $\lim_{x\to 0^+} \sqrt{x}\ln x$
8. $\lim_{x\to 0^+} x \ln \sin x$
9. $\lim_{\alpha\to 0^+} \alpha \cot \alpha$
10. $\lim_{x\to +\infty} x \sin(1/x)$
11. $\lim_{x\to 0^+} x^a \ln x$
12. $\lim_{x\to 0^+} x^x$

13. $\lim_{x\to 0^+} (\sin x)^x$

14. $\lim_{x\to 0^+} (\tan x)^x$

15. $\lim_{x\to 0^+} (\sinh x)^x$

16. $\lim_{x\to 0^+} x^{2x}$

17. $\lim_{x\to 0^+} x^{x^2}$

18. $\lim_{x\to 0^+} (\cot x)^{\tan x}$

19. $\lim_{x\to 0^+} (\csc x)^{\sin x}$

20. $\lim_{x\to 0^+} (\coth x)^x$

21. $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$

22. $\lim_{x\to \infty} \left(1 + \frac{k}{x}\right)^x$

23. $\lim_{x\to 0} (1 + kx)^{\frac{1}{x}}$

24. $\lim_{x\to 0} (1 + x)^{\frac{k}{x}}$

25. $\lim_{x\to \infty} \left(e^{\frac{1}{x}} + \frac{1}{x}\right)^x$

82. Taylor's Theorem.* The theorem of the mean is a special case of a more general theorem which we shall now establish. We recall that, if $f(x)$ is continuous for all values of x in the interval $a \leqq x \leqq b$ (or $b \leqq x \leqq a$ in case $b < a$) and has a derivative in $a < x < b$ (or $b < x < a$), then there exists a value ξ of x between a and b such that

$$f(b) = f(a) + (b - a)f'(\xi)$$

A much more general theorem, known as **Taylor's theorem,** holds if $f(x)$ possesses certain further properties as follows:

If $f(x)$ is a function such that

(1) $f^{(n-1)}(x)$ exists and is continuous in $a \leqq x \leqq b$ (or $b \leqq x \leqq a$)

(2) $f^{(n)}(x)$ exists in $a < x < b$ (or $b < x < a$)

then a value ξ of x between a and b exists such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!} f''(a) + \cdots + \frac{(b - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + \frac{(b - a)^n}{n!} f^{(n)}(\xi) \quad (5)$$

To prove (5), we note that a number Q can be found so that

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!} f''(a) + \cdots + \frac{(b - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + \frac{(b - a)^n}{n!} Q \quad (6)$$

that is, so that

$$-f(b) + f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!} f''(a) + \cdots + \frac{(b - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + \frac{(b - a)^n}{n!} Q = 0 \quad (7)$$

* Named for Brook Taylor (1685–1731).

If we replace a by x, where $a \leqq x \leqq b$ (or $b \leqq x \leqq a$), the left-hand member of (7) becomes a function of x, say $F(x)$, thus:

$$F(x) = -f(b) + f(x) + (b - x)f'(x) + \frac{(b-x)^2}{2!}f''(x) + \cdots + \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{(b-x)^n}{n!}Q \quad (8)$$

Remember always that Q is simply a *constant* defined by (6). Our purpose is now to find an expression for Q.

We note that $F(x)$ exists and is continuous in $a \leqq x \leqq b$ (or $b \leqq x \leqq a$), for since $f^{(n-1)}(x)$ exists and is continuous in this interval, so $f(x)$ and all its derivatives of lower than $(n-1)$st order exist and are continuous in the same interval. Also $F(a) = 0$ by virtue of (7), and $F(b) = 0$ by direct substitution. Furthermore, since $f^{(n)}(x)$ is supposed to exist in $a < x < b$ (or $b < x < a$), then

$$F'(x) = f'(x) - f'(x) + (b-x)f''(x) - (b-x)f''(x) + \frac{(b-x)^2}{2!}f'''(x) - \frac{(b-x)^2}{2!}f'''(x) + \cdots + \frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x) - \frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x) + \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{(b-x)^{n-1}}{(n-1)!}Q$$

exists in this interval.

Note that pairs of terms add to zero, leaving only

$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - Q] \quad (9)$$

Hence, $F(x)$ satisfies the conditions of Rolle's theorem. Therefore $F'(x)$ must be zero for some value of x, say ξ, *between* a and b. Hence, from (9),

$$F'(\xi) = \frac{(b-\xi)^{n-1}}{(n-1)!}[f^{(n)}(\xi) - Q] = 0$$

and therefore, since $\xi \neq b$, $f^{(n)}(\xi) - Q = 0$. Hence

$$Q = f^{(n)}(\xi)$$

This establishes (5). If we take x some number in the interval $a \leqq x \leqq b$ (or $b \leqq x \leqq a$), we may replace b by x in (5) and obtain a very useful form of **Taylor's theorem.**

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi) \quad (10)$$

for ξ between a and x.

This theorem is sometimes called *Taylor's formula, Taylor's expansion,* or *Taylor's series with remainder,* the "remainder" being the term

$\frac{(x-a)^n}{n!}f^{(n)}(\xi)$. We say that $f(x)$ is *expanded* by Taylor's theorem.

A convenient form of Taylor's theorem is obtained by setting

$$x - a = h$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h) \qquad \text{where } 0 < \theta < 1$$

Note that the theorem of the mean is merely formula (10) with $n = 1$.

Taylor's theorem enables us to find a polynomial in $(x - a)$ that approximtes a given function for values of x near to $x = a$, provided the function has continuous derivatives at $x = a$. To be sure, the remainder term cannot, in general, be found since ξ is unknown; however, it is frequently possible to estimate limits for the size of this remainder. Of course, if $f(x)$ is itself a polynomial, this remainder term will be zero if the degree of $f(x)$ is k and n is taken to be $k + 1$ or greater. Taylor's expansion will be further discussed in Chap. 19.

Example 1. Approximate $\ln x$ by a polynomial of fifth degree by use of Taylor's theorem with $a = 1$. We have

$$\begin{aligned}
f(x) &= \ln x & &\text{and} & f(1) &= 0\\
f'(x) &= \frac{1}{x} & &\text{and} & f'(1) &= 1\\
f''(x) &= -\frac{1}{x^2} & &\text{and} & f''(1) &= -1\\
f'''(x) &= \frac{2}{x^3} & &\text{and} & f'''(1) &= 2\\
f^{(4)}(x) &= -\frac{2\cdot 3}{x^4} & &\text{and} & f^{(4)}(1) &= -3!\\
f^{(5)}(x) &= \frac{2\cdot 3\cdot 4}{x^5} & &\text{and} & f^{(5)}(1) &= 4!\\
f^{(6)}(x) &= -\frac{2\cdot 3\cdot 4\cdot 5}{x^6} & &\text{and} & f^{(6)}(\xi) &= -\frac{5!}{\xi^6}
\end{aligned}$$

Hence

$$\begin{aligned}
\ln x &= (x-1)\cdot 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-3!)\\
&\quad + \frac{(x-1)^5}{5!}(4!) + \frac{(x-1)^6}{6!}\left(-\frac{5!}{\xi^6}\right)\\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}\cdot\frac{1}{\xi^6}
\end{aligned}$$

where ξ is between 1 and x.

Example 2. Compute $\ln 1.2$ by the expression of Example 1, and estimate the maximum numerical error involved in using the first five terms for the computation.

Substituting $x = 1.2$ in the formula of Example 1, we have

$$\ln 1.2 = (0.2) - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6}\cdot\frac{1}{\xi^6}$$

where $1 < \xi < 1.2$. Taking the first five terms,

$$\begin{aligned} 0.2 &= 0.20000000 & -\frac{(0.2)^2}{2} &= -0.02000000 \\ \frac{(0.2)^3}{3} &= 0.00266667 & -\frac{(0.2)^4}{4} &= -0.00040000 \\ \frac{(0.2)^5}{5} &= 0.00006400 & & \\ & \overline{0.20273067} & & \overline{-0.02040000} \end{aligned}$$

Therefore, the sum of the first five terms is 0.18233067, correct to seven places of decimals. Now $1/\xi^6$ cannot have a value as great as $1/1^6$ or as small as $1/(1.2)^6$. Hence, the maximum numerical value of the term $\frac{(0.2)^6}{6}\cdot\frac{1}{\xi^6}$ is less than

$$\frac{(0.2)^6}{6}\cdot 1 = 0.0000106 < 0.000011$$

Consequently, neglecting the sixth term and using only the first five terms in this expansion involves an error that does not exceed 0.000011 in numerical value. The first five terms give 0.1823307, and the sixth term can affect the fifth place of decimals by not more than 1. Hence, log $1.2 = 0.1823$, correct to four places. The student may verify this result by consulting a table of natural logarithms.

83. Maclaurin's Theorem.* A useful special case of Taylor's theorem is obtained by setting $a = 0$. The result is known as **Maclaurin's theorem,** or **Maclaurin's series with remainder:**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\xi)$$

where ξ is between 0 and x. This requires, of course, that $f(x)$ have its first $n - 1$ derivatives continuous in an interval 0 to x, inclusive, and possess a derivative of order n *within* this interval.

Example 1. Approximate $\sin x$ with a polynomial of seventh degree by use of Maclaurin's theorem. We have

$$\begin{aligned} f(x) &= \sin x && \text{and} & f(0) &= 0 \\ f'(x) &= \cos x && \text{and} & f'(0) &= 1 \\ f''(x) &= -\sin x && \text{and} & f''(0) &= 0 \\ f'''(x) &= -\cos x && \text{and} & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x && \text{and} & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x && \text{and} & f^{(5)}(0) &= 1 \end{aligned}$$

* Named for Colin Maclaurin (1698–1746).

$$f^{(6)}(x) = -\sin x \quad \text{and} \quad f^{(6)}(0) = 0$$
$$f^{(7)}(x) = -\cos x \quad \text{and} \quad f^{(7)}(0) = -1$$
$$f^{(8)}(x) = \sin x \quad \text{and} \quad f^{(8)}(\xi) = \sin \xi$$

Therefore
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^8}{8!}\sin \xi$$

where ξ is between 0 and x. The term $\frac{x^8}{8!}\sin \xi$ is the remainder or "error."

Example 2. Calculate the sine of 0.3 radian (that is, of 17°11′19.4″ approximately), and estimate the maximum numerical error involved in using eight terms of Maclaurin's series. (Note that, since terms in even powers of x have zero coefficients, only four terms will actually appear. The ninth term of Maclaurin's series is the fifth term in the above expansion of $\sin x$ and is the "error term," which we shall estimate.) We have

$$\sin 0.3 = 0.3 - \frac{(0.3)^3}{3!} + \frac{(0.3)^5}{5!} - \frac{(0.3)^7}{7!} + \frac{(0.3)^8}{8!}\sin \xi \qquad \text{where } 0 < \xi < 0.3$$

$$\begin{aligned} 0.3 &= 0.30000000 \\ -\frac{(0.3)^3}{3!} &= -0.00450000 \\ \frac{(0.3)^5}{5!} &= 0.00002025 \\ -\frac{(0.3)^7}{7!} &= -0.00000004 \\ & \overline{0.29552021} \end{aligned}$$

Now, since the maximum value of $\sin \xi$ for $0 < \xi < 0.3$ is less than $\sin (0.3) < 1$, the maximum numerical error cannot exceed

$$\frac{(0.3)^8}{8!} = 0.000000002$$

which might affect the eighth place of decimals. Therefore, surely $\sin 0.3 = 0.295520$ is correct to six places of decimals by use of the first four terms of the expansion of $\sin x$ (actually the first eight terms of Maclaurin's series).

Example 3. How many terms of the expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^n}{n!}\cos \xi \qquad \text{where } 0 < \xi < x$$

should be used in calculating $\sin 0.5$ to ensure accuracy to five places of decimals (0.5 radian = 28°38′52.4″ approximately)? Since the maximum error will be $\frac{(0.5)^n}{n!}\cos \xi < \frac{(0.5)^n}{n!}\cdot 1$, we need only find n so that $\frac{(0.5)^n}{n!} < 0.000005$. This value of n can be found by trial. If $n = 5$, then $\frac{(0.5)^5}{5!} < 0.00026$ which is too large. We try $n = 7$, obtaining $\frac{(0.5)^7}{7!} < 0.000002$. This will suffice; and therefore, if we use $\sin 0.5 = 0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!}$, the result will be correct to five places of decimals. Note that, since terms containing even powers of x will have zero coefficients and therefore drop out, it is unnecessary to consider $n = 6$.

EXERCISES

1. Find four terms of Taylor's expansion of $f(x) = \sqrt{x}$, using $a = 4$. Use this to find $\sqrt{4.1}$, and estimate the remainder term. Check by reference to a table of square roots.

2. Assuming $\sin 45° = \cos 45° = \sqrt{2}/2 = 0.70711$ to be known, find four terms of Taylor's expansion for sin 46°. Use $a = 45° = \frac{\pi}{4}$, $b = 46° = \frac{\pi}{4} + \frac{\pi}{180} = \frac{46\pi}{180}$. Use this to calculate sin 46°, and estimate the remainder. Check by reference to a table of sines.

3. Assuming $e^2 = 7.3891$ to be known, find $e^{2.1}$ by using four terms of Taylor's expansion. Check by reference to a table of powers of e.

4. Assuming $e^{-1} = 0.36788$ to be known, find $e^{-1.1}$ by using four terms of Taylor's expansion. Check by reference to a table of powers of e. Estimate the remainder.

5. In Exercise 4, how many terms should be taken to ensure a remainder numerically less than 0.0001?

6. Expand e^x by Maclaurin's theorem, and show that

$$|R| < \frac{|x^n|}{n!} \qquad \text{for } x < 0$$

$$|R| < \frac{x^n}{n!}e^x \qquad \text{for } x > 0$$

7. Calculate e by use of 10 terms of Maclaurin's expansion, and compare with the value given in any available table. Estimate the error (use the fact that $e < 3$).

8. How many terms in the Maclaurin's expansion for e (Exercise 7) are needed to ensure an error less than 0.0001?

9. Approximate $\cos x$ with a polynomial of sixth degree.

10. Calculate the cosine of 0.2 radian by use of the result of Exercise 9. Check by reference to an appropriate table of cosines. Estimate the error.

11. How many terms of the Maclaurin's expansion for cos 0.2 are needed to ensure accuracy to the fifth place of decimals?

12. Use the expansion of Exercise 9 to calculate cos 9°. Use $9° = \pi/20 = 0.157$ radian. Check by reference to a table of cosines.

13. Expand $\ln(1 + x)$ by Maclaurin's theorem, and show that

$$|R| < \frac{|x^n|}{n(1+x)^n} \qquad \text{for } -1 < x < 0$$

$$|R| < \frac{x^n}{n} \qquad \text{for } x > 0$$

14. Use five terms of the expansion of Exercise 13 to calculate ln 1.02. Check by reference to a table of natural logarithms.

84. Maxima and Minima. In Chap. 5, we considered methods for finding maximum and minimum values of a function but based our reasoning largely upon geometrical intuition. We are now in a position to develop certain of these rules independently of geometry. Let us suppose that the nth derivative of $f(x)$ is continuous throughout an interval $b \leqq x \leqq c$. This ensures that $f(x)$ and its first $n - 1$ derivatives are continuous throughout the same interval. Suppose that $x = a$ is a point

within this interval (that is, $b < a < c$). We restate our definitions (Art. 32) of maximum and minimum values of $f(x)$:

If, for all values of x within some interval

$$a - h \leqq x \leqq a + h$$

(1) $f(x) - f(a) < 0$, $f(a)$ is a maximum value of $f(x)$ (Fig. 100);

(2) $f(x) - f(a) > 0$, $f(a)$ is a minimum value of $f(x)$ (Fig. 101);

(3) $f(x) - f(a)$ has one sign for $x < a$ and opposite sign for $x > a$, then $f(a)$ is neither a maximum nor a minimum. (Fig. 102).

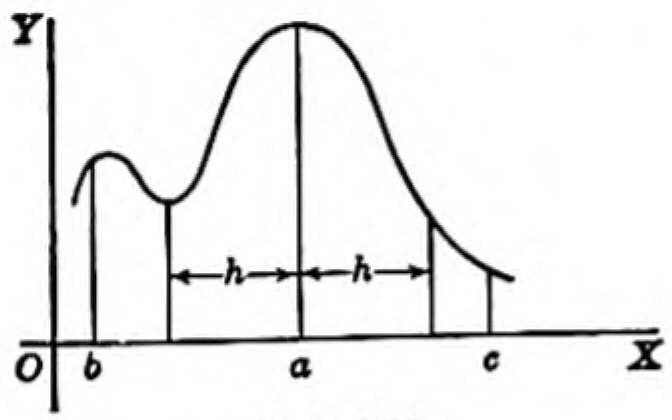


FIG. 100.

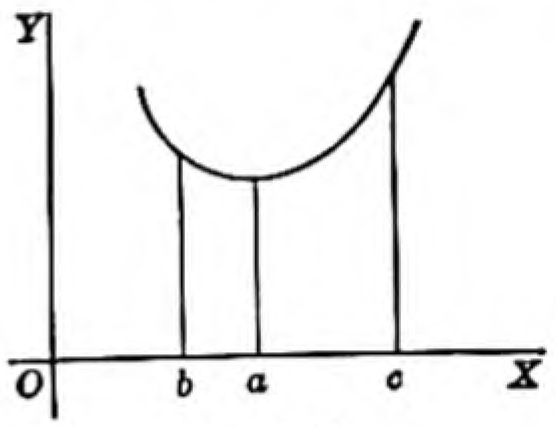


FIG. 101.

Evidently, if $f(x) - f(a)$ is zero in the interval $a - h \leqq x \leqq a + h$, the function $f(x)$ is a constant in this interval.

Case I. Suppose $f'(a) \neq 0$. We have, using the theorem of the mean,

$$f(x) - f(a) = (x - a)f'(\xi) \tag{11}$$

for some ξ between a and x. Now, since $f'(x)$ is supposed continuous and $f'(a) \neq 0$, it is clear from the definition of continuity (Art. 11) that there is some interval

$$a - h \leqq x \leqq a + h$$

of width $2h$ throughout which $f'(x)$ has the same sign as $f'(a)$. Within this interval, therefore, $f'(\xi)$ has the same sign as $f'(a)$. Since $x - a$ is negative for $x < a$ and positive for $x > a$, we conclude at once from (11) that $f(x) - f(a)$ has one sign for $x < a$ and opposite sign for $x > a$.

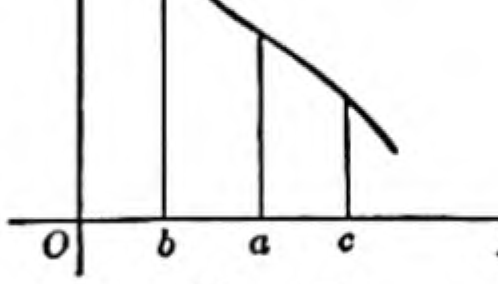


FIG. 102.

Hence, $f(x)$ has neither a maximum nor a minimum value for $x = a$. This establishes the fact that if $f(x)$ has a continuous derivative throughout an interval, then $f'(x) = 0$ is necessary for a maximum or a minimum.

Case II. Suppose $f'(a) = 0$ and $f''(a) \neq 0$. We have, using Taylor's theorem with $n = 2$,

$$\begin{aligned} f(x) - f(a) &= (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(\xi) \\ &= \frac{(x - a)^2}{2!}f''(\xi) \end{aligned}$$

for some ξ between a and x, since $f'(a) = 0$. We have supposed $f''(x)$

continuous throughout $b \leqq x \leqq c$; hence, since

$$f''(a) \neq 0$$

there must be some interval $a - h \leqq x \leqq a + h$ throughout which $f''(x)$, and therefore $f''(\xi)$, has the same sign as $f''(a)$. Furthermore, since $(x - a)^2$ is always positive (except at $x = a$ when it is zero), $f(x) - f(a)$ has the same sign as $f''(\xi)$ throughout the interval $a - h \leqq x \leqq a + h$ (it is, of course, zero at $x = a$). Hence, $f(a)$ is a maximum if $f''(a) < 0$ and a minimum if $f''(a) > 0$. This agrees with the rule, already given in Sec. 34, whose development was made to depend largely upon geometrical intuition.

Case III (*General Case*). Suppose that

$$f'(a) = f''(a) = \cdot \cdot \cdot = f^{(n-1)}(a) = 0$$

but that $f^{(n)}(a) \neq 0$. Applying Taylor's theorem as before, we have

$$f(x) - f(a) = \frac{(x - a)^n}{n!} f^{(n)}(\xi)$$

for some ξ between a and x. Again, since $f^{(n)}(x)$ has been supposed continuous, there is some h such that, throughout the interval $a - h \leqq x \leqq a + h$, $f^{(n)}(x)$, and therefore $f^{(n)}(\xi)$, has the same sign as $f^{(n)}(a)$. Consequently, if n is *even*, $(x - a)^n$ is positive (for all $x \neq a$), and $f(x) - f(a)$ has the same sign as $f^{(n)}(a)$. Therefore, $f(a)$ is a maximum if $f^{(n)}(a)$ is negative, a minimum if $f^{(n)}(a)$ is positive. On the other hand, if n is *odd*, then $(x - a)^n$ will be negative for $x < a$ and positive for $x > a$, and therefore $f(x) - f(a)$ will have one sign for $x < a$ and opposite sign for $x > a$. Therefore, $f(a)$ is neither a maximum nor a minimum. These results are embodied in the following test:

Calculate the successive derivatives (supposed continuous throughout an interval containing the point $x = a$) of $f(x)$, and evaluate them for $x = a$. If the first of these derivatives which is not zero at $x = a$ is of *even* order, then $f(a)$ is a maximum if this derivative is negative, a minimum if it is positive. If the first nonzero derivative is of *odd* order, $f(a)$ is neither a maximum nor a minimum.

Example 1. If $y = f(x) = x^4$ (Fig. 103), find any maximum or minimum points. We have

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$
$$f'''(x) = 24x$$
$$f^{(4)}(x) = 24$$

Hence, all the derivatives are continuous, and

$$f'(0) = f''(0) = f'''(0) = 0$$

Hence, the first derivative that does not vanish for $x = 0$ is $f^{(4)}(0) = 24$. This is of *even order* (fourth); and since it

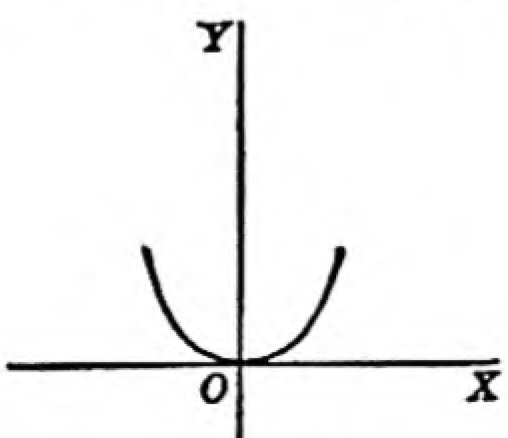


FIG. 103.

is positive, $x = 0$ gives a *minimum* value. Note that it would have been equally simple to observe that $f'(x)$ changes sign from minus to plus as x increases through zero and that, hence, $x = 0$ gives a minimum.

Example 2. If $y = f(x) = x^5$ (Fig. 104), find any maximum or minimum points. We have

$$\begin{aligned} f'(x) &= 5x^4 \\ f''(x) &= 20x^3 \\ f'''(x) &= 60x^2 \\ f^{(4)}(x) &= 120x \\ f^{(5)}(x) &= 120 \end{aligned}$$

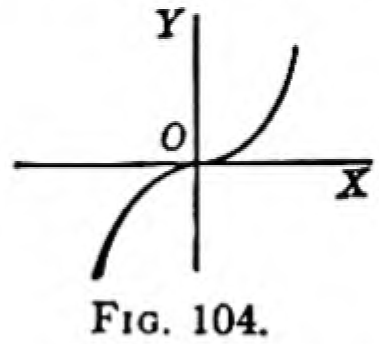


FIG. 104.

The first four derivatives are all zero for $x = 0$, and the first nonzero derivative, $f^{(5)}(0) = 120$, is of odd order. Hence $x = 0$ does not give a maximum or minimum. Furthermore, since $f'(x)$ is continuous for all x and zero only for $x = 0$, there can be no other possible maxima or minima.

MISCELLANEOUS EXERCISES

Investigate the following functions for maximum and minimum values by the method of Art. 84 (Ex. 1 to 7).

1. $y = x^3$

2. $y = \frac{1}{6}x^6 - \frac{2}{5}x^5 + \frac{1}{4}x^4 + 1$

3. $f(x) = \frac{1}{4}x^4 + 2x^3 + 6x^2 + 8x + 1$

4. $f(x) = x + \sin x$

5. $f(\theta) = \frac{1}{2}\theta^2 + \cos\theta$

6. $f(r) = 3r^4 - 8r^3 + 6r^2 + 4$

7. $f(u) = u - \sinh u$

8. Suppose that the first derivative of $f(x)$ that does not vanish at $x = a$ is of odd order n. Show that $f(x)$ is *increasing* at $x = a$ if $f^{(n)}(a)$ is *positive* and *decreasing* if $f^{(n)}(a)$ is *negative*.

Evaluate the following limits (Ex. 9 to 22):

9. $\lim\limits_{\theta \to 0} \dfrac{\theta - \sin\theta}{\theta^3}$

10. $\lim\limits_{x \to a} \dfrac{\cos x - \cos a}{x - a}$

11. $\lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{x^2}$

12. $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cot\theta + \csc\theta - 1}{\cot\theta - \csc\theta + 1}$

13. $\lim\limits_{x \to \infty} \dfrac{\ln x}{e^x}$

14. $\lim\limits_{x \to \infty} \dfrac{\ln \cosh x}{x}$

15. $\lim\limits_{\theta \to 0^+} \dfrac{\ln \sin\theta}{\ln \tan\theta}$

16. $\lim\limits_{\varphi \to \frac{\pi^-}{2}} (\sec\varphi - \tan\varphi)$

17. $\lim\limits_{y \to 0} \left(\dfrac{1}{y} - \dfrac{1}{1 - e^y}\right)$

18. $\lim\limits_{x \to 0^+} x \ln \sinh x$

19. $\lim\limits_{x \to \infty} x \sin (k/x)$

20. $\lim\limits_{x \to 0^+} x^{\tan x}$

21. $\lim\limits_{x \to \frac{\pi^-}{2}} (\sec x)^{\cos x}$

22. $\lim\limits_{x \to 0^+} (\coth x)^{\sinh x}$

23. Show that $x^{\frac{k}{\ln x}} \to e^k$ for $x \to \infty$, for $x \to 0^+$, and for $x \to 1$.

24. Assuming $e^2 = 7.3891$ to be known, find $e^{1.9}$ by using four terms of Taylor's expansion. Check by reference to a table of powers of e.

25. Assuming $\cos 30° = 0.86603$ and $\sin 30° = 0.50000$ as known, use four terms of Taylor's expansion to find $\cos 32°$. Check by reference to a table of cosines.

26. Approximate $\sinh x$ by a polynomial of seventh degree.

27. Use the result of Exercise 26 to calculate $\sinh \frac{1}{2}$. Check by reference to a table of hyperbolic functions. Estimate the remainder.

28. Approximate $\cosh x$ by a polynomial of sixth degree.

29. Use the result of Exercise 28 to calculate $\cosh 1$. Check by reference to a table of hyperbolic functions. Estimate the remainder.

30. Let $f(x) = (a + x)^n$, where a is any constant. Use Maclaurin's expansion to prove the binomial theorem for n a positive integer.

CHAPTER 12

INTEGRATION, STANDARD FORMS

85. Integral of a Function. So far, we have been concerned largely with problems whose solution involved finding the derivative of a *given function*. The student has very likely thought of the question "If the *derivative is given*, can we find a function having this derivative?" In a large number of cases, the answer is in the affirmative. For example, if $\frac{dy}{dx} = 5x^4$, then we may take $y = x^5$. Instead of asking for a function whose *derivative* is given, we might ask for a function whose *differential* is given. For instance, if $dy = 5x^4\,dx$, then we may take $y = x^5$.

Now it is perfectly clear that if $\frac{dy}{dx} = 5x^4$ or if $dy = 5x^4\,dx$, then we may take $y = x^5 + 2$. Or we could take, in fact, $y = x^5 + C$ where C is any constant whatever. In any case, we get $5x^4$ for the derivative with respect to x, or $5x^4\,dx$ for the differential, since the derivative (or the differential) of a constant is zero. Since the value of C is perfectly arbitrary, we call C an *arbitrary constant*. We employ the following notation to mean that $x^5 + C$ is a function whose differential is $5x^4\,dx$ (or whose derivative with respect to x is $5x^4$):

$$\int 5x^4\,dx = x^5 + C$$

Similarly

$$\int e^{2t}\,dt = \tfrac{1}{2}e^{2t} + C$$

means that $\frac{1}{2}e^{2t} + C$ is a function whose differential is $e^{2t}\,dt$, or whose derivative with respect to t is e^{2t}. In general

$$\int f(x)\,dx = F(x) + C$$

means that $F(x) + C$ is *a function whose differential is* $f(x)\,dx$, or whose derivative with respect to x is $f(x)$. For any particular value of C, say $C = C_1$, $F(x) + C_1$ is called *an integral* of $f(x)$, a *primitive function* for $f(x)$, or an *antiderivative* of $f(x)$, and C is called the *constant of integration*. The symbol $\int$ is called the *sign of integration*, $f(x)$ is called the *integrand*, and the process of finding $F(x)$ is called *integration*. The symbol $\int f(x)\,dx$ is itself termed the *integral of* $f(x)\,dx$.

Example 1. Find $\int \sin 2x\,dx$. We must find a function whose differential is $\sin 2x\,dx$, that is, one whose derivative with respect to x is $\sin 2x$. Evidently, we

must differentiate $\cos 2x$ to get an expression involving $\sin 2x$. But the derivative of $\cos 2x$ is $-2 \sin 2x$. So, to avoid the factor -2, we take

$$\int \sin 2x \, dx = -\tfrac{1}{2} \cos 2x + C$$

Let the student check this by differentiating $-\frac{1}{2} \cos 2x + C$.

Example 2. Find $\displaystyle\int \frac{dx}{\sqrt{x}}$. We must find a function whose derivative is

$$1/\sqrt{x} = x^{-1/2}$$

Evidently, we must differentiate $x^{1/2}$ to get an expression containing $x^{-1/2}$. But the derivative of $x^{1/2}$ is $\frac{1}{2}x^{-1/2}$. Hence, we take $\displaystyle\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$ to avoid the factor $\frac{1}{2}$.

Let the student show by differentiation that this result is correct.

From these examples, two things are clear. First, integration involves as an essential feature guessing correctly the function whose derivative will be the integrand. Second, the result can be checked by differentiation. It is therefore necessary for the student to be thoroughly familiar with the formulas for differentiating the various functions already studied. He will do well to make a review of these before taking up the study of integration. However, it will be necessary to "guess" the function whose differential is given in only a few cases. These will provide us with certain *standard forms*, and our study of integration will then consist of a systematic treatment of methods by which given integrands can be transformed to standard forms. But we must not expect success in every case, for only certain classes of integrands can be so transformed. For example, no transformation will reduce $\int e^{-x^2} dx$ to a standard form or to a combination of a finite number of standard forms. Yet the integrand is a comparatively simple function. In other words, there are functions whose primitive functions, or antiderivatives, cannot be expressed in finite form in terms of the so-called *elementary functions*, that is, in terms of algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

Before taking up these standard forms, we shall establish certain important fundamental properties of integrals.

86. Fundamental Properties of Integrals. (1) It has been noted that $\int f(x)\,dx = F(x) + C$. Hence, the difference between any two of these integrals of $f(x)$ is a constant. For example, $x^5 + 5$ and $x^5 - 11$ are both integrals of $5x^4$, and their difference is $(x^5 + 5) - (x^5 - 11) = 16$, a constant. The question arises, "Could there be any other integral of $f(x)$ that is some entirely different function and that could not be obtained by assigning some special value to C?" That the answer is no is guaranteed by the following **theorem**: *If two functions of x have the same derivative in an interval $a \leq x \leq b$, their difference is a constant throughout that interval.*

In other words, if $F_1(x) = \int f(x)\,dx$ and $F_2(x) = \int f(x)\,dx$, then

$$F_1(x) - F_2(x) = k$$

a constant. The proof is as follows:

Given that, for all $a \leqq x \leqq b$, $F_1'(x) = F_2'(x) = f(x)$.

Let $$\varphi(x) = F_1(x) - F_2(x)$$
Then $$\varphi'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

throughout the interval. Hence (page 189), $\varphi(x)$ is a constant, that is, $F_1(x) - F_2(x) = k$, as was to be proved.

(2) The **converse of the theorem** just proved is sometimes useful, namely: *If the difference of two functions is a constant, they have the same derivative, provided that the derivative exists.* For let $F_1(x) - F_2(x) = C$ throughout an interval $a \leqq x \leqq b$. Then $F_1'(x) - F_2'(x) = 0$, and

$$F_1'(x) = F_2'(x)$$

(3) If du is the differential of a function, then

$$\int du = u + C$$

This is simply the definition of integration expressed in symbols.

(4) $$\int k\,du = k\int du = ku + C \qquad (k \text{ constant})$$

Here we assume that the arbitrary constants of integration are properly adjusted; for

$$\int k\,du = ku + C_1$$
$$k\int du = k(u + C_2) = ku + kC_2$$

If we take $C_1 = C$ and $C_2 = C/k$, the formula is established. It asserts that *any* integral of $k\,du$ is k times *an* integral of du. This formula allows us to take a *constant factor* outside the sign of integration. The student must, however, be very careful never to take a *variable* factor outside the sign of integration.

(5) If $du, dv, \ldots, dw$ are the differentials of a finite number of functions, then

$$\int (du + dv + \cdots + dw) = \int du + \int dv + \cdots + \int dw$$
$$= u + v + \cdots + w + C$$

Again, we assume that the arbitrary constants of integration are properly adjusted. For, by the definition of an integral, the first member is $u + v + \cdots + w + C_0$, and the second member is

$$(u + C_1) + (v + C_2) + \cdots + (w + C_n) = u + v + \cdots + w + (C_1 + C_2 + \cdots + C_n)$$

If we take $C_0 = C$ and $C_1 + C_2 + \cdots + C_n = C$,* the formula is

* For instance, let $C_1 = C_2 = \cdots = C_n = C/n$.

established. Stated in words, we have the following: *An integral of the sum of a finite number of functions is the sum of their integrals.*

Example 1. Find $\int 17x^2\,dx$. Since 17 is a constant factor of the integrand, we may take it outside the sign of integration. The integral then becomes

$$\int 17x^2\,dx = 17\int x^2\,dx = 17(\tfrac{1}{3}x^3 + C') = \tfrac{17}{3}x^3 + C$$

Example 2. Find $\int(3x^2 + 5x - 11\sqrt{x})\,dx$. By (5), this is equal to

$$\int 3x^2\,dx + \int 5x\,dx + \int(-11)\sqrt{x}\,dx$$

But $\int 3x^2\,dx = x^3 + C_1$, $\int 5x\,dx = 5\int x\,dx = \frac{5}{2}x^2 + C_2$, and

$$\int(-11)\sqrt{x}\,dx = -11\int\sqrt{x}\,dx = (-11)\cdot\tfrac{2}{3}x^{3/2} + C_3$$

Hence, the original integral equals

$$x^3 + C_1 + \tfrac{5}{2}x^2 + C_2 - \tfrac{22}{3}x^{3/2} + C_3 = x^3 + \tfrac{5}{2}x^2 - \tfrac{22}{3}x^{3/2} + C$$

EXERCISES

Find by inspection the following integrals. Check by differentiation (Ex. 1 to 22).

1. $\displaystyle\int 4x^3\,dx$
2. $\displaystyle\int 3\,dx$
3. $\displaystyle\int 7x^6\,dx$
4. $\displaystyle\int x^2\,dx$
5. $\displaystyle\int (x + 3x^2)\,dx$
6. $\displaystyle\int e^x\,dx$
7. $\displaystyle\int e^{3x}\,dx$
8. $\displaystyle\int \frac{dx}{x^2}$
9. $\displaystyle\int \frac{dy}{y}$
10. $\displaystyle\int \sqrt{y}\,dy$
11. $\displaystyle\int \frac{dz}{\sqrt{z^3}}$
12. $\displaystyle\int \frac{dz}{1+z}$
13. $\displaystyle\int \frac{dz}{1+z^2}$
14. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$
15. $\displaystyle\int \sin x\,dx$
16. $\displaystyle\int \cos 3x\,dx$
17. $\displaystyle\int \sec^2\theta\,d\theta$
18. $\displaystyle\int \csc^2\theta\,d\theta$
19. $\displaystyle\int \sec\alpha\tan\alpha\,d\alpha$
20. $\displaystyle\int \sinh t\,dt$
21. $\displaystyle\int \cosh t\,dt$
22. $\displaystyle\int \sin x\cos x\,dx$
23. Show that the functions $x^4 - 8x^2 - 1$ and $(x^2 - 4)^2$ differ by a constant.
24. Show that the functions $2\sin^2 x - 5$ and $3 - \cos 2x$ differ by a constant.

87. The Constant of Integration. In expressing the antiderivative, or integral, we have seen that an arbitrary additive constant is involved. Now, it may very well happen that the function to be found must satisfy another condition besides the requirement that it possess a given derivative. In general, this condition can be met by assigning a suitable value to the constant of integration. This is best made clear by examples.

Example 1. The slope of a given curve at any point is equal to twice the abscissa of that point. Furthermore, the curve passes through the point (3,5). Find the equation of the curve. We have, at any point (x,y) of the plane, $\frac{dy}{dx} = 2x$. Therefore, y is a function whose derivative with respect to x is $2x$. In our integral notation, $y = \int 2x\, dx$. Hence

$$y = x^2 + C \qquad (1)$$

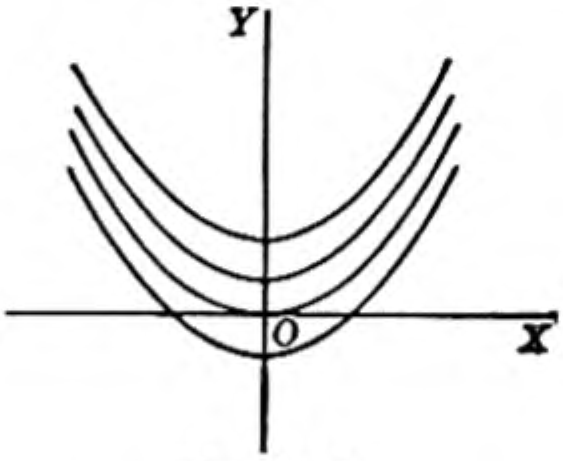


FIG. 105.

Therefore, the curve is a parabola, and with each value of C we have a different parabola. Equation (1) represents, in fact, a *family of parabolas* with axes on the y axis (Fig. 105). But only one of these parabolas passes through the point (3,5). So we must pick out this particular parabola, and this is done by a proper determination of a value for C. The coordinates (3,5) must satisfy equation (1); consequently, $5 = 9 + C$, and $C = -4$. Therefore, $y = x^2 - 4$ is the required curve.

Example 2. A projectile is thrown vertically upward with initial velocity of 256 ft. per second. Assuming that its acceleration due to gravity is $j = -32$ ft./sec.^{2} and neglecting all forces other than gravity find how high it will go and when it will strike the ground. Let y represent the distance of the projectile above the starting point t sec. after starting (Fig. 106). Note that the acceleration is negative since gravity imparts an acceleration downward.

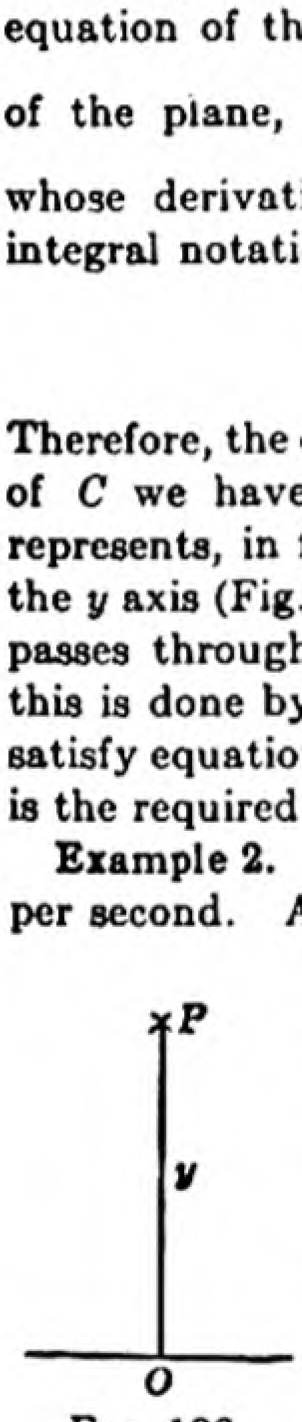


FIG. 106.

We have certain *initial* or *starting conditions* given, namely: (1) At time $t = 0$, the velocity is 256 ft. per second (positive since the projectile starts upward). (2) At time $t = 0$, the projectile is at the starting point, and $y = 0$. Briefly stated, at $t = 0$, $v = 256$, $y = 0$. Now, since $j = \frac{dv}{dt}$, the velocity is a function of t whose derivative is -32. Consequently $v = \int(-32)\,dt = -32\int dt = -32t + C_1$. But, in addition to this requirement, v must have the value 256 when $t = 0$. Hence $256 = (-32)\cdot 0 + C_1$, and $C_1 = 256$. Therefore $v = -32t + 256$. Now $\frac{dy}{dt} = v$, and consequently y is a function of t whose derivative with respect to t is v. In integral notation, we have

$$y = \int v\,dt = \int(-32t + 256)\,dt = -16t^2 + 256t + C_2$$

However, y must satisfy the additional condition that, when $t = 0$, $y = 0$. Substituting, we obtain $C_2 = 0$. Therefore

$$y = -16t^2 + 256t$$

This is the *law of motion* of the projectile.

It is easy to answer the remaining questions. To find when y will be a maximum,

we set $v = 0$, so that $-32t + 256 = 0$, which gives $t = 8$. The maximum value of y will be, therefore

$$-16 \cdot (64) + 256 \cdot (8) = 1024 \text{ ft.}$$

Also, $y = 0$ if $-16t^2 + 256t = -16t(t - 16) = 0$, that is, when $t = 0$ or when $t = 16$. Hence, the projectile will rise to a height of 1024 ft. and will strike the ground 16 sec. after starting.

Example 3. A projectile is fired from a gun into the air at an angle of elevation α and with an initial velocity (muzzle velocity) v_0. Taking the origin at the muzzle of the gun (Fig. 107) and the positive direction of the y axis upward, find the path of the projectile. Ignore forces other than gravity.

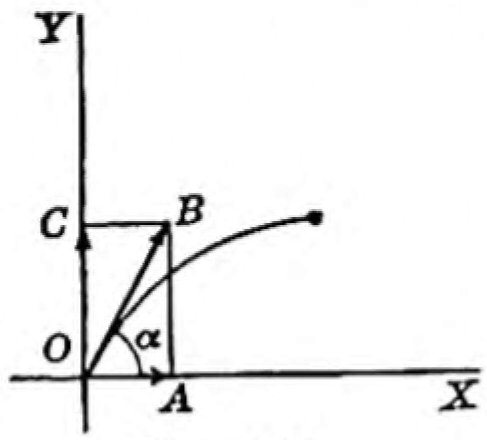


FIG. 107.

We shall first describe the initial conditions. Let the vector OB represent the initial velocity v_0. Then vectors $OA = v_0 \cos \alpha$ and $OC = v_0 \sin \alpha$ represent, respectively, the horizontal and vertical components of the initial velocity. Measuring time from the instant when the projectile leaves the point O, we also have the following: When $t = 0$, $x = 0$ and $y = 0$.

Let $P(x,y)$ be the position of the projectile at any time t. Since gravity is the only force acting, the horizontal component of acceleration* is zero, and the vertical component is $-g$. Thus

$$j_x = \frac{dv_x}{dt} = 0 \qquad \text{and} \qquad j_y = \frac{dv_y}{dt} = -g$$

Integrating, we have

$$v_x = C_1 \qquad \text{and} \qquad v_y = -gt + C_2$$

When $t = 0$, v_x must be $v_0 \cos \alpha$, and v_y must be $v_0 \sin \alpha$. Therefore

$$C_1 = v_0 \cos \alpha \qquad \text{and} \qquad C_2 = v_0 \sin \alpha$$

Thus $$\frac{dx}{dt} = v_x = v_0 \cos \alpha \qquad \text{and} \qquad \frac{dy}{dt} = v_y = -gt + v_0 \sin \alpha$$

Integrating to find x and y, we have

$$x = (v_0 \cos \alpha)t + C_3 \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t + C_4$$

Again using the initial conditions, we see that C_3 and C_4 are both zero.

Therefore $$x = v_0 t \cos \alpha \qquad y = -\tfrac{1}{2}gt^2 + v_0 t \sin \alpha$$

are the parametric equations of the path of the projectile.

It must be emphasized that these equations hold only if air resistance is neglected. As a matter of fact, air resistance is of major importance. These equations alone would give a quite erroneous description of the flight of a projectile fired from a gun into the air, but this complicated problem of *exterior ballistics* cannot be taken up here.

EXERCISES

1. The slope of a given curve at any point (x,y) is $2x + 1$. It passes through $(-4,2)$. Find its equation.

2. Find the equation of the curve for which $\dfrac{dy}{dx} = x^2 - 1$ and which passes through $(3,8)$.

* See Art. 67.

3. Describe the family of curves for which $\frac{dy}{dx} = 3x^2$ at any point.

4. Describe the family of curves for which $\frac{dy}{dx} = \frac{1}{x}$ at any point.

5. Find the equation of the curve for which $\frac{d^2y}{dx^2} = 1$ and which has slope -2 at $(3,-1)$.

6. Find the equation of the curve for which $\frac{d^2y}{dx^2} = -2$ and which is tangent to the line $3x - y - 5 = 0$ at $(2,1)$.

7. Find the equation of the curve for which $y'' = 6x$ and which is tangent to the line $2x + y - 6 = 0$ at $(1,4)$.

8. Find the equation of the curve for which $y'' = 4$ and which has a horizontal tangent at the point $(-2,-1)$.

9. Find the equation of the curve for which $y'' = 6x - 4$ and which passes through the points $(1,2)$ and $(-2,5)$.

10. Find the equation of the curve for which $y'' = 2 - 12x$ and which passes through the points $(2,-2)$ and $(1,3)$.

11. A projectile is thrown vertically upward from the ground with initial velocity of 128 ft. per second. If $j = g = -32$ ft./sec.^{2} and no forces other than gravity are considered, find the law of motion, the height to which the projectile will rise, and the speed with which it will strike the ground.

12. Same as Exercise 11 with initial velocity 512 ft. per second

13. An object falls from a window 144 ft. above the ground. How long will it take to reach the ground, and how fast will it be going when it strikes the ground?

14. A stone is hurled straight downward from a window 256 ft. above the ground with an initial speed of 96 ft. per second. Find the law of motion. When will the stone strike the ground, and at what speed?

15. A stone is thrown vertically upward from a window 144 ft. above the ground with initial speed of 128 ft. per second. How high will the stone rise? How fast will it be going when it strikes the ground?

16. A ball is thrown vertically upward from the ground with initial speed of 96 ft. per second. Will it reach a window 150 ft. above the ground?

17. Use the results of Example 3, Art. 87, to find the cartesian equation of the path of a projectile fired at an angle α with the horizontal.

18. Using the results of Example 3, Art. 87, find the point where the projectile strikes the ground (assumed level with the muzzle of the gun), and so determine the *range*. What angle gives maximum range?

19. In Exercise 18, find the height to which the projectile will rise.

20. In Exercise 18, find the time of flight of the projectile.

88. Integral of a Power of a Function. We begin our study of the *standard forms* with consideration of

$$\text{(I)} \qquad \int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

The formula is established at once by differentiating the right-hand side; its differential is $\frac{(n+1)u^n}{n+1}\,du = u^n\,du$. Note that this holds for any n *except* $n = -1$, since the denominator $n + 1 = 0$ for $n = -1$. If

$n = -1$, we have

$$\text{(II)} \qquad \int \frac{du}{u} = \ln |u| + C$$

To verify this note that, if $u > 0$, then $d(\ln u + C) = \dfrac{du}{u}$. If $u < 0$, let $u = -v$ where $v > 0$. Then

$$d(\ln |u| + C) = d(\ln v + C) = \frac{dv}{v} = \frac{-du}{-u} = \frac{du}{u}$$

We shall refer to (I) and (II) as the *power form.*

Example 1. Using formula (I),

$$\int (x^5 + x^4 - x^3 - x^2 + x - 1)\,dx = \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + C$$

Example 2. Using formulas (I) and (II),

$$\begin{aligned}\int \left(17x^4 - 6x^2 + \frac{5}{x}\right) dx &= 17 \int x^4\,dx - 6 \int x^2\,dx + 5 \int \frac{dx}{x} \\ &= 17 \cdot \frac{x^5}{5} - 6 \cdot \frac{x^3}{3} + 5 \ln |x| + C \\ &= \tfrac{17}{5}x^5 - 2x^3 + 5 \ln |x| + C\end{aligned}$$

Example 3. Find $\int (x^2 + 16)^2 x\,dx$. If we first multiply out, we get

$$\begin{aligned}\int (x^2 + 16)^2 x\,dx &= \int (x^4 + 32x^2 + 256)x\,dx \\ &= \int (x^5 + 32x^3 + 256x)\,dx \\ &= \frac{x^6}{6} + 8x^4 + 128x^2 + C\end{aligned}$$

Now suppose we wish to find $\int (x^2 + 16)^{11} x\,dx$. This integrand could, of course, be expanded by the binomial theorem as was done in Example 3. However, this would require prolonged calculations. Notice that $(x^2 + 16)$ is a function of x which is raised to the eleventh power. If we were to let $u = x^2 + 16$, then $(x^2 + 16)^{11} = u^{11}$. Furthermore, $du = 2x\,dx$. Fortunately, the integrand contains the factor $x\,dx$. This could be replaced by its equal, $\frac{1}{2}\,du$. The integral becomes

$$\int (x^2 + 16)^{11} x\,dx = \int u^{11} \cdot \frac{1}{2}\,du = \frac{1}{2} \int u^{11}\,du = \frac{1}{2} \cdot \frac{u^{12}}{12} + C = \frac{u^{12}}{24} + C$$

Now, writing for u its value in terms of x, we have our final result

$$\int (x^2 + 16)^{11} x\,dx = \tfrac{1}{24}(x^2 + 16)^{12} + C$$

Let us apply this method to the integral of Example 3, as follows:

$$\begin{aligned}\int (x^2 + 16)^2 x\,dx &= \int u^2 \cdot \frac{1}{2}\,du = \frac{1}{2}\int u^2\,du = \frac{1}{2}\cdot\frac{u^3}{3} + C' \\ &= \tfrac{1}{6}u^3 + C' \\ &= \tfrac{1}{6}(x^2 + 16)^3 + C'\end{aligned}$$

This result is easily checked by differentiation. We may compare with the result given in Example 3 by expanding

$$\begin{aligned}\tfrac{1}{6}(x^2 + 16)^3 + C' &= \tfrac{1}{6}(x^6 + 48x^4 + 768x^2 + 4096) + C' \\ &= \tfrac{1}{6}x^6 + 8x^4 + 128x^2 + C\end{aligned}$$

where $C = \frac{2048}{3} + C'$.

In these examples, note particularly that in each case the original integrand was the *power of a function, times* a factor which differed from the differential of the function only by a constant factor. If we had had $\int(x^2 + 16)^{11}\,dx$, we could not have called this $\int u^{11}\,du$, since, if $u = x^2 + 16$, $du = 2x\,dx$. The *presence of the factor* x was what made possible the use of formula (I) without expansion by the binomial theorem. Note that the constant factor 2 played an unimportant role. Since we had only $x\,dx$ instead of $2x\,dx$, we had $\frac{1}{2}\,du$ instead of du; but the factor $\frac{1}{2}$ could be taken outside the sign of integration.

Example 4. Find $\int x^2\sqrt{x^3 + 8}\,dx$. Evidently we cannot reduce this to a standard form by expanding $(x^3 + 8)^{1/2}$ by the binomial theorem, for this expansion gives a never-ending series. But we do have the $\frac{1}{2}$ power of a function. Is this multiplied by the differential of the function? To find out, set $u = x^3 + 8$; then $du = 3x^2\,dx$. Fortunately, we have $x^2\,dx$. Hence, $\frac{1}{3}\,du = x^2\,dx$, and

$$\int x^2\sqrt{x^3 + 8}\,dx = \int u^{1/2}\cdot\tfrac{1}{3}\,du = \tfrac{1}{3}\int u^{1/2}\,du$$

This is easily integrated by (I), and we obtain

$$\frac{1}{3}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3 + 8)^{3/2} + C$$

as our result. Again notice that having $x^2\,dx$ instead of $3x^2\,dx$ is unimportant.

Example 5. Find $\int \sin 3\theta \cos^4 3\theta\,d\theta$. This is evidently a power form, for we have a power of $\cos 3\theta$ times a constant multiple of $-3\sin 3\theta\,d\theta$, the differential of $\cos 3\theta$. In fact, if $u = \cos 3\theta$, then $du = -3\sin 3\theta\,d\theta$, and we have $-\frac{1}{3}\,du = \sin 3\theta\,d\theta$.

Hence
$$\begin{aligned}\int \sin 3\theta\cos^4 3\theta\,d\theta &= -\frac{1}{3}\int u^4\,du = -\frac{1}{3}\cdot\frac{u^5}{5} + C \\ &= -\tfrac{1}{15}\cos^5 3\theta + C\end{aligned}$$

The student should check this by differentiation.

Example 6. Find $\displaystyle\int \frac{dx}{\sqrt{2x + 5}}$. Here, if $u = 2x + 5$,

$$du = 2\,dx \qquad \tfrac{1}{2}\,du = dx$$

and we have

$$\int \frac{dx}{\sqrt{2x+5}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{u} + C$$
$$= \sqrt{2x+5} + C$$

Example 7. Find $\int \frac{5x\,dx}{x^2-16}$. Here if $u = x^2 - 16$,

$$du = 2x\,dx \qquad \tfrac{1}{2}\,du = x\,dx$$

and we have

$$\int \frac{5x\,dx}{x^2-16} = \frac{5}{2}\int \frac{du}{u} = \frac{5}{2}\ln|u| + C = \frac{5}{2}\ln|x^2-16| + C$$

Example 8. Find $\int \frac{3x^2-2x}{x+4}\,dx$. We are here confronted with an integral that is certainly not, as it stands, of the form $\int u^n\,du$. But we can reduce it to a sum of such integrals by carrying out the indicated division, thus

$$\frac{3x^2-2x}{x+4} = 3x - 14 + \frac{56}{x+4}$$

We then have

$$\int \frac{3x^2-2x}{x+4}\,dx = \frac{3}{2}x^2 - 14x + 56\ln|x+4| + C$$

We note that $\frac{3x^2-2x}{x+4}$ is a *rational fraction* (Art. 6). We state the following rule which will have frequent application: *As a preliminary step toward integrating any rational fraction in which the degree of the numerator is the same as, or higher than, the degree of the denominator, carry out the indicated division until the remainder is of lower degree than the denominator.*

EXERCISES

Find the following integrals, and check by differentiation:

1. $\int (3x^5 - x^4 + 5x^2 - 1)\,dx$
2. $\int (15x^3 + 7x^2 - 11x + 17)\,dx$
3. $\int \left(x - \frac{1}{x}\right)dx$
4. $\int \left(2y^2 + \frac{3}{y^2}\right)dy$
5. $\int (\sqrt{y} - 9y^{3/2})\,dy$
6. $\int \sqrt{2y}\,dy$
7. $\int \left(\frac{3}{\sqrt{5z}} + \frac{1}{z^{3/4}}\right)dz$
8. $\int (z+4)^2\,dz$

9. $\int (1 - t)^2 \, dt$

10. $\int (t^2 + 1)^2 \, dt$

11. $\int t(t^2 + 1)^2 \, dt$

12. $\int \sqrt{x - 6} \, dx$

13. $\int x(x^2 - 9)^{3/2} \, dx$

14. $\int \frac{x \, dx}{x^2 + 16}$

15. $\int \frac{dy}{\sqrt{2 - y}}$

16. $\int \frac{y \, dy}{(y^2 + 25)^2}$

17. $\int \frac{(2x + 1)}{x^2 + x + 3} \, dx$

18. $\int (8 - x)^{3/2} \, dx$

19. $\int \frac{x^2 + 4}{x} \, dx$

20. $\int \frac{x}{x^2 + 4} \, dx$

21. $\int \frac{dx}{x^2 - 4x + 4}$

22. $\int \cos^2 \theta \sin \theta \, d\theta$

23. $\int \frac{\sin 2\theta}{\cos^4 2\theta} \, d\theta$

24. $\int \frac{\cos 3\theta}{\sqrt{\sin 3\theta}} \, d\theta$

25. $\int \tan \frac{1}{2}\theta \sec^2 \frac{1}{2}\theta \, d\theta$

26. $\int \sqrt{\cot \theta} \csc^2 \theta \, d\theta$

27. $\int \frac{\sec \theta \tan \theta \, d\theta}{1 + \sec \theta}$

28. $\int \sin^{3/2} \alpha \cos \alpha \, d\alpha$

29. $\int \frac{\sec^2 \theta \, d\theta}{(1 + \tan \theta)^3}$

30. $\int \frac{u \, du}{(a^2 - u^2)^2}$

31. $\int \frac{t^3 \, dt}{t^4 + 16}$

32. $\int t^5(2t^6 + 1)^4 \, dt$

33. $\int \frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}} \, dx$

34. $\int (x^{3/2} + a^{3/2})^2 \sqrt{x} \, dx$

35. $\int \frac{\ln x}{x} \, dx$

36. $\int \frac{\sqrt{\ln x}}{x} \, dx$

37. $\int \frac{dx}{x \ln x}$

38. $\int \frac{dx}{x \ln^2 x}$

39. $\int \frac{(1 + \ln x)^{3/2}}{x} \, dx$

40. $\int \frac{\ln \ln x}{x \ln x} \, dx$

41. $\int e^x(1 + e^x)^2 \, dx$

42. $\int \frac{e^{3x}}{1 + e^{3x}} \, dx$

43. $\int \frac{\sqrt{1 + e^{-x}}}{e^x} \, dx$

44. $\int \frac{e^t \, dt}{(1 + e^t)^{3/2}}$

45. $\int \frac{x^2 + 4}{x - 1} \, dx$

46. $\int \frac{x + a}{x - a} \, dx$

47. $\int \frac{(x^2 + 3x + 4)}{x + 3} \, dx$

48. $\int \tan \theta \, d\theta$

49. $\int \cot \frac{1}{3}\alpha \, d\alpha$

50. $\int \frac{(x^2 - a^2)^2}{x^2} \, dx$

51. $\int \frac{(x^3 + 3x^2 + 2x - 7) \, dx}{2x + 1}$

52. $\int \frac{\csc^2 2\theta}{4 - 3 \cot 2\theta} \, d\theta$

53. $\displaystyle\int \frac{\sin \frac{1}{2}\alpha \, d\alpha}{(1 + 5 \cos \frac{1}{2}\alpha)^2}$

54. $\displaystyle\int \frac{\arcsin x}{\sqrt{1 - x^2}} \, dx$

55. $\displaystyle\int \frac{\arctan 2x}{1 + 4x^2} \, dx$

56. $\displaystyle\int \sinh y \cosh^2 y \, dy$

57. $\displaystyle\int \frac{\cosh y}{\sinh^3 y} \, dy$

58. $\displaystyle\int \frac{\ln \sinh z \, dz}{\tanh z}$

59. $\displaystyle\int \tanh 5x \, dx$

60. $\displaystyle\int \frac{\operatorname{sech}^2 x}{1 + \tanh x} \, dx$

89. Integral of the Exponential Function. The standard formulas

$$\text{(III)} \qquad \int e^u \, du = e^u + C$$

and

$$\text{(IV)} \qquad \int a^u \, du = \frac{a^u}{\ln a} + C = a^u \log_a e + C$$

are easily verified by differentiation. Note that the exponent of e (or of a) is a function whose differential appears in the integrand.

Example 1. Find $\int e^{2x} \, dx$. If we let $u = 2x$, then $du = 2 \, dx$. Since the integrand contains only dx, we have $\frac{1}{2} \, du = dx$, and so

$$\int e^{2x} \, dx = \tfrac{1}{2}\int e^u \, du = \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{2x} + C$$

Example 2. Find $\int xe^{-x^2} \, dx$. Here, if we set $u = -x^2$, then

$$du = -2x \, dx$$

We have $x \, dx$, which is just $-\frac{1}{2} \, du = x \, dx$. Hence

$$\int xe^{-x^2} \, dx = -\tfrac{1}{2}\int e^u \, du = -\tfrac{1}{2}e^u + C = -\tfrac{1}{2}e^{-x^2} + C$$

Example 3. Find $\displaystyle\int \frac{e^x \, dx}{1 + e^x}$. This is simply the power form. Let $u = 1 + e^x$. Then $du = e^x \, dx$, and our integral is

$$\int \frac{du}{u} = \ln |u| + C = \ln |1 + e^x| + C = \ln (1 + e^x) + C$$

The student should be constantly alert to recognize the power form in any given case.

Example 4. Find $\displaystyle\int \frac{dx}{1 + e^x}$. Note the difference from the integral of Example 3.

This is *not* a power form. Neither is it of the form $\displaystyle\int e^u \, du$. But a simple transformation will reduce it to standard forms. Carry out the indicated division, obtaining $\dfrac{1}{1 + e^x} = 1 - \dfrac{e^x}{1 + e^x}$. The process should be stopped as soon as a remainder occurs that can be integrated. We have

$$\int \frac{dx}{1 + e^x} = \int \left(1 - \frac{e^x}{1 + e^x}\right) dx = x - \ln (1 + e^x) + C$$

Example 5. Find $\int \frac{e^{3x}\,dx}{1+e^x}$. Write this fraction as $\frac{e^{3x}}{e^x+1}$, and carry out the indicated division until a remainder appears that can be integrated,

$$\frac{e^{3x}}{e^x+1} = e^{2x} - e^x + \frac{e^x}{e^x+1}$$

We obtain

$$\int \frac{e^{3x}\,dx}{1+e^x} = \int \left(e^{2x} - e^x + \frac{e^x}{e^x+1}\right) dx = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + C$$

Example 6. The Law of Natural Growth. Suppose bacteria are allowed to grow naturally in a culture that provides a sufficient food supply. Then the increase per unit of time in the number of bacteria in a unit of volume (that is, the rate of growth) is proportional to the number present in that unit. If x represents the number of bacteria present* at time t, then this rate of growth may be represented by $\frac{dx}{dt}$. This law of growth of the population of the culture can therefore be expressed in the following way:

$$\frac{dx}{dt} = kx$$

To get all expressions containing x on the left-hand side and those containing t on the right-hand side, we multiply both sides of this equation by $\frac{dt}{x}$; we get $\frac{dx}{x} = k\,dt$. Integrating, $\ln|x| = kt + \ln C$, which reduces to

$$x = Ce^{kt}$$

This is often called the *law of natural growth.*

EXERCISES

Find the following integrals, and check by differentiation (Ex. 1 to 24):

1. $\int e^{-3x}\,dx$
2. $\int xe^{x^2}\,dx$
3. $\int e^{2x-1}\,dx$
4. $\int e^{1-4x}\,dx$
5. $\int y^3e^{y^4}\,dy$
6. $\int e^{\sin y}\cos y\,dy$
7. $\int \frac{e^{2x}\,dx}{4+e^{2x}}$
8. $\int \frac{e^{3x}}{4+e^x}\,dx$
9. $\int \frac{1+e^x}{1-e^x}\,dx$
10. $\int \frac{e^{4x}\,dx}{1+e^{2x}}$

* We shall assume x to vary continuously, although, in interpreting the final result of the calculation in a given problem, we shall take the appropriate integral value of x. Compare the remarks in Example 4, Art. 37.

11. $\int e^x(1 + e^x)^2\,dx$

12. $\int (1 + e^x)^2\,dx$

13. $\int \frac{e^x\,dx}{e^{\frac{1}{2}x} + 1}$

14. $\int (e^{x^2} + 16)xe^{x^2}\,dx$

15. $\int \frac{dx}{1 + e^{\frac{1}{2}x}}$

16. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

17. $\int e^{\tan 2\theta} \sec^2 2\theta\,d\theta$

18. $\int 10^{x^2}x\,dx$

19. $\int a^{bx}\,dx$

20. $\int \frac{10^x - 1}{10^x + 1}\,dx$

21. $\int (a^x + x^a)\,dx$

22. $\int 10^{\cot 3\theta} \csc^2 3\theta\,d\theta$

23. $\int \frac{(e^{4x} - 2e^{3x} + 5e^x - 2)\,dx}{e^x + 1}$

24. $\int \frac{e^x\,dx}{\sqrt{e^x + 8}}$

25. One hundred bacteria were present in a certain culture. Three hours later the number was found to have increased to 280. If the rate of increase was proportional to the number present, find the law of growth.

26. Sugar decomposes at a rate proportional to the amount present. If 100 g. becomes 40 g. in 3 hr., find when 0.1 g. will remain.

27. A town had a population of 10,000 in 1940. By 1950, this had increased to 17,000. If the population growth follows the law of natural growth, find the population (nearest thousand) that may be expected for 1960.

28. If a principal of P dollars is invested at an interest rate r (that is, $100r$ per cent) per year compounded k times per year, the amount A to which the investment will accumulate at the end of t years is

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

Show that, if interest is compounded continuously, or "instantaneously" (that is, if $k \to \infty$), then $A = Pe^{rt}$. For this reason the law of natural growth is sometimes called the *compound interest law*.

29. An endowment fund is increasing continuously at the rate of 2 per cent per year. In what time will the amount be doubled (see Exercise 28)?

30. What rate of interest continuously compounded is equivalent to 4 per cent compounded annually (see Exercise 28)?

31. The deceleration of a ship in still water is proportional to its velocity. If the velocity is v_0 ft. per second at the time the power is shut off, show that the distance the ship travels in the next t seconds is $s = \frac{v_0}{k}(1 - e^{-kt})$, where k is a constant of proportionality.

90. Integrals of Trigonometric Functions. The following standard forms are suggested by the formulas for derivatives and are easily verified by differentiation:

(V) $\int \sin u\,du = -\cos u + C$

(VI) $\int \cos u\,du = \sin u + C$

(VII) $$\int \sec^2 u\, du = \tan u + C$$

(VIII) $$\int \csc^2 u\, du = -\cot u + C$$

(IX) $$\int \sec u \tan u\, du = \sec u + C$$

(X) $$\int \csc u \cot u\, du = -\csc u + C$$

It is not difficult to find the integrals of the tangent, cotangent, secant and cosecant. We have

$$\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du$$

Here we may set $v = \cos u$; then $dv = -\sin u\, du$, and

$$\int \tan u\, du = -\int \frac{dv}{v} = -\ln |v| + C = -\ln |\cos u| + C$$
$$= \ln |\sec u| + C$$

We therefore have

(XI) $$\int \tan u\, du = \ln |\sec u| + C$$

Similarly, the student can show that

(XII) $$\int \cot u\, du = \ln |\sin u| + C$$

To find $\int \sec u\, du$, multiply the integrand by

$$\frac{\sec u + \tan u}{\sec u + \tan u}$$

This gives

(XIII) $$\int \sec u\, du = \int \frac{\sec^2 u + \tan u \sec u}{\tan u + \sec u}\, du$$
$$= \ln |\sec u + \tan u| + C$$

Similarly, the student can show that

(XIV) $$\int \csc u\, du = -\ln |\csc u + \cot u| + C$$
$$= \ln |\csc u - \cot u| + C$$

Since the correctness of any integral can be tested by differentiation, the student should verify each of these results, and also the following, in this way.

Example 1. To find $\int \sin 4x\, dx$, notice that we have the sine of a function times the differential of that function; for if $u = 4x$, $du = 4\, dx$, and $\frac{1}{4}\, du = dx$. Hence

$$\int \sin 4x\, dx = \tfrac{1}{4}\int \sin u\, du = -\tfrac{1}{4} \cos u + C$$
$$= -\tfrac{1}{4} \cos 4x + C$$

Example 2. To find $\int \sec^2 \left(\frac{x}{2} + \frac{\pi}{4}\right) dx$, note that, if $u = \frac{x}{2} + \frac{\pi}{4}$, then $du = \frac{1}{2}\, dx$, and $2\, du = dx$. Hence

$$\int \sec^2 \left(\frac{x}{2} + \frac{\pi}{4}\right) dx = 2\int \sec^2 u\, du = 2 \tan u + C = 2 \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) + C$$

Example 3. To find $\int x \csc^2 (3x^2 + 1)\, dx$, note that, if $u = 3x^2 + 1$, then

$$du = 6x\, dx$$

and $\frac{1}{6}\, du = x\, dx$. Therefore

$$\begin{aligned}\int x \csc^2 (3x^2 + 1)\, dx &= \tfrac{1}{6}\int \csc^2 u\, du = -\tfrac{1}{6} \cot u + C \\ &= -\tfrac{1}{6} \cot (3x^2 + 1) + C\end{aligned}$$

EXERCISES

Find the following integrals, and check by differentiation:

1. $\displaystyle\int \cos 3\theta\, d\theta$

2. $\displaystyle\int \sin (\theta/2)\, d\theta$

3. $\displaystyle\int \csc^2 \left(\frac{x}{3} + \frac{\pi}{4}\right) dx$

4. $\displaystyle\int \sec^2 (2x + 1)\, dx$

5. $\displaystyle\int \sec(\alpha/\pi) \tan (\alpha/\pi)\, d\alpha$

6. $\displaystyle\int \csc \left(\frac{2\alpha}{3} + \frac{\pi}{8}\right) \cot \left(\frac{2\alpha}{3} + \frac{\pi}{8}\right) d\alpha$

7. $\displaystyle\int x \sin x^2\, dx$

8. $\displaystyle\int \frac{1}{x^2} \sin \frac{\pi}{x}\, dx$

9. $\displaystyle\int e^x \cos e^x\, dx$

10. $\displaystyle\int \frac{\tan \sqrt{x}\, dx}{\sqrt{x}}$

11. $\displaystyle\int \cot (x/k)\, dx$

12. $\displaystyle\int \frac{dy}{\cos^2 y}$

13. $\displaystyle\int \frac{\sin z\, dz}{\cos^2 z}$

14. $\displaystyle\int \tan 3\theta \sec^2 3\theta\, d\theta$

15. $\displaystyle\int x \sin^2 x^2 \cos x^2\, dx$

16. $\displaystyle\int \sec (3x/4)\, dx$

17. $\displaystyle\int \csc \left(5\theta + \frac{\pi}{7}\right) d\theta$

18. $\displaystyle\int \sin 3\theta \cos^5 3\theta\, d\theta$

19. $\displaystyle\int \frac{d\beta}{\cos \beta \cot \beta}$

20. $\displaystyle\int \frac{d\beta}{\sin 2\beta}$

21. $\displaystyle\int \frac{\cos z + \sin z}{\sin^2 z}\, dz$

22. $\displaystyle\int \frac{\sin z + \cos z}{\sin z}\, dz$

23. $\displaystyle\int \csc^2 2x \cot^2 2x\, dx$

24. $\displaystyle\int x \sec x^2\, dx$

25. $\displaystyle\int x^3 \cos 5x^4\, dx$

26. $\displaystyle\int x^2 \sin 2x^3\, dx$

27. $\displaystyle\int \frac{x}{\sin^2 x^2}\, dx$

28. $\displaystyle\int \sec 4\theta \tan 4\theta\, d\theta$

29. $\displaystyle\int x^2 \tan (x^3/2)\, dx$

30. $\displaystyle\int \frac{d\theta}{\sin 3\theta \tan 3\theta}$

31. $\int (1 + \sin 5y)^2 \cos 5y \, dy$

32. $\int \cos (3x/2) \csc (3x/2) \, dx$

33. $\int \frac{(1 + \tan 2z)^2}{\cos^2 2z} dz$

34. $\int \frac{\cos x^{2/3}}{\sqrt[3]{x}} dx$

35. $\int \frac{\cos^3 x}{1 + \sin x} dx$

36. $\int \frac{\sin^3 x}{1 - \cos x} dx$

37. $\int \frac{\tan^3 x}{\sec x - 1} dx$

38. $\int \frac{\cot^3 x}{1 - \csc x} dx$

91. Integrals Leading to Inverse Trigonometric Functions. The following standard formulas are easily verified as indicated:

(XV)
$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

for
$$\frac{d}{du}\left(\arcsin \frac{u}{a}\right) = \frac{1/a}{\sqrt{1 - \frac{u^2}{a^2}}} = \frac{1}{\sqrt{a^2 - u^2}}$$

Evidently,

(XVa)
$$\int \frac{du}{\sqrt{a^2 - u^2}} = -\operatorname{arccos} \frac{u}{a} + C$$

(XVI)
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

for
$$\frac{d}{du}\left(\frac{1}{a} \arctan \frac{u}{a}\right) = \frac{1}{a} \cdot \frac{1/a}{1 + \frac{u^2}{a^2}} = \frac{1}{a^2 + u^2}$$

Evidently,

(XVIa)
$$\int \frac{du}{a^2 + u^2} = -\frac{1}{a} \operatorname{arccot} \frac{u}{a} + C$$

(XVII)
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$$

for
$$\frac{d}{du}\left(\frac{1}{a} \operatorname{arcsec} \frac{u}{a}\right) = \frac{1}{a} \cdot \frac{1/a}{\frac{u}{a}\sqrt{\frac{u^2}{a^2} - 1}} = \frac{1}{u\sqrt{u^2 - a^2}}$$

Evidently,

(XVIIa)
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = -\frac{1}{a} \operatorname{arccsc} \frac{u}{a} + C$$

In each of the above formulas, the principal value of the function is to be used.

The student should verify by differentiation the results in the following examples.

Example 1. To find $\int \frac{dx}{\sqrt{16 - 25x^2}}$, let $u = 5x$, $du = 5\,dx$, $\frac{1}{5}\,du = dx$. Then

$$\int \frac{dx}{\sqrt{16 - 25x^2}} = \frac{1}{5}\int \frac{du}{\sqrt{16 - u^2}} = \frac{1}{5}\arcsin\frac{u}{4} + C = \frac{1}{5}\arcsin\frac{5x}{4} + C$$

Example 2. We have

$$\int \frac{dx}{11 + 7x^2} = \frac{1}{\sqrt{7}}\int \frac{du}{11 + u^2}$$
$$= \frac{1}{\sqrt{7}}\cdot\frac{1}{\sqrt{11}}\arctan\frac{u}{\sqrt{11}} + C = \frac{1}{\sqrt{77}}\arctan\sqrt{\tfrac{7}{11}}\,x + C$$

if we let $u = \sqrt{7}\,x$, $du = \sqrt{7}\,dx$, and $\frac{1}{\sqrt{7}}\,du = dx$.

Example 3. We have

$$\int \frac{dx}{x\sqrt{4x^2 - 49}} = \int \frac{\frac{1}{2}\,du}{\frac{1}{2}u\sqrt{u^2 - 49}} = \int \frac{du}{u\sqrt{u^2 - 49}}$$
$$= \frac{1}{7}\operatorname{arcsec}\frac{u}{7} + C = \frac{1}{7}\operatorname{arcsec}\frac{2x}{7} + C$$

if we let $u = 2x$, $du = 2\,dx$, and $\frac{1}{2}\,du = dx$.

Example 4. To find $\int \frac{dx}{\sqrt{21 + 12x - 9x^2}}$, we note that the expression under the radical sign can be made the *difference* of two squares by adding and subtracting 4, thus:

$$21 + 12x - 9x^2 = 25 - 4 + 12x - 9x^2 = 25 - (4 - 12x + 9x^2)$$
$$= 25 - (2 - 3x)^2 = 25 - (3x - 2)^2$$

Hence, the integral becomes

$$\int \frac{dx}{\sqrt{25 - (3x - 2)^2}} = \frac{1}{3}\int \frac{du}{\sqrt{25 - u^2}} = \frac{1}{3}\arcsin\frac{u}{5} + C = \frac{1}{3}\arcsin\frac{3x - 2}{5} + C$$

if $u = 3x - 2$, $du = 3\,dx$, and $\frac{1}{3}\,du = dx$. Notice that we reach the same result if we use

$$\int \frac{dx}{\sqrt{25 - (2 - 3x)^2}} = -\frac{1}{3}\int \frac{dv}{\sqrt{25 - v^2}} = -\frac{1}{3}\arcsin\frac{v}{5} + C$$
$$= -\frac{1}{3}\arcsin\frac{2 - 3x}{5} + C = \frac{1}{3}\arcsin\frac{3x - 2}{5} + C$$

Example 5. To find $\int \frac{dx}{\sqrt{3 - 5x - 2x^2}}$, we first remove the factor $\sqrt{2}$ from the denominator, thus: $\sqrt{2}\sqrt{\frac{3}{2} - \frac{5}{2}x - x^2}$. Next, we complete the square:

$$\tfrac{3}{2} - \tfrac{5}{2}x - x^2 = \tfrac{3}{2} - (x^2 + \tfrac{5}{2}x + \tfrac{25}{16}) + \tfrac{25}{16} = \tfrac{49}{16} - (x + \tfrac{5}{4})^2$$

Hence, our integral becomes

$$\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{49}{16} - (x + \frac{5}{4})^2}} = \frac{1}{\sqrt{2}}\arcsin\frac{x + \frac{5}{4}}{\frac{7}{4}} + C = \frac{1}{\sqrt{2}}\arcsin\frac{4x + 5}{7} + C$$

Observe that, in Examples 4 and 5, we were able to reduce the expression $ax^2 + bx + c$ to the form $k^2 - u^2$ because a was negative.

Example 6. To find $\int \frac{dx}{3x^2 + 4x + 5}$, we first factor 3 out of the denominator and then complete the square, obtaining

$$\frac{1}{3}\int \frac{dx}{x^2 + \frac{4}{3}x + \frac{5}{3}} = \frac{1}{3}\int \frac{dx}{(x^2 + \frac{4}{3}x + \frac{4}{9}) - \frac{4}{9} + \frac{5}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{(x + \frac{2}{3})^2 + \frac{11}{9}} = \frac{1}{3}\cdot\frac{3}{\sqrt{11}} \arctan\left(\frac{x + \frac{2}{3}}{\frac{\sqrt{11}}{3}}\right) + C$$

$$= \frac{1}{\sqrt{11}} \arctan \frac{3x + 2}{\sqrt{11}} + C$$

In this integral, fractions can be avoided as follows: Multiply numerator and denominator by 3.

$$\int \frac{dx}{3x^2 + 4x + 5} = \int \frac{3\,dx}{9x^2 + 12x + 15} = \int \frac{3\,dx}{(9x^2 + 12x + 4) + 11}$$

$$= \int \frac{3\,dx}{(3x + 2)^2 + 11} = \int \frac{du}{u^2 + 11}$$

$$= \frac{1}{\sqrt{11}} \arctan \frac{u}{\sqrt{11}} + C = \frac{1}{\sqrt{11}} \arctan \frac{3x + 2}{\sqrt{11}} + C$$

Such preliminary transformations are often very effective in reducing the work of computation.

It was possible, in Example 6, to reduce the quadratic expression $ax^2 + bx + c$ to a *sum* of squares because $b^2 - 4ac$ was *negative*. In general, $\int \frac{dx}{ax^2 + bx + c}$ is an arctangent if $b^2 - 4ac < 0$.

EXERCISES

Find the following integrals and check by differentiation.

1. $\int \frac{dx}{\sqrt{16 - x^2}}$
2. $\int \frac{dx}{25 + x^2}$
3. $\int \frac{dy}{\sqrt{1 - 4y^2}}$
4. $\int \frac{dx}{\sqrt{25 - 36x^2}}$
5. $\int \frac{dz}{16 + 81z^2}$
6. $\int \frac{dy}{\sqrt{11 - 5y^2}}$
7. $\int \frac{dx}{x\sqrt{x^2 - 9}}$
8. $\int \frac{dx}{12 + 5x^2}$
9. $\int \frac{z^2\,dz}{\sqrt{64 - 9z^6}}$
10. $\int \frac{dz}{z\sqrt{4z^2 - 25}}$
11. $\int \frac{e^{2x}\,dx}{25 + e^{4x}}$
12. $\int \frac{dy}{y\sqrt{49y^2 - 36}}$

13. $\int \frac{e^x\,dx}{\sqrt{1 - e^{2x}}}$

14. $\int \frac{x^2\,dx}{36 + 49x^6}$

15. $\int \frac{dx}{\sqrt{4x - x^2}}$

16. $\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

17. $\int \frac{dx}{x^2 + 2x + 5}$

18. $\int \frac{dy}{\sqrt{-y^2 - 8y - 7}}$

19. $\int \frac{dy}{y^2 + 7y + 15}$

20. $\int \frac{x\,dx}{\sqrt{16 - 5x^2}}$

21. $\int \frac{dx}{\sqrt{x - x^2}}$

22. $\int \frac{dx}{4x^2 + 12x + 13}$

23. $\int \frac{3x^2 + 12x - 2}{x^2 + 4}\,dx$

24. $\int \frac{dx}{x\sqrt{x - 1}}$

25. $\int \frac{dx}{\sqrt{7 - 6x - 9x^2}}$

26. $\int \frac{(x - 1)\,dx}{\sqrt{3 + 2x - x^2}}$

27. $\int \frac{dx}{(1 + x)\sqrt{x}}$

28. $\int \frac{dx}{\sqrt{e^{2x} - 1}}$

29. $\int \frac{(x - 1)\,dx}{x^2 + 6x + 13}$

30. $\int \frac{(2x - 1)\,dx}{\sqrt{3 - 2x - x^2}}$

92. Integrals Leading to Logarithmic Functions. The following formulas resemble those of the preceding section. They are readily derived by methods to be considered in the next chapter; but, for the present, we content ourselves with their verification.

(XVIII) $$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\,(u + \sqrt{u^2 + a^2}) + C$$

for

$$\frac{d}{du}[\ln\,(u + \sqrt{u^2 + a^2})] = \frac{1 + \dfrac{u}{\sqrt{u^2 + a^2}}}{u + \sqrt{u^2 + a^2}} = \frac{1}{\sqrt{u^2 + a^2}}$$

(XIX) $$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\,|u + \sqrt{u^2 - a^2}| + C$$

Let the student verify this result.

(XX) $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right| + C$$

(XX*a*) $$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u + a}{u - a}\right| + C$$

Let the student verify these results.

Example 1. We have

$$\int \frac{dx}{\sqrt{16x^2 + 25}} = \frac{1}{4}\int \frac{du}{\sqrt{u^2 + 25}} = \frac{1}{4}\ln\,(u + \sqrt{u^2 + 25}) + C$$
$$= \tfrac{1}{4}\ln\,(4x + \sqrt{16x^2 + 25}) + C$$

where we take $u = 4x$.

Example 2. To find $\int \frac{dx}{\sqrt{3x^2 + 4x + 5}}$, we may factor $\sqrt{3}$ out of the denominator and complete the square, obtaining

$$\begin{aligned}
\frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{4}{3}x + \frac{5}{3}}} &= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x^2 + \frac{4}{3}x + \frac{4}{9}) - \frac{4}{9} + \frac{5}{3}}} \\
&= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x + \frac{2}{3})^2 + \frac{11}{9}}} = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2 + \frac{11}{9}}} \\
&= \frac{1}{\sqrt{3}} \ln \left(u + \sqrt{u^2 + \tfrac{11}{9}}\right) + C \\
&= \frac{1}{\sqrt{3}} \ln \left(x + \frac{2}{3} + \sqrt{\left(x + \frac{2}{3}\right)^2 + \frac{11}{9}}\right) + C \\
&= \frac{1}{\sqrt{3}} \ln \left(3x + 2 + \sqrt{(3x + 2)^2 + 11}\right) - \frac{1}{\sqrt{3}} \log 3 + C \\
&= \frac{1}{\sqrt{3}} \ln \left(3x + 2 + \sqrt{9x^2 + 12x + 15}\right) + C' \qquad \text{where } u = x + \frac{2}{3}
\end{aligned}$$

Computation will be simplified if we first multiply numerator and denominator by $\sqrt{3}$, obtaining

$$\begin{aligned}
\int \frac{\sqrt{3}\,dx}{\sqrt{9x^2 + 12x + 15}} &= \sqrt{3} \int \frac{dx}{\sqrt{(9x^2 + 12x + 4) + 11}} \\
&= \sqrt{3} \int \frac{dx}{\sqrt{(3x + 2)^2 + 11}} = \frac{1}{3}\sqrt{3} \int \frac{dv}{\sqrt{v^2 + 11}} \qquad \text{where } v = 3x + 2 \\
&= \frac{1}{\sqrt{3}} \ln \left(v + \sqrt{v^2 + 11}\right) + C \\
&= \frac{1}{\sqrt{3}} \ln \left(3x + 2 + \sqrt{9x^2 + 12x + 15}\right) + C
\end{aligned}$$

Example 3. To find $\int \frac{dx}{2x^2 + 5x - 1}$, we may factor 2 out of the denominator and complete the square, obtaining

$$\begin{aligned}
\frac{1}{2} \int \frac{dx}{x^2 + \frac{5}{2}x - \frac{1}{2}} &= \frac{1}{2} \int \frac{dx}{x^2 + \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} - \frac{1}{2}} \\
&= \frac{1}{2} \int \frac{dx}{(x + \frac{5}{4})^2 - \frac{33}{16}} \\
&= \frac{1}{2} \cdot \frac{2}{\sqrt{33}} \ln \left| \frac{x + \frac{5}{4} - \frac{\sqrt{33}}{4}}{x + \frac{5}{4} + \frac{\sqrt{33}}{4}} \right| + C \\
&= \frac{1}{\sqrt{33}} \ln \left| \frac{4x + 5 - \sqrt{33}}{4x + 5 + \sqrt{33}} \right| + C
\end{aligned}$$

Or we may multiply numerator and denominator by 8, and thus avoid fractions.

Note that an expression of the form $ax^2 + bx + c$ can be expressed* as the difference of two squares if $b^2 - 4ac > 0$.

EXERCISES

Find the following integrals, and check by differentiation:

1. $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 16}}$

2. $\displaystyle\int \frac{dx}{\sqrt{4x^2 - 25}}$

3. $\displaystyle\int \frac{dx}{\sqrt{16 - x^2}}$

4. $\displaystyle\int \frac{dx}{\sqrt{25x^2 + 1}}$

5. $\displaystyle\int \frac{dy}{\sqrt{25y^2 - 1}}$

6. $\displaystyle\int \frac{dx}{5 - 3x^2}$

7. $\displaystyle\int \frac{dy}{\sqrt{3y^2 + 7}}$

8. $\displaystyle\int \frac{dz}{\sqrt{11z^2 - 6}}$

9. $\displaystyle\int \frac{du}{9u^2 - 17}$

10. $\displaystyle\int \frac{dz}{\sqrt{11 + 3z^2}}$

11. $\displaystyle\int \frac{dz}{64z^2 + 81}$

12. $\displaystyle\int \frac{dy}{\sqrt{25 - 4y^2}}$

13. $\displaystyle\int \frac{dx}{\sqrt{25x + 36}}$

14. $\displaystyle\int \frac{dx}{x^2 + 4x + 3}$

15. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x + 2}}$

16. $\displaystyle\int \frac{dy}{\sqrt{y^2 - 4y}}$

17. $\displaystyle\int \frac{dx}{\sqrt{6x - x^2 - 8}}$

18. $\displaystyle\int \frac{dy}{\sqrt{y^2 + 6y + 25}}$

19. $\displaystyle\int \frac{x\,dx}{3x^2 - 16}$

20. $\displaystyle\int \frac{x\,dx}{\sqrt{x^4 + 1}}$

21. $\displaystyle\int \frac{e^x\,dx}{\sqrt{e^{2x} + 16}}$

22. $\displaystyle\int \frac{e^x\,dx}{\sqrt{3e^x + 4}}$

23. $\displaystyle\int \frac{x\,dx}{x^4 - 9}$

24. $\displaystyle\int \frac{dx}{\sqrt{3x^2 + 2x + 1}}$

25. $\displaystyle\int \frac{dx}{11 + 3x - 2x^2}$

93. Hyperbolic and Inverse Hyperbolic Functions. The following formulas are readily verified by differentiation:

$$\text{(XXI)} \qquad \int \sinh u\,du = \cosh u + C$$

$$\text{(XXII)} \qquad \int \cosh u\,du = \sinh u + C$$

* For $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$
$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a^2}(b^2 - 4ac)\right].$

(XXIII) $$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

(XXIV) $$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

(XXV) $$\int \tanh u\, du = \int \frac{\sinh u}{\cosh u}\, du = \ln \cosh u + C$$

(XXVI) $$\int \coth u\, du = \ln |\sinh u| + C$$

(XXVII) $$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

(XXVIII) $$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

The integrals of Art. 92 can be expressed in terms of inverse hyperbolic functions in the following way (see Art. 54):

(XXIX) $$\int \frac{du}{\sqrt{u^2 + a^2}} = \operatorname{argsinh} \frac{u}{a} + C$$

(XXX) $$\int \frac{du}{\sqrt{u^2 - a^2}} = \operatorname{argcosh} \frac{u}{a} + C$$

(XXXI) $$\int \frac{du}{u^2 - a^2} = -\frac{1}{a} \operatorname{argcoth} \frac{u}{a} + C \qquad \text{for } u^2 > a^2$$

or $$\int \frac{du}{u^2 - a^2} = -\frac{1}{a} \operatorname{argtanh} \frac{u}{a} + C \qquad \text{for } u^2 < a^2$$

(XXXIa) $$\int \frac{du}{a^2 - u^2} = \frac{1}{a} \operatorname{argcoth} \frac{u}{a} + C \qquad \text{for } u^2 > a^2$$

or $$\int \frac{du}{a^2 - u^2} = \frac{1}{a} \operatorname{argtanh} \frac{u}{a} + C \qquad \text{for } u^2 < a^2$$

EXERCISES

Find the following integrals, and check by differentiation:

1. $\int \sinh 2x\, dx$

2. $\int \cosh 4x\, dx$

3. $\int \operatorname{csch}^2 (3x + 1)\, dx$

4. $\int \tanh (x/3)\, dx$

5. $\int z \operatorname{sech}^2 z^2\, dz$

6. $\int x \sinh x^2\, dx$

7. $\int \frac{\operatorname{sech} \sqrt{x} \tanh \sqrt{x}}{\sqrt{x}}\, dx$

8. $\int \frac{dz}{5z^2 - 1}$

9. $\int \sinh (y/3)\, dy$

10. $\int \frac{\tanh 3x}{\cosh 3x}\, dx$

11. $\displaystyle\int \frac{\tanh \sqrt{x}}{\sqrt{x}}\,dx$

12. $\displaystyle\int \frac{dx}{x^2 + 2x}$

13. $\displaystyle\int \frac{dy}{\sqrt{16y^2 - 9}}$

14. $\displaystyle\int e^x \cosh e^x\,dx$

15. $\displaystyle\int z^3 \coth z^4\,dz$

16. $\displaystyle\int y \operatorname{csch}^2 (y^2 - 9)\,dy$

17. $\displaystyle\int \frac{x\,dx}{\sqrt{x^4 + a^4}}$

18. $\displaystyle\int x^3 \sinh x^4\,dx$

19. $\displaystyle\int \frac{x^2\,dx}{16 - x^6}$

20. $\displaystyle\int \frac{dz}{\sinh z \tanh z}$

21. $\displaystyle\int \cosh (y/2)\,dy$

22. $\displaystyle\int \frac{e^x\,dx}{a - e^{2x}}$

94. Integration by Parts. A type of integral that appears very frequently has for its integrand the product of one function of x by the differential of some other function of x. Thus, if u and v are functions of x, we wish to find a means of evaluating $\int u\,dv$. We recall that

$$d(uv) = u\,dv + v\,du$$

Therefore

$$uv = \int u\,dv + \int v\,du$$

and consequently

★(XXXII) $$\int u\,dv = uv - \int v\,du$$

We are therefore able to find the integral of $u\,dv$ provided that we can find the integral of $v\,du$. The method is best made clear by some examples.

Example 1. Find $\int x \sin 2x\,dx$. This is clearly neither the power form nor a standard trigonometric form. We can, however, regard it as $\int u\,dv$ where $u = x$ and $dv = \sin 2x\,dx$. We then have $du = dx$ and $v = -\frac{1}{2}\cos 2x$. Hence, by applying (XXXII),

$$\begin{aligned}\int x \sin 2x\,dx &= -\tfrac{1}{2}x \cos 2x + \tfrac{1}{2}\int \cos 2x\,dx \\ &= -\tfrac{1}{2}x \cos 2x + \tfrac{1}{4}\sin 2x + C\end{aligned}$$

Two important points should be noted:

1. In choosing "parts," that is, in deciding what is to be called u and what dv, it is essential to take for dv a differential whose integral can be found.

2. With this first condition in mind, we should, if possible, choose u so that $\int v\,du$ can be evaluated by some means or other. For instance, in Example 1, if we had chosen $dv = x\,dx$, we could find $v = \frac{1}{2}x^2$ easily enough, but then $u = \sin 2x$, and $du = 2 \cos 2x\,dx$, so that

$$\int v\,du = \int x^2 \cos 2x\,dx$$

which is more complicated than the original integral. When it is not possible to choose parts so that $\int v\,du$ is a standard form, this integral may itself yield to integration by parts or to some of the transformations which will be discussed in the next chapter.

Example 2. Find $\int x^2 \arcsin x\,dx = I$. Evidently, a simplification will be effected by setting

$$u = \arcsin x \qquad dv = x^2\,dx$$
$$du = \frac{dx}{\sqrt{1 - x^2}} \qquad v = \tfrac{1}{3}x^3$$

Then
$$I = \frac{1}{3}x^3 \arcsin x - \frac{1}{3}\int \frac{x^3\,dx}{\sqrt{1 - x^2}}$$

Clearly, $\int v\,du$ is not a standard form. But integrals of this type often yield to integration by parts. Consider $\int \frac{x^3\,dx}{\sqrt{1 - x^2}} = \int x^2 \cdot \frac{x\,dx}{\sqrt{1 - x^2}}$. If we set $dv = \frac{x\,dx}{\sqrt{1 - x^2}}$, we have at once $v = -\sqrt{1 - x^2}$. This requires $u = x^2$; therefore, $du = 2x\,dx$. Hence

$$\begin{aligned}\int \frac{x^3\,dx}{\sqrt{1 - x^2}} &= -x^2\sqrt{1 - x^2} + \int \sqrt{1 - x^2}\cdot 2x\,dx \\ &= -x^2\sqrt{1 - x^2} - \tfrac{2}{3}(1 - x^2)^{3/2} + C' \\ &= -\sqrt{1 - x^2}\,[x^2 + \tfrac{2}{3}(1 - x^2)] + C' \\ &= -\tfrac{1}{3}\sqrt{1 - x^2}\,(x^2 + 2) + C'\end{aligned}$$

Therefore, finally

$$\int x^2 \arcsin x\,dx = \tfrac{1}{3}x^3 \arcsin x + \tfrac{1}{9}(x^2 + 2)\sqrt{1 - x^2} + C$$

It may happen that $\int v\,du$ cannot be integrated directly. In such cases, it is sometimes possible to express this integral in such a way that the original integral can be found by solving an equation.

Example 3. Find $\int e^{2x} \sin x\,dx = I$. If we choose

$$u = e^{2x} \qquad dv = \sin x\,dx$$
$$du = 2e^{2x}\,dx \qquad v = -\cos x$$

we have
$$I = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \tag{2}$$

Evidently, $\int e^{2x}\cos x\,dx$ is of the same type as I, and no more readily integrated. However, if we apply integration by parts to this integral, choosing

$$u = e^{2x} \qquad dv = \cos x\,dx$$
$$du = 2e^{2x}\,dx \qquad v = \sin x$$

we obtain
$$\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx$$

Hence, from (2),
$$\begin{aligned} I &= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx \\ &= -e^{2x}\cos x + 2e^{2x}\sin x - 4I + C'\end{aligned}$$

It should be noted that the integral $\int e^{2x}\sin x\,dx$ that appears on the right-hand side may differ by an arbitrary constant from the integral written on the left-hand side;

hence, the introduction of the constant C'. Therefore

$$\begin{aligned} 5I &= -e^{2x}\cos x + 2e^{2x}\sin x + C' \\ &= e^{2x}(2\sin x - \cos x) + C' \\ I &= \tfrac{1}{5}e^{2x}(2\sin x - \cos x) + C \end{aligned}$$

Alternative Method. Another method, which is often convenient, is as follows: Choose parts as before, obtaining equation (2) as before:

$$I = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \tag{2}$$

Now, in the original integral, $\int e^{2x}\sin x\,dx$, let

$$\begin{aligned} u &= \sin x & dv &= e^{2x}\,dx \\ du &= \cos x\,dx & v &= \tfrac{1}{2}e^{2x} \end{aligned}$$

This different choice of parts leads to a second expression for I, namely,

$$I = \tfrac{1}{2}e^{2x}\sin x - \tfrac{1}{2}\int e^{2x}\cos x\,dx \tag{3}$$

Now, eliminate $\int e^{2x}\cos x\,dx$ between (2) and (3), remembering that this integral in (3) may differ by a constant from the integral in (2). It is sufficient to multiply (3) by 4 and add to (2), thus

$$\begin{aligned} 5I &= -e^{2x}\cos x + 2e^{2x}\sin x + C' \\ I &= \tfrac{1}{5}e^{2x}(2\sin x - \cos x) + C \end{aligned}$$

The student may have wondered why the arbitrary constant was not introduced during the finding of v from dv. That this is unnecessary is clear; for if we use $v + C$ instead of v only, we have

$$\begin{aligned} \int u\,dv &= u(v + C) - \int(v + C)\,du \\ &= uv + Cu - \int v\,du - C\int du \\ &= uv + Cu - \int v\,du - C(u + C') \\ &= uv - \int v\,du + C'' \end{aligned}$$

The constant C'' may be combined with the constant arising from integrating $\int v\,du$.

Example 4. Find $\int \ln x\,dx$.

Let $$u = \ln x \qquad dv = dx$$

Then $$du = \frac{dx}{x} \qquad v = x$$

and $$\begin{aligned} \int \ln x\,dx &= x\ln x - \int dx = x\ln x - x + C \\ &= x(\ln x - 1) + C \end{aligned}$$

Observe that the method of integration by parts enables us to reduce the problem of integrating the transcendental function $\ln x$ to the problem of integrating an algebraic function (in this case, simply 1). Integrals involving transcendental functions having algebraic functions for their derivatives, namely, the inverse trigonometric, inverse hyperbolic, and logarithmic functions, can often be very advantageously handled by integration by parts. Simply let the transcendental function be the part that is differentiated (that is, u).

EXERCISES

Find the following integrals:

1. $\int \theta \sin \theta \, d\theta$

2. $\int \theta \cos \theta \, d\theta$

3. $\int xe^x \, dx$

4. $\int x^2 e^{3x} \, dx$

5. $\int x \sec^2 x \, dx$

6. $\int x \csc^2 3x \, dx$

7. $\int xe^{x^2} \, dx$

8. $\int x^3 e^{-x^2} \, dx$

9. $\int y^2 \cos y \, dy$

10. $\int x \sqrt{x + 1} \, dx$

11. $\int x^3 \sqrt{x^2 + 1} \, dx$

12. $\int x^2 \sinh 2x \, dx$

13. $\int x^3 \cosh x \, dx$

14. $\int x \tanh x^2 \, dx$

15. $\int x \operatorname{sech}^2 x \, dx$

16. $\int \frac{x^3 \, dx}{(4 + x^2)^2}$

17. $\int \frac{x^3 \, dx}{(x^2 + 1)^{3/2}}$

18. $\int \arcsin x \, dx$

19. $\int \arctan x \, dx$

20. $\int x \operatorname{arcsec} x \, dx$

21. $\int x \ln x \, dx$

22. $\int \ln^2 x \, dx$

23. $\int \frac{\ln^2 x \, dx}{x}$

24. $\int \frac{x^5 \, dx}{(x^3 + 8)^2}$

25. $\int \frac{x^7 \, dx}{\sqrt{(x^4 - 16)}}$

26. $\int \frac{x^5 \, dx}{(1 - x^6)^{3/2}}$

27. $\int \operatorname{argsinh} x \, dx$

28. $\int \operatorname{argtanh} x \, dx$

29. $\int e^x \sin x \, dx$

30. $\int e^{3x} \sin 2x \, dx$

31. $\int e^{-\theta} \cos \theta \, d\theta$

32. $\int e^{-4\theta} \cos 3\theta \, d\theta$

33. $\int e^{ax} \sin bx \, dx \qquad b \neq 0$

34. $\int e^{ax} \cos bx \, dx \qquad a$ and b not both zero

35. $\int x^n \ln x \, dx \qquad n \neq -1$

36. $\int \sec^3 x \, dx$

37. $\int \csc^3 x \, dx$

Summary. The following are the most frequently employed *standard forms* (the numbers correspond to those in the text):

$$\left.\begin{array}{ll}\text{(I)} & \displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C \\ & \qquad\text{where } n \neq -1 \\ \text{(II)} & \displaystyle\int \frac{du}{u} = \ln|u| + C\end{array}\right\}\text{ power form}$$

(III) $$\int e^u\,du = e^u + C$$

(V) $$\int \sin u\,du = -\cos u + C$$

(VI) $$\int \cos u\,du = \sin u + C$$

(XI) $$\int \tan u\,du = \ln|\sec u| + C$$

(XII) $$\int \cot u\,du = \ln|\sin u| + C$$

(XIII) $$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

(XIV) $$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

(VII) $$\int \sec^2 u\,du = \tan u + C$$

(IX) $$\int \sec u \tan u\,du = \sec u + C$$

(XV) $$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$$

(XVI) $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

(XVII) $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

(XVIII) $$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left(u + \sqrt{u^2 + a^2}\right) + C$$

(XIX) $$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$$

(XX) $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

(XXXII) $$\int u\,dv = uv - \int v\,du \qquad \text{(integration by parts)}$$

MISCELLANEOUS EXERCISES

Find the following integrals and check by differentiation (Ex. 1 to 43):

1. $\int x\sqrt{x^2+16}\,dx$

2. $\int \frac{dx}{\sqrt{5-3x^2}}$

3. $\int \frac{dx}{\sqrt{3x^2-5}}$

4. $\int \frac{\ln^2 x\,dx}{x}$

5. $\int \frac{dx}{\sqrt{2x-x^2}}$

6. $\int \frac{dy}{y^2+8y+20}$

7. $\int \frac{y\,dy}{y^2+8y+12}$

8. $\int \frac{t^3\,dt}{(t^4-1)^{3/2}}$

9. $\int (x^2+4)^2\,dx$

10. $\int \frac{(2y+1)\,dy}{y^2-4y+5}$

11. $\int xe^{2x}\,dx$

12. $\int x\sin 3x\,dx$

13. $\int x^2(x^3-8)^4\,dx$

14. $\int \log(x+2)\,dx$

15. $\int 10^{2x}\,dx$

16. $\int \frac{\sin\sqrt{x}\,dx}{\sqrt{x}}$

17. $\int \frac{\sqrt{1+\ln x}}{x}\,dx$

18. $\int \sin^4\theta\cos\theta\,d\theta$

19. $\int e^x\cos e^x\,dx$

20. $\int \frac{(x-2)\,dx}{x^2+x+1}$

21. $\int y^2\sqrt{y+1}\,dy$

22. $\int y\sqrt{y^2+1}\,dy$

23. $\int \sec(2\theta+1)\,d\theta$

24. $\int \operatorname{sech}^2(4x-1)\,dx$

25. $\int \tan^2 3\theta\sec^2 3\theta\,d\theta$

26. $\int \frac{\ln x^2}{x}\,dx$

27. $\int \frac{(x^3+4x+3)}{x^2+1}\,dx$

28. $\int \frac{x\,dx}{\sqrt{x^2-8x+15}}$

29. $\int \frac{(x+1)\,dx}{\sqrt{25-x^2}}$

30. $\int \arcsin 3y\,dy$

31. $\int \operatorname{argtanh} 2y\,dy$

32. $\int \frac{e^x\,dx}{4e^x+1}$

33. $\int x\cosh x^2\,dx$

34. $\int e^x\sinh x\,dx$

35. $\int \frac{(2y+1)\,dy}{3y^2-4y+1}$

36. $\int x(10^x)\,dx$

37. $\int \sin 3\theta\sec 3\theta\,d\theta$

38. $\int \frac{\cot^2\theta\,d\theta}{\csc\theta-1}$

39. $\int \frac{(e^x-4)}{e^x+4}\,dx$

40. $\int x^2\arccos x\,dx$

41. $\int x^2 \arcsin 3x \, dx$

42. $\int \sec^5 \theta \, d\theta$

43. $\int \csc^5 \theta \, d\theta$

44. Verify that $\int \sec^{2n+1} \theta \, d\theta = \frac{1}{2n}\left[\sec^{2n-1} \theta \tan \theta + (2n - 1) \int \sec^{2n-1} \theta \, d\theta\right]$.

In Exercises 45 to 47, find the integral by first transforming the integrand to an immediately integrable form as suggested. Find such transformations for Exercises 48 to 50.

45. $\int \frac{x\,dx}{(x + 4)^5} = \int \frac{(x + 4) - 4}{(x + 4)^5}\,dx$

46. $\int \frac{e^{3x}\,dx}{4 - e^{2x}} = \int \frac{4e^x - e^x(4 - e^{2x})}{4 - e^{2x}}\,dx$

47. $\int \frac{x\,dx}{\sqrt{16 - 3x}} = \int \frac{16 - (16 - 3x)}{3\sqrt{16 - 3x}}\,dx$

48. $\int \frac{e^{2x}\,dx}{e^x + 1}$

49. $\int \frac{x^3\,dx}{(x^2 - 4)^3}$

50. $\int \frac{x\,dx}{\sqrt{5x + 1}}$

CHAPTER 13

METHODS OF INTEGRATION

95. Trigonometric Integrands. Various types of integrals whose integrands contain trigonometric functions can be reduced to standard forms by use of familiar trigonometric identities. We shall consider three types.

1. Integrals of the type $\int \sin^m x \cos^n x\,dx$ where either m or n is a positive *odd* integer. Suppose, for instance, that n is a positive odd integer greater than 1 (if $n = 1$, we may integrate at once). We may then write

$$\sin^m x \cos^n x = \sin^m x \cos^{n-1} x \cos x$$

Now, $n - 1$ is *even;* hence, $\cos^{n-1} x$ can be expressed as some positive integral power of $\cos^2 x$. Since

$$\cos^2 x = 1 - \sin^2 x$$

$\sin^m x \cos^{n-1} x$ becomes a sum of powers of $\sin x$. Hence, the integrand becomes a sum of powers of $\sin x$ each multiplied by $\cos x\,dx$, and the integral can be evaluated at once. If m is a positive odd integer, similar treatment reduces the integrand to a sum of powers of $\cos x$ each multiplied by $\sin x\,dx$, and the integral is readily evaluated. If both m and n are even integers, either both positive, or one positive and the other zero,* other identities must be used [see (3)]. Integration by parts can also be used effectively for integrals of these types (see Art. 102).

Example 1

$$\begin{aligned}\int \sin^2 x \cos^5 x\,dx &= \int \sin^2 x \cos^4 x \cos x\,dx\\ &= \int \sin^2 x(1 - \sin^2 x)^2 \cos x\,dx\\ &= \int \sin^2 x(1 - 2\sin^2 x + \sin^4 x) \cos x\,dx\\ &= \int(\sin^2 x - 2\sin^4 x + \sin^6 x)\cos x\,dx\\ &= \tfrac{1}{3}\sin^3 x - \tfrac{2}{5}\sin^5 x + \tfrac{1}{7}\sin^7 x + C\end{aligned}$$

Example 2

$$\begin{aligned}\int \sin^3 x \cos^{2/3} x\,dx &= \int \sin^2 x \cos^{2/3} x \sin x\,dx = \int(1 - \cos^2 x)\cos^{2/3} x \sin x\,dx\\ &= \int(\cos^{2/3} x - \cos^{8/3} x)\sin x\,dx\\ &= -\tfrac{3}{5}\cos^{5/3} x + \tfrac{3}{11}\cos^{11/3} x + C\end{aligned}$$

2. Integrals of the type $\int \tan^m x \sec^n x\,dx$ or $\int \cot^m x \csc^n x\,dx$ can be integrated readily by use of one of the identities

$$\sec^2 x = 1 + \tan^2 x \qquad \csc^2 x = 1 + \cot^2 x$$

* Zero is an *even* integer.

provided that n is any positive even integer, or m any positive integer and n zero, or m and n both positive odd integers.

Example 3

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan^2 x \tan x\,dx = \int(\sec^2 x - 1)\tan x\,dx\\ &= \int \tan x \sec^2 x\,dx - \int \tan x\,dx = \tfrac{1}{2}\tan^2 x + \ln|\cos x| + C\end{aligned}$$

Example 4

$$\begin{aligned}\int \cot^4 x\,dx &= \int \cot^2 x \cot^2 x\,dx = \int \cot^2 x(\csc^2 x - 1)\,dx\\ &= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx\\ &= -\tfrac{1}{3}\cot^3 x - \int(\csc^2 x - 1)\,dx = -\tfrac{1}{3}\cot^3 x + \cot x + x + C\end{aligned}$$

Example 5

$$\begin{aligned}\int \tan^3 x \sec^4 x\,dx &= \int \tan^3 x \sec^2 x \sec^2 x\,dx\\ &= \int \tan^3 x(1 + \tan^2 x)\sec^2 x\,dx\\ &= \int \tan^3 x \sec^2 x\,dx + \int \tan^5 x \sec^2 x\,dx\\ &= \tfrac{1}{4}\tan^4 x + \tfrac{1}{6}\tan^6 x + C\end{aligned}$$

Example 6

$$\begin{aligned}\int \tan^3 x \sec^5 x\,dx &= \int \tan^2 x \sec^4 x \tan x \sec x\,dx\\ &= \int(\sec^2 x - 1)\sec^4 x \tan x \sec x\,dx\\ &= \int \sec^6 x \tan x \sec x\,dx - \int \sec^4 x \tan x \sec x\,dx\\ &= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C\end{aligned}$$

If we have a positive even (or zero) power of $\tan x$ multiplied by a positive odd power of $\sec x$, this method will not reduce the integral to a standard form.

Example 7

$$\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx$$

which does not reduce to standard forms by using $\sec^2 x = 1 + \tan^2 x$. However, this can be integrated by parts* as follows:

Let $\qquad u = \sec x \qquad dv = \sec^2 x\,dx$

$\qquad du = \sec x \tan x\,dx \qquad v = \tan x$

Then

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx\\ &= \sec x \tan x - \int(\sec^2 x - 1)\sec x\,dx\\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx\end{aligned}$$

Therefore, solving for $\int \sec^3 x\,dx$, we have

$$\begin{aligned}2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx\\ \int \sec^3 x\,dx &= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

3. Integrals of the type $\int \sin^m x \cos^n x\,dx$ where both m and n are even integers, either both positive or one positive and one zero. Such integrals may be handled conveniently by use of the identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$
$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Example 8

$$\begin{aligned}\int \sin^4 x\,dx &= \int(\sin^2 x)^2\,dx\\ &= \tfrac{1}{4}\int(1 - \cos 2x)^2\,dx = \tfrac{1}{4}\int(1 - 2\cos 2x + \cos^2 2x)\,dx\end{aligned}$$

* See Exercise 36, page 235.

But since $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$, we have

$$\begin{aligned}\int \sin^4 x\,dx &= \tfrac{1}{4}\int(1 - 2\cos 2x + \tfrac{1}{2} + \tfrac{1}{2}\cos 4x)\,dx\\ &= \tfrac{1}{4}\int \tfrac{3}{2}\,dx - \tfrac{1}{2}\int \cos 2x\,dx + \tfrac{1}{8}\int \cos 4x\,dx\\ &= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C\end{aligned}$$

Example 9

$$\begin{aligned}\int \sin^2 x \cos^4 x\,dx &= \int(\sin x\cos x)^2\cos^2 x\,dx = \int \tfrac{1}{4}\sin^2 2x \cdot \tfrac{1}{2}(1 + \cos 2x)\,dx\\ &= \tfrac{1}{8}\int \sin^2 2x\,dx + \tfrac{1}{8}\int \sin^2 2x\cos 2x\,dx\\ &= \tfrac{1}{16}\int(1 - \cos 4x)\,dx + \tfrac{1}{8}\int \sin^2 2x\cos 2x\,dx\\ &= \tfrac{1}{16}x - \tfrac{1}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x + C\end{aligned}$$

EXERCISES

Find the following integrals:

1. $\displaystyle\int \sin^3 \theta\,d\theta$
2. $\displaystyle\int \sin^2 \theta \cos^3 \theta\,d\theta$
3. $\displaystyle\int \cos^2 2x \sin 2x\,dx$
4. $\displaystyle\int \cos^5 x\,dx$
5. $\displaystyle\int \sin^3 \theta \cos^{3/5} \theta\,d\theta$
6. $\displaystyle\int \frac{\cos^5 \theta\,d\theta}{\sin^{3/5} \theta}$
7. $\displaystyle\int \tan^3 3\theta\,d\theta$
8. $\displaystyle\int \cot^3 (\theta/2)\,d\theta$
9. $\displaystyle\int \tan^4 \theta\,d\theta$
10. $\displaystyle\int \tan x \sec^4 x\,dx$
11. $\displaystyle\int \cot^2 2x \csc^4 2x\,dx$
12. $\displaystyle\int \tan^5 \alpha\,d\alpha$
13. $\displaystyle\int \tan^7 \alpha \sec^4 \alpha\,d\alpha$
14. $\displaystyle\int \tan^3 4\theta \sec 4\theta\,d\theta$
15. $\displaystyle\int \cot^5 \theta \csc^5 \theta\,d\theta$
16. $\displaystyle\int \sin^2 u\,du$
17. $\displaystyle\int \cos^2 u\,du$
18. $\displaystyle\int \cos^4 x\,dx$
19. $\displaystyle\int \cos^7 3x\,dx$
20. $\displaystyle\int \sin^4 3x \cos^6 3x\,dx$
21. $\displaystyle\int \sin^3 x \cos^2 x\,dx$
22. $\displaystyle\int \cos^6 x\,dx$
23. $\displaystyle\int \sin^4 x \cos^4 x\,dx$
24. $\displaystyle\int \cos^{3/4} 2\theta \sin 2\theta\,d\theta$
25. $\displaystyle\int \sin^4 3\theta \cos 3\theta\,d\theta$
26. $\displaystyle\int \tan 5x \sec^7 5x\,dx$
27. $\displaystyle\int x \sin^2 x\,dx$
28. $\displaystyle\int x \cos^2 x \sin x\,dx$

29. $\int x \sin^2 x \, dx$

30. $\int x \tan^2 x \, dx$

31. $\int \cot^4 2x \, dx$

32. $\int x \sin 2x \cos 2x \, dx$

33. $\int e^x \sin^2 x \, dx$

34. $\int x \sin x^2 \cos x^2 \, dx$

35. $\int \sin mx \sin nx \, dx \qquad m^2 \neq n^2$

36. $\int \cos mx \cos nx \, dx \qquad m^2 \neq n^2$

37. $\int \sin mx \cos nx \, dx \qquad m^2 \neq n^2$

38. $\int \sin 2x \cos 4x \, dx$

39. $\int \sin 5x \sin 3x \, dx$

40. $\int \cos x \cos 7x \, dx$

96. Integration by Substitution. If $\int f(x)\, dx$ is not readily integrated by the methods so far discussed, it may be possible to transform the integrand to a form that can be so integrated by replacing x by some function of a new variable, say z. Of course, dx must also be expressed in terms of z and dz. Thus we write

$$x = \varphi(z) \qquad dx = \varphi'(z)\, dz$$

and

$$\int f(x)\, dx = \int f[\varphi(z)]\varphi'(z)\, dz = \int \psi(z)\, dz$$
$$= \Psi(z) + C = F(x) + C'$$

We express $\Psi(z)$ in terms of x by means of the equation $x = \varphi(z)$, and we have the desired result.

In our work so far, we have actually been making use of the method of integration by substitution, for we have determined by inspection what function of x should be called u. But many cases arise in which a suitable substitution is not so easily recognized, although it is still possible to reduce the integral to a standard form by a proper choice of a new variable. A successful choice depends largely upon the ingenuity and experience of the worker; though no rules can be given that cover all cases, it is worthwhile to suggest various substitutions that prove desirable in the situations illustrated. Frequently, several different substitutions can be used, any one of which will succeed. It is hardly necessary to emphasize the importance of first making sure that the given integral is not already in a standard form before trying a substitution. Careless students waste much time in making totally unnecessary, though perhaps not incorrect, substitutions.

97. Algebraic Substitutions. An important use of the method of substitution is in the rationalization of irrational integrands. The resulting integral may then be in a standard form or may be integrated by the method to be discussed in Art. 99. We consider four such cases and add some remarks about two other types of integrals.

1. *Integrand containing fractional powers of $a + bx$.* Substitution of a suitable power of z for $a + bx$ will rationalize the integrand.

Example 1. $\int \frac{x^2\,dx}{(2+3x)^{2/3}}$. If we let $z^3 = 2 + 3x$, then $z^2 = (2+3x)^{2/3}$, $3z^2\,dz = 3\,dx$, and $z^2\,dz = dx$. Furthermore $x = \frac{1}{3}(z^3 - 2)$, and thus $x^2 = \frac{1}{9}(z^3-2)^2$. Consequently

$$\int \frac{x^2\,dx}{(2+3x)^{2/3}} = \frac{1}{9}\int \frac{(z^3-2)^2z^2\,dz}{z^2} = \frac{1}{9}\int (z^6 - 4z^3 + 4)\,dz$$
$$= \frac{1}{9}\left(\frac{1}{7}z^7 - z^4 + 4z\right) + C = \frac{z}{63}(z^6 - 7z^3 + 28) + C$$

We must replace z by its value in terms of x, thus

$$z = (2+3x)^{1/3}$$

Therefore

$$\int \frac{x^2\,dx}{(2+3x)^{2/3}} = \frac{(2+3x)^{1/3}}{63}[(2+3x)^2 - 7(2+3x) + 28] + C$$
$$= \frac{(2+3x)^{1/3}}{63}(9x^2 - 9x + 18) + C$$
$$= \tfrac{1}{7}(2+3x)^{1/3}(x^2 - x + 2) + C$$

Example 2. $\int \frac{dx}{4+\sqrt{4+5x}}$. Let $z^2 = 4 + 5x$, $2z\,dz = 5\,dx$, $\frac{2}{5}z\,dz = dx$.

Therefore

$$\int \frac{dx}{4+\sqrt{4+5x}} = \frac{2}{5}\int \frac{z\,dz}{4+z} = \frac{2}{5}\int\left(1 - \frac{4}{4+z}\right)dz$$
$$= \tfrac{2}{5}[z - 4\ln|4+z|] + C$$
$$= \tfrac{2}{5}\sqrt{4+5x} - \tfrac{8}{5}\ln(4+\sqrt{4+5x}) + C$$

2. *Integrand containing fractional powers of $a + bx^n$.* Substitution of a suitable power of z for $a + bx^n$ *may* rationalize the integrand.

Example 3. $\int \frac{(x^2-a^2)^{3/2}}{x}\,dx$. Let $z^2 = x^2 - a^2$; then $z\,dz = x\,dx$. But we need, not $x\,dx$, but $\frac{dx}{x}$. To get this, divide both sides of $z\,dz = x\,dx$ by x^2, noting that $x^2 = z^2 + a^2$, thus: $\frac{z\,dz}{z^2+a^2} = \frac{dx}{x}$. Now

$$\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \int \frac{z^3 \cdot z\,dz}{z^2+a^2} = \int \frac{z^4\,dz}{z^2+a^2}$$
$$= \int\left(z^2 - a^2 + \frac{a^4}{z^2+a^2}\right)dz$$
$$= \frac{1}{3}z^3 - a^2z + a^3\arctan\frac{z}{a} + C$$
$$= \frac{1}{3}(x^2-a^2)^{3/2} - a^2\sqrt{x^2-a^2} + a^3\arctan\frac{\sqrt{x^2-a^2}}{a} + C$$

If desired, we may write (Fig. 108)

$$\arctan \frac{\sqrt{x^2 - a^2}}{a} = \arccos \frac{a}{x} = \frac{\pi}{2} - \arcsin \frac{a}{x}$$

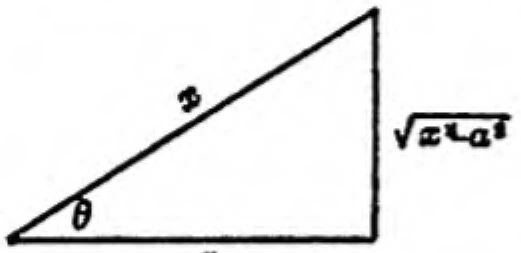


FIG. 108.

The $\pi/2$ may be included in the constant of integration, giving as a result

$$\frac{1}{3}(x^2 - a^2)^{3/2} - a^2 \sqrt{x^2 - a^2} - a^3 \arcsin \frac{a}{x} + C'$$

Example 4. $\int x^5 \sqrt{2 + x^3}\, dx$. Let $z^2 = 2 + x^3$; then, $2z\, dz = 3x^2\, dx$. But we want $x^5\, dx$; so multiply both sides of this last equation by $\frac{1}{3}x^3$, remembering that $x^3 = z^2 - 2$, thus: $\frac{2}{3}(z^2 - 2)z\, dz = x^5\, dx$. Therefore,

$$\begin{aligned}
\int x^5 \sqrt{2 + x^3}\, dx &= \tfrac{2}{3}\int z(z^2 - 2)z\, dz \\
&= \tfrac{2}{3}\int (z^4 - 2z^2)\, dz = \tfrac{2}{3}(\tfrac{1}{5}z^5 - \tfrac{2}{3}z^3) + C \\
&= \frac{2}{3}\left(\frac{3z^5 - 10z^3}{15}\right) + C = \frac{2}{45} z^3(3z^2 - 10) + C \\
&= \tfrac{2}{45}(2 + x^3)^{3/2}[3(2 + x^3) - 10] + C \\
&= \tfrac{2}{45}(2 + x^3)^{3/2}(3x^3 - 4) + C
\end{aligned}$$

3. *A general theorem for binomial differentials.* A differential of the form $x^m(a + bx^n)^{\frac{p}{q}}\, dx$ where $q > 0$, $n \neq 0$, m, and p are all integers is called a *binomial differential.* We have the following **theorem:** *If, in $\int x^m(a + bx^n)^{\frac{p}{q}}\, dx$, q is a positive integer and p, m, n are any integers ($n \neq 0$), then $x^m(a + bx^n)^{\frac{p}{q}}\, dx$ will be rationalized by the substitutions listed below:*

Case I. The substitution $z^q = a + bx^n$, provided that $\dfrac{m + 1}{n}$ is an integer (positive, negative, or zero).

Case II. The substitution $z^q x^n = a + bx^n$, provided $\dfrac{m + 1}{n} + \dfrac{p}{q}$ is an integer (positive, negative, or zero). The proof is left as an exercise for the reader.

If the integral in question has a binomial differential for its integrand, the suggestions of (1) and (2) are included in this theorem. The preceding examples would be included as follows:

Example 1. $m = 2$, $n = 1$, $p = -2$, $q = 3$; therefore, $\dfrac{m + 1}{n} = 3$, and we use $z^3 = 2 + 3x$.

Example 2. Not a binomial differential.

Example 3. $m = -1$, $n = 2$, $p = 3$, $q = 2$; therefore, $\dfrac{m + 1}{n} = 0$, and we use $z^2 = x^2 - a^2$.

Example 4. $m = 5$, $n = 3$, $p = 1$, $q = 2$; therefore, $\dfrac{m + 1}{n} = 2$, and we use $z^2 = 2 + x^3$.

Example 5. Find $\displaystyle\int \frac{dx}{x^2(16+x^4)^{3/4}} = \int x^{-2}(16+x^4)^{-3/4}\,dx$. Here, $m = -2$, $n = 4$, $p = -3$, $q = 4$; therefore

$$\frac{m+1}{n} + \frac{p}{q} = -\frac{1}{4} - \frac{3}{4} = -1$$

So we use $z^4x^4 = 16 + x^4$. This gives

$$x^4 = 16(z^4-1)^{-1}$$

Hence
$$16 + x^4 = z^4x^4 = 16z^4(z^4-1)^{-1}$$

Also $x^2 = 4(z^4-1)^{-1/2}$, and $x = 2(z^4-1)^{-1/4}$, so that $dx = -2(z^4-1)^{-5/4}z^3\,dz$. Our integral becomes

$$-\int \tfrac{1}{4}(z^4-1)^{1/2}\cdot\tfrac{1}{8}z^{-3}(z^4-1)^{3/4}\cdot 2(z^4-1)^{-5/4}\cdot z^3\,dz$$
$$= -\frac{1}{16}\int dz = -\frac{1}{16}z + C = -\frac{(16+x^4)^{1/4}}{16x} + C$$

4. *Integrand of the form* $\dfrac{1}{(hx+k)\sqrt{ax^2+bx+c}}\cdot$ Let $hx + k = 1/z$.

Example 6. To evaluate $\displaystyle\int \frac{dx}{(3x+2)\sqrt{7x^2+6x+1}} = I$, we let $3x + 2 = \dfrac{1}{z}\cdot$ Then, $dx = -\dfrac{dz}{3z^2}\cdot$ and $x = \dfrac{1}{3}\left(\dfrac{1}{z} - 2\right) = \dfrac{1-2z}{3z}\cdot$ This gives

$$7x^2 + 6x + 1 = \frac{z^2 - 10z + 7}{9z^2}$$

Therefore

$$I = \int \frac{-\dfrac{dz}{3z^2}}{\dfrac{1}{z}\cdot\dfrac{1}{3z}\sqrt{z^2-10z+7}} = -\int \frac{dz}{\sqrt{z^2-10z+25-18}}$$

$$= -\int \frac{dz}{\sqrt{(z-5)^2-18}} = -\ln|z - 5 + \sqrt{z^2-10z+7}| + C$$

Replacing z by its equal, $\dfrac{1}{3x+2}$, this reduces to

$$I = -\ln\left|\frac{(-3)(5x+3-\sqrt{7x^2+6x+1})}{3x+2}\right| + C$$
$$= \ln\left|\frac{3x+2}{5x+3-\sqrt{7x^2+6x+1}}\right| + C'$$

where $C' = C - \ln 3$. Let the reader check this result by differentiation.

5. *Integrand rational.* The substitutions suggested in (1), (2), and (3) are often useful when the integrand involves $a + bx$ or $a + bx^n$ rationally.

Example 7. To find $\int \frac{x^3\,dx}{(x^2-a^2)^2}$, let $z = x^2 - a^2$; then, $dz = 2x\,dx$, and

$$\tfrac{1}{2}(a^2 + z)\,dz = x^3\,dx$$

Hence

$$\int \frac{x^3\,dx}{(x^2-a^2)^2} = \frac{1}{2}\int \frac{(a^2+z)\,dz}{z^2} = \frac{a^2}{2}\int \frac{dz}{z^2} + \frac{1}{2}\int \frac{dz}{z}$$

$$= -\frac{a^2}{2z} + \frac{1}{2}\ln|z| + C$$

$$= -\frac{a^2}{2(x^2-a^2)} + \frac{1}{2}\ln|x^2-a^2| + C$$

This integral can also be found quickly by integration by parts, by letting $u = x^2$, $dv = \frac{x\,dx}{(x^2-a^2)^2}$. The reader should complete the work and compare the results.

Example 8. The integral $\int \frac{dx}{x^2(a^2+x^2)}$ can be evaluated by using the "reciprocal substitution," $x = a/z$. We have

$$dx = -\frac{a\,dz}{z^2}$$

and

$$\int \frac{dx}{x^2(a^2+x^2)} = -\int \frac{\frac{a\,dz}{z^2}}{\frac{a^2}{z^2}\left(a^2+\frac{a^2}{z^2}\right)} = -\frac{1}{a^3}\int \frac{z^2\,dz}{z^2+1}$$

$$= -\frac{1}{a^3}\int dz + \frac{1}{a^3}\int \frac{dz}{z^2+1}$$

$$= -\frac{1}{a^3}z + \frac{1}{a^3}\arctan z + C = -\frac{1}{a^2x} + \frac{1}{a^3}\arctan\frac{a}{x} + C$$

6. *Other types of integrands.* Sometimes a substitution will reduce an integrand involving transcendental functions to standard forms. As already remarked, no rule can be given to cover all cases, but it is always possible to try out a substitution that seems promising. If one does not avail, another may perhaps succeed. The following two examples illustrate the method:

Example 9. To find $\int \frac{dx}{\sqrt{e^{2x}+1}}$, we may try $z^2 = e^{2x} + 1$. This choice is prompted by the hope of eliminating the radical. We have $z\,dz = e^{2x}\,dx$. To find dx, divide both sides of this equation by $e^{2x} = z^2 - 1$, obtaining $\frac{z}{z^2-1}\,dz = dx$. We have

$$\int \frac{dx}{\sqrt{e^{2x}+1}} = \int \frac{z\,dz}{(z^2-1)z} = \int \frac{dz}{z^2-1} = \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + C$$

$$= \frac{1}{2}\ln\frac{\sqrt{e^{2x}+1}-1}{\sqrt{e^{2x}+1}+1} + C = \frac{1}{2}\ln\frac{(\sqrt{e^{2x}+1}-1)^2}{e^{2x}} + C$$

$$= \ln\frac{\sqrt{e^{2x}+1}-1}{e^x} + C = \ln(\sqrt{e^{2x}+1}-1) - x + C$$

Example 10. To find $\int \frac{\sin x \cos x\, dx}{5 + 4 \sin x}$, we may let $z = 5 + 4 \sin x$. Then $dz = 4 \cos x\, dx$ so that $\frac{1}{4}\, dz = \cos x\, dx$. We also have $\sin x = \frac{1}{4}(z - 5)$. Hence, our integral becomes

$$\begin{aligned}\frac{1}{16} \int \frac{(z - 5)\, dz}{z} = \frac{1}{16} \int \left(1 - \frac{5}{z}\right) dz &= \tfrac{1}{16} z - \tfrac{5}{16} \ln |z| + C \\ &= \tfrac{1}{16}(5 + 4 \sin x) - \tfrac{5}{16} \ln (5 + 4 \sin x) + C \\ &= \tfrac{1}{4} \sin x - \tfrac{5}{16} \ln (5 + 4 \sin x) + C'\end{aligned}$$

combining $\frac{5}{16}$ with the constant of integration.

EXERCISES

Find the following integrals:

1. $\int \frac{dx}{1 + \sqrt{x}}$

2. $\int \frac{x\, dx}{\sqrt{x + 3}}$

3. $\int \frac{x\, dx}{3 + \sqrt{x}}$

4. $\int \frac{x\, dx}{(1 + 2x)^{3/2}}$

5. $\int \frac{\sqrt{9 - 4x}}{x}\, dx$

6. $\int \frac{x^2\, dx}{(1 + x)^{1/3}}$

7. $\int x(1 + x)^{2/3}\, dx$

8. $\int x^2(a + x)^{1/4}\, dx$

9. $\int \frac{x\, dx}{(3 + 5x)^{3/4}}$

10. $\int x \sqrt{a^2 - x^2}\, dx$

11. $\int \frac{x}{\sqrt{a^2 + x^2}}\, dx$

12. $\int x^3 \sqrt{a^2 - x^2}\, dx$

13. $\int \frac{\sqrt{a^2 - x^2}}{x}\, dx$

14. $\int \frac{\sqrt{x^2 - a^2}}{x}\, dx$

15. $\int \frac{(x^2 - a^2)^{5/2}}{x}\, dx$

16. $\int x^5 \sqrt{1 + x^3}\, dx$

17. $\int x^8 \sqrt{1 + x^3}\, dx$

18. $\int \frac{x^4\, dx}{x^2 + 4}$

19. $\int \frac{x^3\, dx}{(x^2 + 4)^2}$

20. $\int \frac{x^3\, dx}{(x^2 + 4)^4}$

21. $\int \frac{x^5\, dx}{(x^2 - a^2)^3}$

22. $\int \frac{x^3\, dx}{(2 + 3x^2)^{3/2}}$

23. $\int \frac{dx}{x^4 \sqrt{1 + x^2}}$

24. $\int \frac{dx}{(2x - 1) \sqrt{16x^2 - 12x + 3}}$

25. $\int \frac{dx}{(3x + 1) \sqrt{5x^2 + 4x + 1}}$

26. $\int \frac{x^3\, dx}{(a^2 - x^2)^{3/2}}$

27. $\int \frac{dx}{(x^2 + 9)^{3/2}}$ (Let $x = 3/z$.)

28. $\int \frac{dx}{x^2(x^2 + 4)}$

29. $\displaystyle\int \frac{dx}{x^5(x^2+a^2)}$

30. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}$

31. $\displaystyle\int \frac{dx}{4+\sqrt{4+x}}$

32. $\displaystyle\int \sqrt{4+\sqrt{x}}\,dx$

33. $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{x}}}$

34. $\displaystyle\int (1+\sqrt{x})^{1/2}\,dx$

35. $\displaystyle\int \cos\sqrt{x}\,dx$

36. $\displaystyle\int \sec^2\sqrt{x}\,dx$

37. $\displaystyle\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx$

38. $\displaystyle\int \sqrt{e^x+9}\,dx$

39. $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{e^x-4}}$

40. $\displaystyle\int \frac{e^x\,dx}{\sqrt{4-e^{2x}}}$

41. $\displaystyle\int e^{2x}\sqrt{e^x+1}\,dx$

42. $\displaystyle\int \frac{\sin x\cos x\,dx}{2+3\sin x}$

43. $\displaystyle\int \frac{\sin x\cos^2 x\,dx}{1-2\cos x}$

44. $\displaystyle\int \frac{\csc^2\theta\cot^3\theta\,d\theta}{1-\cot^2\theta}$

45. $\displaystyle\int \frac{\sec^2\theta\tan^3\theta\,d\theta}{\sqrt{9+\tan^2\theta}}$

98. Trigonometric Substitutions. Many integrals can be reduced to standard forms by substituting a trigonometric function for the variable of integration. The three following are especially helpful since they serve to reduce a sum or difference of two squares to the square of a single function:

If the integrand contains $a^2 - x^2$, try $x = a\sin\theta$.
If the integrand contains $a^2 + x^2$, try $x = a\tan\theta$.
If the integrand contains $x^2 - a^2$, try $x = a\sec\theta$.

There are many other possibilities for effecting an integration by use of a trigonometric substitution, and some of these will be illustrated.

Example 1. In $\displaystyle\int \frac{(x^2-a^2)^{3/2}}{x}\,dx$, set $x = a\sec\theta$: then

$$dx = a\sec\theta\tan\theta\,d\theta$$

and the integral becomes

$$\int \frac{(a^2\sec^2\theta - a^2)^{3/2}}{a\sec\theta}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= \int (a^2\tan^2\theta)^{3/2}\tan\theta\,d\theta = a^3\int \tan^4\theta\,d\theta$$

$$= a^3\int \tan^2\theta(\sec^2\theta - 1)\,d\theta$$

$$= a^3\int \tan^2\theta\sec^2\theta\,d\theta - a^3\int \tan^2\theta\,d\theta$$

$$= a^3 \int \tan^2 \theta \sec^2 \theta \, d\theta - a^3 \int (\sec^2 \theta - 1) \, d\theta$$
$$= \frac{a^3}{3} \tan^3 \theta - a^3 \tan \theta + a^3 \theta + C$$

Since $\sec \theta = x/a$, we have $\theta = \text{arcsec } (x/a) = \arccos (a/x) = \pi/2 - \arcsin (a/x)$, as is easily seen by reference to the triangle, Fig. 108. Hence, referring to Fig. 108, we have

$$\tan \theta = \frac{\sqrt{x^2 - a^2}}{a}$$

and

$$\int \frac{(x^2 - a^2)^{3/2}}{x} dx = \frac{a^3}{3} \left(\frac{\sqrt{x^2 - a^2}}{a} \right)^3 - a^3 \cdot \frac{\sqrt{x^2 - a^2}}{a} + a^3 \left(\frac{\pi}{2} - \arcsin \frac{a}{x} \right) + C$$
$$= \frac{1}{3} (x^2 - a^2)^{3/2} - a^2 \sqrt{x^2 - a^2} - a^3 \arcsin \frac{a}{x} + C'$$

where we include $a^3(\pi/2)$ in the constant of integration (compare with Example 3, Art. 97).

Example 2. To find $\int \frac{dx}{x^2 \sqrt{25x^2 + 16}}$, we observe that the sum of squares $25x^2 + 16$ will be reduced to the square of a single function if we set $25x^2 = 16 \tan^2 \theta$, that is,

$$x = \tfrac{4}{5} \tan \theta \qquad dx = \tfrac{4}{5} \sec^2 \theta \, d\theta$$

Therefore
$$\int \frac{dx}{x^2 \sqrt{25x^2 + 16}} = \int \frac{\frac{4}{5} \sec^2 \theta \, d\theta}{\frac{16}{25} \tan^2 \theta \sqrt{16 \tan^2 \theta + 16}}$$
$$= \frac{5}{16} \int \frac{\sec^2 \theta \, d\theta}{\tan^2 \theta \sec \theta} = \frac{5}{16} \int \frac{\cos \theta \, d\theta}{\sin^2 \theta}$$
$$= \tfrac{5}{16} \int \cot \theta \csc \theta \, d\theta = -\tfrac{5}{16} \csc \theta + C$$

Since $\tan \theta = 5x/4$, it is easily seen from Fig. 109 that

$$\csc \theta = \frac{\sqrt{25x^2 + 16}}{5x}$$

We therefore have

$$\int \frac{dx}{x^2 \sqrt{25x^2 + 16}} = -\frac{\sqrt{25x^2 + 16}}{16x} + C$$

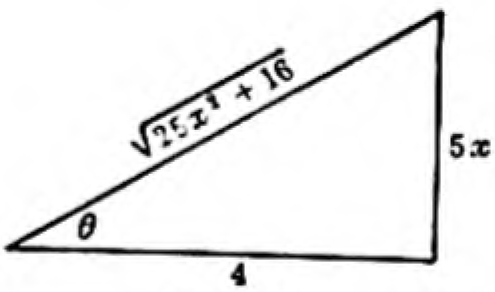


FIG. 109.

Example 3. Find $\int \frac{\sqrt{a - x}}{x^{3/2}} dx$. We may reduce $a - x$ to the square of a single function by setting

$$x = a \sin^2 \theta$$

and therefore
$$dx = 2a \sin \theta \cos \theta \, d\theta$$

We have

$$\int \frac{\sqrt{a-x}}{x^{3/2}}\,dx = \int \frac{\sqrt{a}\cos\theta \cdot 2a\sin\theta\cos\theta\,d\theta}{a^{3/2}\sin^3\theta}$$
$$= 2\int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = 2\int \cot^2\theta\,d\theta$$
$$= 2\int (\csc^2\theta - 1)\,d\theta = -2\cot\theta - 2\theta + C$$

Since $\sin^2\theta = x/a$, $\sin\theta = \sqrt{x/a}$; also (regarding θ as a first quadrant angle), $\csc^2\theta = \frac{a}{x}$, and therefore $\cot^2\theta = \frac{a}{x} - 1 = \frac{a-x}{x}$. Consequently

$$\int \frac{\sqrt{a-x}}{x^{3/2}}\,dx = -2\sqrt{\frac{a-x}{x}} - 2\arcsin\sqrt{\frac{x}{a}} + C$$

Example 4. Find

$$\int \frac{dx}{(x^2-4x+13)^2} = \int \frac{dx}{(x^2-4x+4+9)^2} = \int \frac{dx}{[(x-2)^2+9]^2}$$

Here we may set $x - 2 = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, obtaining for our integral

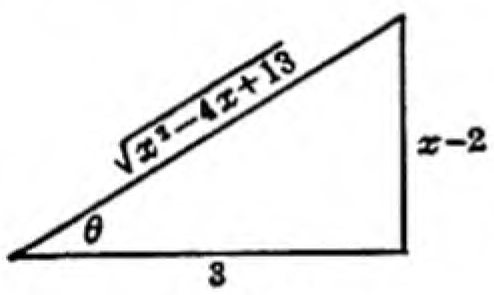


FIG. 110.

$$\int \frac{3\sec^2\theta\,d\theta}{(9\tan^2\theta+9)^2} = \frac{1}{27}\int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{54}\int (1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{54}(\theta + \tfrac{1}{2}\sin 2\theta) + C$$

Note that $\tan\theta = \frac{x-2}{3}$, so that $\sin\theta$ and $\cos\theta$ are easily found from Fig. 110. However, to express $\sin 2\theta$ in terms of x, we use the identity $\sin 2\theta = 2\sin\theta\cos\theta$. Hence

$$\int \frac{dx}{(x^2-4x+13)^2} = \frac{1}{54}(\sin\theta\cos\theta + \theta) + C$$
$$= \frac{x-2}{18(x^2-4x+13)} + \frac{1}{54}\arctan\frac{x-2}{3} + C$$

EXERCISES

Find the following integrals by use of a trigonometric substitution (Ex. 1 to 32):

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

2. $\displaystyle\int \frac{dx}{(25-x^2)^{3/2}}$

3. $\displaystyle\int \sqrt{a^2-x^2}\,dx$

4. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx$

5. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}}$

6. $\displaystyle\int \frac{dx}{(a^2+x^2)^2}$

7. $\int \sqrt{a^2 + x^2}\, dx$

8. $\int \frac{dx}{x\sqrt{a^2 + x^2}}$

9. $\int \frac{dx}{x(x^2 - a^2)}$

10. $\int \frac{\sqrt{x^2 - a^2}}{x}\, dx$

11. $\int \frac{(x^2 - 7)^{3/2}}{x}\, dx$

12. $\int \frac{dx}{x(9 + 4x^2)}$

13. $\int \frac{dx}{(9x^2 + 25)^{3/2}}$

14. $\int \frac{x^2\, dx}{(2 - x^2)^{5/2}}$

15. $\int \sqrt{5 - 3x^2}\, dx$

16. $\int \frac{dx}{(x^2 - 2x + 2)^2}$

17. $\int \frac{dx}{(x^2 + 6x + 34)^2}$

18. $\int \frac{dx}{(x^2 + 4x + 8)^3}$

19. $\int \frac{\sqrt{a - x}}{\sqrt{x}}\, dx$

20. $\int \frac{\sqrt{x}\, dx}{a + x}$

21. $\int \frac{dx}{x(x + a)}$

22. $\int \frac{dx}{x^2(x + a)}$

23. $\int \frac{\sqrt{x}\, dx}{(1 + 2x)^2}$

24. $\int \frac{\sqrt{x - a}}{x}\, dx$

25. $\int \frac{dx}{1 - e^x}$

26. $\int \frac{dx}{1 + e^{2x}}$

27. $\int \frac{dx}{\sqrt{e^x - 1}}$

28. $\int \frac{dx}{\sqrt{1 + e^{2x}}}$

29. $\int \frac{dx}{x(a^4 + x^4)}$

30. $\int \frac{dx}{x^3(a^4 + x^4)}$

31. $\int \frac{dx}{x\sqrt{4x + x^2}}$ (Let $x = 4\tan^2\theta$.)

32. $\int \frac{dx}{x\sqrt{4x - x^2}}$

Find the following integrals (Ex. 33 to 35) by use of a "hyperbolic substitution" as indicated (see Chap. 7).

33. $\int \frac{dx}{(a^2 + x^2)^{3/2}}$ (Let $x = a \sinh z$.)

34. $\int \frac{\sqrt{x^2 - a^2}}{x^2}\, dx$ (Let $x = a \cosh z$.)

35. $\int \frac{dx}{a^2 - x^2}$ (Let $x = a \tanh z$.)

99. Integration of Rational Fractions. We recall (Art. 6) that a rational fraction is a quotient of two polynomials, say

$$\frac{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n} = \frac{P_m(x)}{P_n(x)}$$

Keep in mind that we are dealing only with real functions, so that the polynomials are assumed to have real coefficients. Suppose that $P_m(x)$ and $P_n(x)$ have no common factor other than 1. The fraction is then said to be in its lowest terms. If the degree of the numerator is greater than or equal to the degree of the denominator, we may carry out the division so that $P_m(x)/P_n(x)$ is expressed as a polynomial plus a rational "proper" fraction in which the degree of the numerator is less than the degree of the denominator.

$$\frac{P_m(x)}{P_n(x)} = P(x) + \frac{R(x)}{P_n(x)}$$

We can integrate $P(x)$ at once, and so we need consider only the integration of proper rational fractions that are in lowest terms (see closing paragraph, Art. 88).

In order to effect the integration of such a rational fraction, we shall express it as a sum of simpler so-called *partial fractions*. To do this, we express the denominator $P_n(x)$ as a product of real factors of the forms $(x - \alpha)$ and $(ax^2 + bx + c)$ where $b^2 - 4ac < 0$. That this is always possible when we can solve the equation $P_n(x) = 0$ and that the partial fractions are uniquely determined is shown in algebra.* A factor of the type $x - \alpha$ is called a *linear factor* and, of course, corresponds to a real root of $P_n(x)$. If α is a root of multiplicity r, then $(x - \alpha)^r$ will appear in the factorization of $P_n(x)$, and we say that $(x - \alpha)$ is a *repeated linear factor*. A factor of the type $(ax^2 + bx + c)$ where $b^2 - 4ac < 0$ corresponds to conjugate complex roots of $P_n(x)$ and is called a *quadratic factor*. If $(ax^2 + bx + c)^s$ occurs, we say that $ax^2 + bx + c$ is a *repeated quadratic factor*. We proceed to indicate, chiefly by examples, methods by which partial fractions can be found.

1. *All the factors of the denominator linear and none repeated.* Corresponding to each factor $(x - \alpha)$, assume one partial fraction of the form $\dfrac{A}{x - \alpha}$ where A is some constant to be determined. If the degree of the denominator is n, there must, therefore, be assumed n partial fractions, and there will be n constants to be determined. Such a fraction is integrated at once, thus:

$$\int \frac{A}{x - \alpha}\, dx = A \ln |x - \alpha| + C$$

Example 1. Find $\displaystyle\int \frac{2x + 1}{x^3 - 7x + 6}\, dx$. First, find the factors of the denominator. To do this, we use the factor theorem; recall that any integral roots of $x^3 - 7x + 6$ must be divisors of 6. Trying $x = 1$, we get $1^3 - 7 \cdot 1 + 6 = 0$; hence, $x = 1$ is a

* See, for example, G. Chrystal, *Algebra*, Chap. 8, A. & C. Black, Ltd., London, 1904.

root, and $x - 1$ a factor of $x^3 - 7x + 6$. It is easy to see that

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$$

Therefore
$$\frac{2x + 1}{(x - 1)(x - 2)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x + 3}$$

where A, B, and C must be determined. Since the left- and right-hand members are to be equal for all values of x (except $x = 1$, $x = 2$, $x = -3$ for which the fractions are not defined) we must have

$$\frac{2x + 1}{(x - 1)(x - 2)(x + 3)} = \frac{A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)}{(x - 1)(x - 2)(x + 3)}$$

Hence
$$2x + 1 = A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)$$

for all values of x *including* $x = 1, 2, -3$ since the two sides of this identity are polynomials. Therefore, coefficients of like powers of x on the two sides of the identity must be equal. Thus

Coefficients of x^2	$0 = A + B + C$
Coefficients of x	$2 = A + 2B - 3C$
Coefficients of x^0	$1 = -6A - 3B + 2C$

Solving simultaneously for A, B, and C, we obtain

$$A = -\tfrac{3}{4} \qquad B = 1 \qquad C = -\tfrac{1}{4}$$

Therefore
$$\begin{aligned}\int \frac{2x + 1}{x^3 - 7x + 6}\,dx &= -\frac{3}{4}\int \frac{dx}{x - 1} + \int \frac{dx}{x - 2} - \frac{1}{4}\int \frac{dx}{x + 3}\\ &= -\frac{3}{4}\ln|x - 1| + \ln|x - 2| - \frac{1}{4}\ln|x + 3| + C'\\ &= \frac{1}{4}\ln\left|\frac{(x - 2)^4}{(x - 1)^3(x + 3)}\right| + C'\end{aligned}$$

Alternative Method. A convenient device for determining A, B, and C uses in a slightly different way the fact that

$$2x + 1 = A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)$$

for *all values of* x. If we set $x = 1$, we get

$$3 = -4A \qquad \text{and} \qquad A = -\tfrac{3}{4}$$

If $x = 2$, then

$$5 = 5B \qquad \text{and} \qquad B = 1$$

If $x = -3$, then

$$-5 = 20C \qquad \text{and} \qquad C = -\tfrac{1}{4}$$

2. *All the factors of the denominator linear but some repeated.* For each repeated factor $(x - \alpha)^r$, assume partial fractions $\dfrac{A}{(x - \alpha)} + \dfrac{B}{(x - \alpha)^2}$

$+ \cdots + \dfrac{R}{(x-\alpha)^r}$ where A, B, . . . , R are constants to be determined. In general, they are different from zero, although some, but *not* R, may be zero. With each nonrepeated linear factor $(x - \beta)$, assume a partial fraction $\dfrac{M}{x-\beta}$ as before. Note again that, if the degree of the denominator is n, we assume n partial fractions and we must determine n constants. The fractions such as $\dfrac{R}{(x-\alpha)^r}$ give integrands of the power form, $\displaystyle\int \frac{R\,dx}{(x-\alpha)^r} = -\frac{R}{(r-1)(x-\alpha)^{r-1}} + C.$

Example 2. Find $\displaystyle\int \frac{4x^3 - 2x^2 + x + 1}{(x-2)(x+1)^3}\,dx$. Assume partial fractions as follows:

$$\frac{4x^3 - 2x^2 + x + 1}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Note that the degree of the denominator is 4 and that four partial fractions are assumed. We must have

$$\begin{aligned} 4x^3 &- 2x^2 + x + 1 \\ &= A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2) \\ &= A(x^3 + 3x^2 + 3x + 1) + B(x^3 - 3x - 2) + C(x^2 - x - 2) + D(x-2) \qquad (1) \end{aligned}$$

Equating coefficients of like powers of x,

$$\begin{aligned} 4 &= A + B \\ -2 &= 3A + C \\ 1 &= 3A - 3B - C + D \\ 1 &= A - 2B - 2C - 2D \end{aligned}$$

Solving these equations, we find $A = 1$, $B = 3$, $C = -5$, $D = 2$. Our integral therefore reduces to the sum

$$\begin{aligned} \int \frac{dx}{x-2} &+ 3\int \frac{dx}{x+1} - 5\int \frac{dx}{(x+1)^2} + 2\int \frac{dx}{(x+1)^3} \\ &= \ln|x-2| + 3\ln|x+1| + \frac{5}{x+1} - \frac{1}{(x+1)^2} + C' \\ &= \ln|(x-2)(x+1)^3| + \frac{5x+4}{(x+1)^2} + C' \end{aligned}$$

Alternative method for finding A, B, C, D. Since (1) is true for all values of x, set $x = 2$. This gives $27 = 27A$, and $A = 1$. Set $x = -1$. This gives $-6 = -3D$, and $D = 2$. No choice of x will reduce all terms on the right except those containing B or C to zero. But any two convenient values of x may be used to secure two equations involving B and C. For instance, take $x = 0$. Then,

$$1 = A - 2B - 2C - 2D$$

and since A and D are already known, this reduces to

$$B + C = -2 \qquad (2)$$

Now let $x = 1$. This gives

$$4 = 8A - 4B - 2C - D$$

which reduces to
$$2B + C = 1 \tag{3}$$

Solving (2) and (3) simultaneously, we get $B = 3$, $C = -5$.

3. *Some of the factors of the denominator quadratic but not repeated.* For each factor $ax^2 + bx + c$, assume the partial fraction $\dfrac{A'x + B'}{ax^2 + bx + c}$. This is, in fact, the sum of two essentially different fractions, one with x in the numerator, the other with a constant numerator. As a matter of convenience, we shall assume the two partial fractions

$$\frac{A(2ax + b)}{ax^2 + bx + c} + \frac{B}{ax^2 + bx + c}$$

to correspond to each nonrepeated quadratic factor of the denominator of the original fraction. The first of these partial fractions integrates into a logarithm and the second into an arctangent. Note again that, if the degree of the denominator of the original fraction is n, we assume n partial fractions and must determine n constants.

Example 3. Find $\displaystyle\int \frac{7x^3 + 20x^2 + 35x - 13}{x^2(x^2 + 4x + 13)}\,dx$. Assume partial fractions as follows:

$$\frac{7x^3 + 20x^2 + 35x - 13}{x^2(x^2 + 4x + 13)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C(2x + 4)}{x^2 + 4x + 13} + \frac{D}{x^2 + 4x + 13}$$

Therefore, we must have

$$\begin{aligned} 7x^3 + 20x^2 + 35x - 13 \\ &= Ax(x^2 + 4x + 13) + B(x^2 + 4x + 13) + C(2x + 4)x^2 + Dx^2 \end{aligned}$$

Equating coefficients of like powers of x,

$$\begin{aligned} 7 &= A + 2C \\ 20 &= 4A + B + 4C + D \\ 35 &= 13A + 4B \\ 13 &= 13B \end{aligned}$$

Solving these equations, we find $A = 3$, $B = -1$, $C = 2$, $D = 1$. Therefore, our integral reduces to

$$3\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{(2x + 4)\,dx}{x^2 + 4x + 13} + \int \frac{dx}{x^2 + 4x + 13}$$

The last of these integrals is

$$\int \frac{dx}{x^2 + 4x + 13} = \int \frac{dx}{(x + 2)^2 + 9} = \frac{1}{3}\arctan\frac{x + 2}{3} + C'$$

The final result is

$$3 \ln |x| + \frac{1}{x} + 2 \ln (x^2 + 4x + 13) + \frac{1}{3} \arctan \frac{x+2}{3} + C''$$
$$= \frac{1}{x} + \ln |x^3(x^2 + 4x + 13)^2| + \frac{1}{3} \arctan \frac{x+2}{3} + C''$$

4. *Some of the factors of the denominator quadratic and repeated.* For each repeated quadratic factor

$$(ax^2 + bx + c)^s$$

assume $2s$ partial fractions,

$$\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c} + \frac{C(2ax+b)}{(ax^2+bx+c)^2} + \frac{D}{(ax^2+bx+c)^2} + \cdots + \frac{R(2ax+b)}{(ax^2+bx+c)^s} + \frac{S}{(ax^2+bx+c)^s}$$

All the partial fractions whose numerators are multiples of $2ax + b$ give integrands of the power form. For $k = 1$ the fraction $H/(ax^2 + bx + c)^k$ integrates into an arctangent; for $k > 1$, it can be integrated by use of a trigonometric substitution, as will be indicated in Example 4. Again note that, if the degree of the denominator of the original fraction is n, we assume n partial fractions and we must determine n constants.

Example 4. Find $\int \frac{x^2 - x + 4}{(x-1)(x^2+2x+2)^2}\,dx$. Assume partial fractions

$$\frac{x^2 - x + 4}{(x-1)(x^2+2x+2)^2} = \frac{A}{x-1} + \frac{B(2x+2)}{x^2+2x+2} + \frac{C}{x^2+2x+2} + \frac{D(2x+2)}{(x^2+2x+2)^2} + \frac{E}{(x^2+2x+2)^2}$$

Therefore

$$x^2 - x + 4 = A(x^2+2x+2)^2 + B(2x+2)(x-1)(x^2+2x+2) + C(x-1)(x^2+2x+2) + D(2x+2)(x-1) + E(x-1)$$

Equating coefficients of like powers of x,

$$\begin{aligned}
A + 2B \qquad\qquad\qquad\qquad &= 0\\
4A + 4B + C \qquad\qquad\quad &= 0\\
8A + 2B + C + 2D \qquad &= 1\\
8A - 4B \qquad\qquad + E &= -1\\
4A - 4B - 2C - 2D - E &= 4
\end{aligned}$$

Solving these equations, we find

$$A = \tfrac{4}{25} \qquad B = -\tfrac{2}{25} \qquad C = -\tfrac{8}{25} \qquad D = \tfrac{1}{10} \qquad E = -\tfrac{13}{5}$$

Hence, our integral reduces to

$$\frac{4}{25}\int \frac{dx}{x-1} - \frac{2}{25}\int \frac{2x+2}{x^2+2x+2}\,dx - \frac{8}{25}\int \frac{dx}{x^2+2x+2} + \frac{1}{10}\int \frac{2x+2}{(x^2+2x+2)^2}\,dx - \frac{13}{5}\int \frac{dx}{(x^2+2x+2)^2}$$

Here the first two and the fourth integrals are of the power form, and the third integral gives an arctangent. The fifth integral remains for further consideration. We have

$$\int \frac{dx}{(x^2 + 2x + 2)^2} = \int \frac{dx}{(x^2 + 2x + 1 + 1)^2} = \int \frac{dx}{[(x+1)^2 + 1]^2}$$

Let $x + 1 = \tan\theta$; then $dx = \sec^2\theta\, d\theta$, and the integral becomes

$$\int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta + 1)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$
$$= \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta) + C' = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C'$$
$$= \frac{1}{2}\arctan(x+1) + \frac{1}{2}\cdot\frac{x+1}{x^2 + 2x + 2} + C'$$

(See Fig. 111.) The final result is

$$\frac{4}{25}\ln|x-1| - \frac{2}{25}\ln(x^2 + 2x + 2) - \frac{8}{25}\arctan(x+1)$$
$$- \frac{1}{10}\cdot\frac{1}{x^2 + 2x + 2} - \frac{13}{5}\left[\frac{1}{2}\arctan(x+1) + \frac{1}{2}\cdot\frac{x+1}{x^2+2x+2}\right] + C''$$
$$= \frac{1}{25}\ln\frac{(x-1)^4}{(x^2+2x+2)^2} - \frac{81}{50}\arctan(x+1) - \frac{13x + 14}{10(x^2 + 2x + 2)} + C''$$

Compare Example 4, Art. 98.

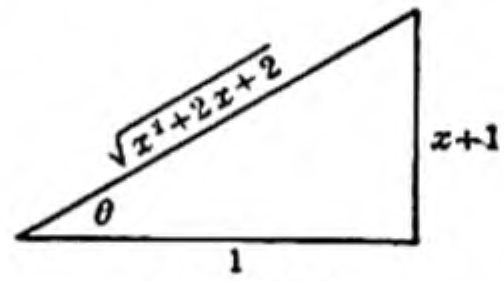


FIG. 111.

The above discussion serves to establish the following **general theorem**: *The integral of every rational function can be expressed in terms of algebraic, logarithmic, and inverse trigonometric functions.*

If the real linear and quadratic factors of the denominator can be determined, we can evaluate the integral. It should be remarked that if we were to consider complex as well as real numbers, we could express the denominator as a product of linear factors and resolve into partial fractions, some of which would have complex denominators.

EXERCISES

Find the following integrals (Ex. 1 to 48):

1. $\int \frac{dx}{x^2 + 3x}$

2. $\int \frac{dx}{x^2 + 2x - 3}$

3. $\int \frac{dx}{x^2 - 9}$ [Compare formula (XX) of Art. 92.]

4. $\int \frac{dx}{x(x^2 - 4)}$

5. $\int \frac{x\, dx}{x^2 - 6x - 7}$

6. $\int \frac{dx}{x^3 + x^2 - 4x - 4}$

7. $\int \frac{dx}{x^4 - 2x^3 - 5x^2 + 6x}$

8. $\int \frac{6x^2 + 10x + 2}{x^3 + 3x^2 + 2x}\, dx$

9. $\int \frac{x^2\, dx}{x^2 - 4x + 3}$

10. $\displaystyle\int \frac{(x^4 + x^3 - 8x^2 - 5x + 6)\,dx}{x^3 + x^2 - 10x + 8}$

11. $\displaystyle\int \frac{x\,dx}{x^4 + 3x^2 - 4}$

12. $\displaystyle\int \frac{x^2\,dx}{x^6 - 5x^3 + 6}$

13. $\displaystyle\int \frac{\cos\theta\,d\theta}{\sin^2\theta - 5\sin\theta}$

14. $\displaystyle\int \frac{\sec^2\theta\,d\theta}{2\tan\theta + \tan^2\theta}$

15. $\displaystyle\int \sec\theta\,d\theta \qquad \left(\textit{Note: } \sec\theta = \frac{1}{\cos\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1 - \sin^2\theta}\right)$

16. $\displaystyle\int \csc\theta\,d\theta$

17. $\displaystyle\int \frac{e^x\,dx}{e^{2x} + e^x - 2}$

18. $\displaystyle\int \frac{dx}{e^x - 1} \qquad \left(\textit{Note: } \frac{1}{e^x - 1} = \frac{e^x}{e^{2x} - e^x}\text{; now let } u = e^x\right)$

19. $\displaystyle\int \frac{dx}{x^3 + x^4}$

20. $\displaystyle\int \frac{dx}{x(x - 1)^2}$

21. $\displaystyle\int \frac{dx}{(x^2 - 4)^2}$

22. $\displaystyle\int \frac{x\,dx}{(x + 1)^2(x - 2)}$

23. $\displaystyle\int \frac{(x^4 + 3x^3 - 5x^2 - 4x + 17)}{x^3 + x^2 - 5x + 3}\,dx$

24. $\displaystyle\int \frac{(2x^5 + 3x^4 - x^3 + 6x^2 - 3x + 2)}{x^4 + 2x^3}\,dx$

25. $\displaystyle\int \frac{dx}{x^2(x^2 - 2x + 1)}$

26. $\displaystyle\int \frac{\cos\theta\,d\theta}{\sin^2\theta(1 - \sin\theta)}$

27. $\displaystyle\int \frac{\cos\theta\,d\theta}{(2 - \sin\theta)^2(1 - \sin\theta)}$

28. $\displaystyle\int \frac{(3x^2 - 22x + 19)\,dx}{(x + 2)(x - 3)^2}$

29. $\displaystyle\int \frac{(2x^4 - 2x + 1)\,dx}{x^4(2x - 1)}$

30. $\displaystyle\int \frac{e^x\,dx}{(e^x - 1)^2(e^{2x} - 1)}$

31. $\displaystyle\int \frac{dx}{e^x(e^x + 1)^2}$

32. $\displaystyle\int \frac{\tan\theta\,d\theta}{1 - \cos\theta}$

33. $\displaystyle\int \frac{\tan\theta\,d\theta}{1 - \sin\theta}$

34. $\displaystyle\int \frac{dx}{x(x^3 + 1)}$

35. $\displaystyle\int \frac{x\,dx}{(x - 1)(x^2 + 4)}$

36. $\displaystyle\int \frac{dx}{x^4 + 9x^2}$

37. $\displaystyle\int \frac{dx}{x^3 + 2x^2 + 2x}$

38. $\displaystyle\int \frac{dx}{x^3 - 5x^2 + 17x - 13}$

39. $\displaystyle\int \frac{(6x^3 + 2x^2 + 18x - 1)\,dx}{x^4 + 5x^2 + 4}$

40. $\displaystyle\int \frac{(x^4 + x^2 + 2x + 4)\,dx}{x^3 + 4x}$

41. $\displaystyle\int \frac{dx}{x^3 + 1}$

42. $\displaystyle\int \frac{x\,dx}{x^3 - 8}$

43. $\displaystyle\int \frac{dx}{x^4 + 4} \qquad [\textit{Hint: } x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - 4x^2]$

44. $\displaystyle\int \frac{dx}{x(x^2 + 1)^2}$

45. $\displaystyle\int \frac{(x^4 + x^3 + 16)\,dx}{x(x^2 + 4)^2}$

46. $\displaystyle\int \frac{(7 + 9x + 9x^2 + x^3 - x^4)\,dx}{(x - 1)(x^2 + 2x + 2)^2}$

47. $\displaystyle\int \frac{dx}{x^2(x^2 + 1)^2}$

48. $\displaystyle\int \frac{(x^4 - 5x^3 - 30x^2 - 73x - 1)\,dx}{(x+2)(x^2+4x+13)^2}$

49. Let $P(x)$ be a polynomial of degree n with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, no two of which are equal. Let $1/P(x)$ be expressed as a sum of partial fractions:

$$\frac{1}{P(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_n}{x-\alpha_n}$$

Using Taylor's theorem (Art. 82), we have

$$\begin{aligned} P(x) &= P(\alpha_i) + (x-\alpha_i)P'(\alpha_i) + \frac{(x-\alpha_i)^2}{2!}P''(\alpha_i) + \cdots + \frac{(x-\alpha_i)^n}{n!}P^{(n)}(\alpha_i) \\ &= (x-\alpha_i)P'(\alpha_i) + (x-\alpha_i)^2Q_i(x) \qquad \text{where } P'(\alpha_i) \neq 0 \end{aligned}$$

for $i = 1, 2, \ldots, n$. Now show that $A_i = \dfrac{1}{P'(\alpha_i)}$.

50. Let $\dfrac{F(x)}{P(x)}$ be a rational function in lowest terms where the degree of $F(x)$ is less than that of $P(x)$, and where $P(x)$ has the properties described in Exercise 49. If the expansion in partial fractions is

$$\frac{F(x)}{P(x)} = \frac{B_1}{x-\alpha_1} + \frac{B_2}{x-\alpha_2} + \cdots + \frac{B_n}{x-\alpha_n}$$

show that $B_i = \dfrac{F(\alpha_i)}{P'(\alpha_i)}$ for $i = 1, 2, \ldots, n$.

100. The Integrals of Rational Functions of x and $\sqrt{p + qx + x^2}$ and of x and $\sqrt{p + qx - x^2}$. It is clear from the theorem of Art. 99 why we wish to rationalize a given integrand. In Art. 97 we sought to find substitutions that would rationalize certain types of integrals. Though it is not advisable to try to discuss in great detail the conditions that will ensure the expressibility of an integral in terms of elementary functions, there are three types of integrals that deserve attention. This and the following section will be devoted to these integrals.

1. Suppose the integrand contains no radical other than $\sqrt{p + qx + x^2}$; that is, suppose it is *rational* in x and this square root. Let the reader show that the integrand reduces to an expression rational in z by use of the substitution

$$(z - x)^2 = p + qx + x^2$$

Example 1. Find $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x+2}}$. We let

$$(z - x)^2 = x^2 + 2x + 2$$

By simple calculations, this gives

$$x = \frac{1}{2}\cdot\frac{z^2 - 2}{z + 1} \qquad dx = \frac{1}{2}\cdot\frac{z^2 + 2z + 2}{(z+1)^2}\,dz$$

$$\sqrt{x^2 + 2x + 2} = \frac{1}{2}\cdot\frac{z^2 + 2z + 2}{z + 1}$$

Substituting into the integrand, we get

$$\int \frac{\frac{1}{2}\cdot\frac{z^2+2z+2}{(z+1)^2}\,dz}{\frac{1}{2}\cdot\frac{z^2-2}{z+1}\cdot\frac{1}{2}\cdot\frac{z^2+2z+2}{z+1}} = 2\int\frac{dz}{z^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{x-\sqrt{2}+\sqrt{x^2+2x+2}}{x+\sqrt{2}+\sqrt{x^2+2x+2}}\right| + C$$

2. Suppose the integrand contains no radical other than $\sqrt{p+qx-x^2}$. We shall confine our discussion to the case where the factors of $p + qx - x^2$ are real.*

Let $x - \alpha$ and $\beta - x$ be the real factors of $p + qx - x^2$. The reader may show that the substitution

$$(x-\alpha)^2z^2 = p + qx - x^2$$

will rationalize the integrand. The substitution

$$(\beta - x)^2z^2 = p + qx - x^2$$

would serve equally well in the reduction of this integrand to a rational form.

Example 2. Find $\displaystyle\int\frac{dx}{x\sqrt{2-x-x^2}} = \int\frac{dx}{x\sqrt{(x+2)(1-x)}}$. We let

$$(x+2)^2z^2 = 2 - x - x^2$$

By simple calculations, this gives

$$x = \frac{1-2z^2}{z^2+1} \qquad dx = \frac{-6z\,dz}{(z^2+1)^2} \qquad \sqrt{2-x-x^2} = \frac{3z}{z^2+1}$$

Substituting into the original integral, we have

$$\int\frac{dx}{x\sqrt{2-x-x^2}} = -2\int\frac{dz}{1-2z^2} = \int\frac{dz}{z^2-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{z-\frac{1}{\sqrt{2}}}{z+\frac{1}{\sqrt{2}}}\right| + C = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\,z-1}{\sqrt{2}\,z+1}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\sqrt{\frac{1-x}{2+x}}-1}{\sqrt{2}\sqrt{\frac{1-x}{2+x}}+1}\right| + C = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2-2x}-\sqrt{2+x}}{\sqrt{2-2x}+\sqrt{2+x}}\right| + C$$

* For if the factors are complex, the expression must be

$$-(x^2-qx-p) = -(x-h-ki)(x-h+ki) = -[(x-h)^2+k^2]$$

which is negative for all values of x, and therefore $\sqrt{p+qx-x^2}$ is imaginary for all x.

101. Integrand a Rational Function of Sin x and Cos x. If the integrand is a rational function of $\sin x$ and $\cos x$, the substitution $z = \tan(x/2)$ will transform it into an integrand that is rational in z. For, since $x/2 = \arctan z$, $dx = \dfrac{2\,dz}{1+z^2}$. Furthermore, $z^2 = \tan^2\dfrac{x}{2} = \dfrac{1-\cos x}{1+\cos x}$ from which we find $\cos x = \dfrac{1-z^2}{1+z^2}$. We also have

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan(x/2)}{\sec^2(x/2)} = \frac{2z}{1+z^2}$$

Therefore, the transformed integrand is a rational function of z, and the integral is expressible in terms of elementary functions. Since $\tan x$, $\cot x$, $\sec x$, $\csc x$ can be expressed rationally in terms of $\sin x$ and $\cos x$, the substitution $z = \tan(x/2)$ will transform a rational function of $\tan x$, $\cot x$, $\sec x$, $\csc x$ into a rational function of z, and the integral is expressible in terms of element. y functions.

Example. If we set $z = \tan(x/2)$, then

$$\int \frac{dx}{3+4\cos x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{3+4\left(\dfrac{1-z^2}{1+z^2}\right)} = 2\int \frac{dz}{7-z^2}$$

$$= \frac{1}{\sqrt{7}}\ln\left|\frac{z+\sqrt{7}}{z-\sqrt{7}}\right| + C = \frac{1}{\sqrt{7}}\ln\left|\frac{\tan\dfrac{x}{2}+\sqrt{7}}{\tan\dfrac{x}{2}-\sqrt{7}}\right| + C$$

EXERCISES

Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+4}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x-4}}$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}}$

5. $\displaystyle\int \frac{\sqrt{x^2+2x}}{x^2}\,dx$

6. $\displaystyle\int \frac{dx}{x\sqrt{2+x-x^2}}$

7. $\displaystyle\int \frac{dx}{(2x-x^2)^{3/2}}$

8. $\displaystyle\int \frac{dx}{x\sqrt{3x-2-x^2}}$

9. $\displaystyle\int \frac{dx}{x\sqrt{4-3x-x^2}}$

10. $\displaystyle\int \frac{\sin x\,dx}{1+\sin x}$

11. $\displaystyle\int \frac{dx}{4+5\cos x}$

12. $\displaystyle\int \frac{dx}{1+2\cos x}$

13. $\displaystyle\int \frac{dx}{3-5\sin x}$

14. $\displaystyle\int \sec x\,dx$

15. $\int \csc x\,dx$

16. $\int \frac{\cot x\,dx}{1 - \cos x}$

17. $\int \frac{dx}{1 + \sin x - \cos x}$

18. $\int \frac{dx}{4 + 2\cos 3x}$

19. $\int \frac{\sin 3x\,dx}{5 + 4\sin 3x}$

20. $\int \frac{\sin 6x\,dx}{1 - 2\cos 6x}$

102. Reduction Formulas. In addition to the methods of integration illustrated in this chapter, many integrals can be very conveniently handled by means of so-called *reduction formulas*. Such a formula expresses the given integral in terms of a simpler integral. We list eight such formulas and indicate their use.

1. *Trigonometric integrand.* Consider

$$I = \int \sin^m x \cos^n x\,dx$$

Integrate by parts, letting

$$u = \cos^{n-1} x \qquad dv = \sin^m x \cos x\,dx$$

$$du = -(n-1)\cos^{n-2} x \sin x\,dx \qquad v = \frac{\sin^{m+1} x}{m+1}$$

Then,

$$I = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\,dx$$

But

$$\begin{aligned}\int \sin^{m+2} x \cos^{n-2} x\,dx &= \int \sin^m x(1 - \cos^2 x)\cos^{n-2} x\,dx \\ &= \int \sin^m x \cos^{n-2} x\,dx - \int \sin^m x \cos^n x\,dx \\ &= \int \sin^m x \cos^{n-2} x\,dx - I\end{aligned}$$

This gives

$$I = \frac{1}{m+1}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx - \frac{n-1}{m+1}\cdot I$$

Solving for I, we get

$$\star(\mathrm{I}) \quad \int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx$$

Note that this formula expresses the given integral in terms of a similar integral in which the exponent of $\cos x$ is reduced by 2. The formula fails when $m + n = 0$. But, in this case, we have an integral of a power of $\tan x$ or $\cot x$ that may be integrated by other methods.

Let the student show that by using $u = \sin^{m-1} x$ and $dv = \cos^n x \sin x \, dx$ and integrating by parts the following formula results:

$$\star\text{(II)} \quad \int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$$

This formula expresses the given integral in terms of a similar integral in which the exponent of $\sin x$ is reduced by 2.

Now solve (I) for the integral on the right-hand side, obtaining

$$\int \sin^m x \cos^{n-2} x \, dx = -\frac{\sin^{m+1} x \cos^{n-1} x}{n-1} + \frac{m+n}{n-1} \int \sin^m x \cos^n x \, dx$$

If we now replace n by $n + 2$, we get

$$\star\text{(III)} \quad \int \sin^m x \cos^n x \, dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} \int \sin^m x \cos^{n+2} x \, dx$$

This formula expresses the given integral in terms of a similar integral in which the exponent of $\cos x$ is increased by 2. It is, therefore, useful if n is negative. It fails when $n + 1 = 0$. In a similar way, from formula (II), we get

$$\star\text{(IV)} \quad \int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x \, dx$$

This fails if $m + 1 = 0$.

Example 1. Using $m = 3$, $n = \frac{2}{3}$ in formula (II), we get

$$\begin{aligned}
\int \sin^3 x \cos^{2/3} x \, dx &= -\frac{\sin^2 x \cos^{5/3} x}{3 + \frac{2}{3}} + \frac{2}{3 + \frac{2}{3}} \int \sin x \cos^{2/3} x \, dx \\
&= -\tfrac{3}{11} \sin^2 x \cos^{5/3} x - \tfrac{6}{11} \cdot \tfrac{3}{5} \cos^{5/3} x + C \\
&= -\tfrac{3}{11}[(1 - \cos^2 x) \cos^{5/3} x + \tfrac{6}{5} \cos^{5/3} x] + C \\
&= -\tfrac{3}{5} \cos^{5/3} x + \tfrac{3}{11} \cos^{11/3} x + C
\end{aligned}$$

Compare with Example 2, Art. 95.

Example 2. If we use $m = 0$, $n = -3$, formula (III) gives

$$\begin{aligned}
\int \sec^3 x \, dx = \int \cos^{-3} x \, dx &= -\frac{\sin x \cos^{-2} x}{-3+1} + \frac{-3+2}{-3+1} \int (\cos x)^{-1} \, dx \\
&= \tfrac{1}{2} \tan x \sec x + \tfrac{1}{2} \int \sec x \, dx \\
&= \tfrac{1}{2} \tan x \sec x + \tfrac{1}{2} \ln |\sec x + \tan x| + C
\end{aligned}$$

Compare with Example 7, Art. 95.

Example 3. If we use $m = 2$, $n = 4$, formula (I) gives

$$\int \sin^2 x \cos^4 x \, dx = \tfrac{1}{6} \sin^3 x \cos^3 x + \tfrac{1}{2}\int \sin^2 x \cos^2 x \, dx \tag{4}$$

Now, if we use $m = 2$, $n = 2$, formula (I) gives, for the integral on the right-hand side,

$$\int \sin^2 x \cos^2 x \, dx = \tfrac{1}{4} \sin^3 x \cos x + \tfrac{1}{4}\int \sin^2 x \, dx \tag{5}$$

Next, with $m = 2$, $n = 0$, formula (II) gives

$$\begin{aligned} \int \sin^2 x \, dx &= -\tfrac{1}{2} \sin x \cos x + \tfrac{1}{2}\int dx \\ &= -\tfrac{1}{2} \sin x \cos x + \tfrac{1}{2}x + C \end{aligned}$$

Combining this with (4) and (5) gives

$$\int \sin^2 x \cos^4 x \, dx = \tfrac{1}{6} \sin^3 x \cos^3 x + \tfrac{1}{8} \sin^3 x \cos x - \tfrac{1}{16} \sin x \cos x + \tfrac{1}{16}x + C'$$

This may be reduced to the result of Example 9, Art. 95, as follows:

$$\begin{aligned} \tfrac{1}{6} \sin^3 x \cos^3 x &= \tfrac{1}{48}(2 \sin x \cos x)^3 = \tfrac{1}{48} \sin^3 2x \\ \tfrac{1}{8} \sin^3 x \cos x - \tfrac{1}{16} \sin x \cos x &= \tfrac{1}{16} \sin x \cos x(2 \sin^2 x - 1) \\ &= -\tfrac{1}{32} \sin 2x \cos 2x = -\tfrac{1}{64} \sin 4x \end{aligned}$$

Hence, the above integral can be written

$$\tfrac{1}{16}x + \tfrac{1}{48} \sin^3 2x - \tfrac{1}{64} \sin 4x + C'$$

Notice how conveniently integrals involving trigonometric integrands can be handled by use of these formulas.

2. *Binomial differentials.* We list four reduction formulas that are helpful in working with the integration of binomial differentials. They may be established by integrating by parts, although we shall omit the proofs. We consider the integral

$$\int x^m(a + bx^n)^r \, dx$$

where m is an integer, n a positive integer, and $r = p/q$, a rational number. This is the integral of (3), Art. 97; but we use r instead of p/q for ease in writing.

$$\star\text{(V)} \quad \int x^m(a + bx^n)^r \, dx = \frac{x^{m-n+1}(a + bx^n)^{r+1}}{(nr + m + 1)b} - \frac{(m - n + 1)a}{(nr + m + 1)b} \int x^{m-n}(a + bx^n)^r \, dx$$

This reduces m by n but fails if $nr + m + 1 = 0$. But then

$$r + \frac{m + 1}{n} = 0$$

That is, $\dfrac{m + 1}{n} + \dfrac{p}{p}$ is an integer, and we can use the method of Art. 97.

$$\star\text{(VI)} \quad \int x^m(a + bx^n)^r \, dx = \frac{x^{m+1}(a + bx^n)^r}{nr + m + 1} + \frac{anr}{nr + m + 1} \int x^m(a + bx^n)^{r-1} \, dx$$

This reduces r by 1 but fails when $nr + m + 1 = 0$. However, in this case, the method of Art. 97 applies.

$$\star\text{(VII)} \quad \int x^m(a + bx^n)^r\,dx = \frac{x^{m+1}(a + bx^n)^{r+1}}{(m + 1)a} - \frac{(nr + n + m + 1)b}{(m + 1)a} \int x^{m+n}(a + bx^n)^r\,dx$$

This increases m by n but fails if $m + 1 = 0$. However, since, in this case, $\dfrac{m + 1}{n} = 0$, an integer, we can use the method of Art. 97.

$$\star\text{(VIII)} \quad \int x^m(a + bx^n)^r\,dx = -\frac{x^{m+1}(a + bx^n)^{r+1}}{n(r + 1)a} + \frac{nr + n + m + 1}{n(r + 1)a} \int x^m(a + bx^n)^{r+1}\,dx$$

This increases r by 1 but fails if $r + 1 = 0$, that is, if $r = -1$. But then the integrand is rational.

EXERCISES

Use the reduction formulas of Art. 102 to evaluate the following integrals:

1. $\int \sin^3\theta \cos^3\theta\,d\theta$

2. $\int \sin^2\theta \cos^3\theta\,d\theta$

3. $\int \sin^3\theta \cos^2\theta\,d\theta$

4. $\int \sin^3\theta\,d\theta$

5. $\int \tan^2\theta \sin\theta\,d\theta$

6. $\int \cot^2\theta \cos\theta\,d\theta$

7. $\int \csc^3\theta\,d\theta$

8. $\int \sin^2 x\,dx$

9. $\int \sin^4 x \cos^2 x\,dx$

10. $\int \sec^4 x\,dx$

11. $\int \sin^4 x\,dx$

12. $\int \sin^{1/2}\theta \cos^3\theta\,d\theta$

13. $\int \sin^3\theta \cos^{1/3}\theta\,d\theta$

14. $\int \dfrac{x^3\,dx}{\sqrt{a^2 - x^2}}$ [Use formula (V).]

15. $\int \sqrt{a^2 + x^2}\,dx$ [Use formula (VI).]

16. $\int \dfrac{dx}{x^3\sqrt{x^2 - a^2}}$ [Use formula (VII).]

17. $\int \dfrac{dx}{(a^2 + x^2)^2}$ [Use formula (VIII).]

18. $\int x^2 \sqrt{a^2 - x^2}\, dx$ [Use formulas (V) and (VI).]

19. $\int \frac{x^5\, dx}{\sqrt{1 - x^2}}$

20. $\int \frac{\sqrt{a^2 + x^2}}{x^2}\, dx$

103. Tables of Integrals. The methods for evaluating integrals that have been studied in this and the preceding chapter will enable the student to handle most of the integrals that he will ordinarily encounter in practice. For other integrals, he should consult a table such as Peirce's *A Short Table of Integrals** in which the integrals have been either evaluated or expressed in terms of simpler integrals. But even in using such a table, which is, after all, merely an extended list of *standard forms*, he will find it necessary to apply many of the principles and methods of integration studied in this and the preceding chapter. He should also remember that there are integrals which cannot be expressed in terms of elementary functions. One such has already been mentioned, namely, $\int e^{-x^2}\, dx$ (Art. 85). Another is $\int \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$ $(0 < k < 1)$ which, in fact, defines a new kind of function; it is called an *elliptic integral* of the first kind. The integrals $\int \sqrt{\frac{1 - k^2x^2}{1 - x^2}}\, dx$ and

$$\int \frac{dx}{(x^2 - a)\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

are called elliptic integrals of the second and third kinds, respectively.

MISCELLANEOUS EXERCISES

Evaluate the following integrals:

1. $\int \cos^3 (x/2)\, dx$

2. $\int \cos^2 5x\, dx$

3. $\int \sin \sqrt{x}\, dx$

4. $\int \sec^4 3x\, dx$

5. $\int \frac{x^2\, dx}{\sqrt{1 + x}}$

6. $\int x(2 + x)^{1/3}\, dx$

7. $\int \tan^{3/2} x \sec^4 x\, dx$

8. $\int \frac{\sqrt{x^2 + a^2}}{x}\, dx$

9. $\int \frac{x^3\, dx}{\sqrt{x^2 - a^2}}$

10. $\int \cot^{3/2} x \csc^4 x\, dx$

11. $\int x^3(1 + 2x^2)^{3/2}\, dx$

12. $\int \frac{dx}{x^2(1 + x^3)^{5/3}}$

* B. O. Peirce, *A Short Table of Integrals*, Ginn & Company, Boston, 1929.

13. $\displaystyle\int \left(\sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\right) dx$

14. $\displaystyle\int \sqrt{\frac{a-x}{a+x}}\, dx$

15. $\displaystyle\int \sin^2 x \cos^6 x\, dx$

16. $\displaystyle\int \sin^5 x \cos^4 x\, dx$

17. $\displaystyle\int x \tan^4 x\, dx$

18. $\displaystyle\int x \cos^2 x\, dx$

19. $\displaystyle\int \frac{dx}{(1+\sqrt{x})^{3/2}}$

20. $\displaystyle\int \sqrt{e^x - 25}\, dx$

21. $\displaystyle\int \frac{e^{4x}\, dx}{\sqrt{1+e^{2x}}}$

22. $\displaystyle\int \frac{e^x\, dx}{\sqrt{1-e^{2x}}}$

23. $\displaystyle\int \sqrt{1+2\sin\theta}\cos\theta\sin\theta\, d\theta$

24. $\displaystyle\int \frac{\sec^4\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta+1}}$

25. $\displaystyle\int \frac{du}{\sqrt{u^2+a^2}}$ [Formula (XVIII) of Art. 92]

26. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}}$ [Formula (XIX) of Art. 92]

27. $\displaystyle\int x^3\sqrt{16-x^2}\, dx$

28. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}$

29. $\displaystyle\int \frac{x^2\, dx}{(x^2+a^2)^2}$

30. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{5-x}}\, dx$

31. $\displaystyle\int \frac{\sqrt{x}\, dx}{(a+bx)}$

32. $\displaystyle\int \frac{dx}{x\sqrt{x^2-2ax}}$ (Let $x = 2a\sec^2\theta$.)

33. $\displaystyle\int \frac{(2x+1)\, dx}{x^3-5x^2+6x}$

34. $\displaystyle\int \frac{(x^2-2x+2)\, dx}{x^3+2x^2-5x-6}$

35. $\displaystyle\int \frac{dx}{2x^2-3x-2}$

36. $\displaystyle\int \frac{(3x^2-13x-59)}{(x+2)^2(x-5)}\, dx$

37. $\displaystyle\int \frac{dx}{x(x^2-4x+4)}$

38. $\displaystyle\int \frac{x^3+4x-4}{x^3(x-1)}\, dx$

39. $\displaystyle\int \frac{x^2\, dx}{(x-4)(x+3)^2}$

40. $\displaystyle\int \frac{du}{u^2-a^2}$ [Formula (XX) of Art. 92]

41. $\displaystyle\int \frac{du}{u(u^2+a^2)}$

42. $\displaystyle\int \frac{dx}{x^5+4x^3}$

43. $\displaystyle\int \frac{dx}{x^3+a^3}$

44. $\displaystyle\int \frac{dx}{x^3+3x^2+3x+2}$

45. $\displaystyle\int \frac{(2x^3+5x^2+26x+15)\, dx}{x^4+2x^3+4x^2-2x-5}$

46. $\displaystyle\int \frac{(2x^3-3x)\, dx}{x^4-3x^2+1}$

47. $\displaystyle\int \frac{(2x+1)\,dx}{\sqrt{x^2+x+1}}$

48. $\displaystyle\int \frac{dx}{x\sqrt{x^2+3x+25}}$

49. $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x-1}}$

50. $\displaystyle\int \frac{dx}{(6x-x^2)^{3/2}}$

51. $\displaystyle\int \frac{\sqrt{x^2+4x}}{x^2}\,dx$

52. $\displaystyle\int \frac{dx}{\sqrt{2+x-x^2}}$

53. $\displaystyle\int \frac{dx}{4+\cos x}$

54. $\displaystyle\int \frac{dx}{2+\sin x}$

55. $\displaystyle\int \frac{dx}{1+\cos x}$

56. $\displaystyle\int \frac{(1+\cos x)\,dx}{\cos x(1+\sin x)}$

CHAPTER 14

THE DEFINITE INTEGRAL

104. Area under a Curve. Suppose we have the graph of the function $y = x^2 = f(x)$ (Fig. 112) and ask what is meant by the *area* bounded by this curve, the x axis, the ordinate at $x = 1$, and the ordinate at $x = 3$. We know what is meant by the area of a rectangle; we measure this area by the product of the base by the altitude. A definite meaning can be attached to the measures of areas of figures bounded by *straight lines* such as triangles, parallelograms, and other polygons. But as yet we have no definition of the area of a figure one (or more) of whose boundaries is a curve. We shall proceed to develop such a definition for the above figure.

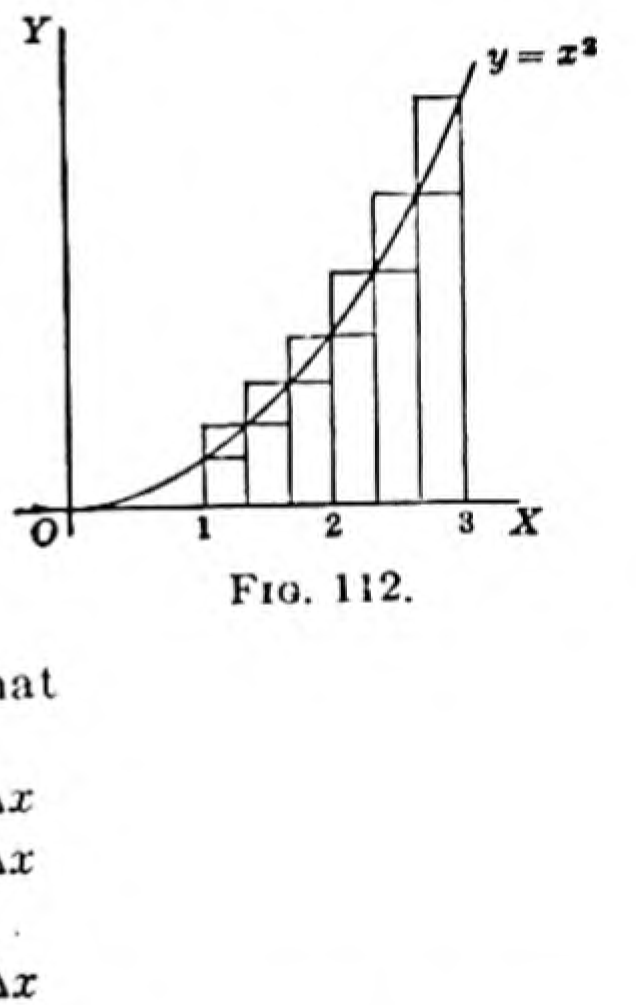


FIG. 112.

Let the interval (Fig. 112) from $x = 1$ to $x = 3$ be divided into 10 equal subintervals, each of length 0.2, by points with abscissas $x_1, x_2, \ldots, x_9$. For simplicity, we shall speak of the points $x_1, x_2, \ldots, x_9$. For convenience, designate 1 by x_0 and 3 by x_{10}. Denote the length of each subinterval by Δx, so that

$$\begin{aligned} x_1 - x_0 = x_1 - 1 &= \Delta x \\ x_2 - x_1 &= \Delta x \\ &\cdots\cdots \\ x_{10} - x_9 = 3 - x_9 &= \Delta x \end{aligned}$$

Let ordinates be erected at each point of subdivision, and draw horizontal line segments forming two sets of rectangles, as indicated in the figure. In one set of rectangles so formed, the left-hand side of each is an ordinate of the curve, and we observe that the entire rectangle is *below* the curve. The area of the first rectangle is

$$f(x_0)\,\Delta x = f(1)\,\Delta x = 1^2(0.2)$$

The area of the second rectangle is

$$f(x_1)\,\Delta x = f(1.2)\,\Delta x = (1.2)^2(0.2)$$

and so on. The total area of this set of rectangles is

$$s(10) = f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_9)\,\Delta x$$
$$= \sum_{i=0}^{9} f(x_i)\,\Delta x^*$$

We shall call this a *lower sum.* The symbol $s(10)$ shall mean a sum of rectangles calculated for a division of the interval from 1 to 3 into 10 subintervals. Each of the rectangles is called an *element of area.*

In the other set of rectangles, the right-hand side of each is an ordinate of the curve, and we observe that a small portion of each rectangle extends above the curve. The area of the first rectangle is

$$f(x_1)\,\Delta x = f(1.2)\,\Delta x = (1.2)^2(0.2)$$

The area of the second rectangle is

$$f(x_2)\,\Delta x = f(1.4)\,\Delta x = (1.4)^2(0.2)$$

and so on. The total area of this set of rectangles is

$$S(10) = f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + \cdots + f(x_{10})\,\Delta x$$
$$= \sum_{i=1}^{10} f(x_i)\,\Delta x$$

We shall call this an *upper sum.* It is easy to calculate these two sums, as indicated in the table:

$y = x^2 = f(x)$

$s(10)$

x	$f(x)$	$f(x)\,\Delta x$
1.0	1.00	0.200
1.2	1.44	0.288
1.4	1.96	0.392
1.6	2.56	0.512
1.8	3.24	0.648
2.0	4.00	0.800
2.2	4.84	0.968
2.4	5.76	1.152
2.6	6.76	1.352
2.8	7.84	1.568
	$s(10) =$	7.880

$S(10)$

x	$f(x)$	$f(x)\,\Delta x$
1.2	1.44	0.288
1.4	1.96	0.392
1.6	2.56	0.512
1.8	3.24	0.648
2.0	4.00	0.800
2.2	4.84	0.968
2.4	5.76	1.152
2.6	6.76	1.352
2.8	7.84	1.568
3.0	9.00	1.800
	$S(10) =$	9.480

* The meaning of this notation is evident: First, let $i = 0$, then $i = 1$, then $i = 2$, etc., and finally stop with $i = 9$; then add the resulting expressions. The symbol $\sum_{i=0}^{9} f(x_i)\,\Delta x$ is read "The sum, i running from 0 to 9, of $f(x_i)\,\Delta x$."

Suppose we take 20 subdivisions, each of length 0.1, and calculate the lower and upper sums. We have

$$\begin{aligned} s(20) &= (1.0)^2(0.1) + (1.1)^2(0.1) + \cdots + (2.9)^2(0.1) \\ &= 8.270^* \\ S(20) &= (1.1)^2(0.1) + (1.2)^2(0.1) + \cdots + (3.0)^2(0.1) \\ &= 9.070 \end{aligned}$$

If we take 200 subdivisions, each of length 0.01, and calculate the lower and upper sums, we get

$$\begin{aligned} s(200) &= (1.00)^2(0.01) + (1.01)^2(0.01) + \cdots + (2.99)^2(0.01) \\ &= 8.6267 \\ S(200) &= (1.01)^2(0.01) + (1.02)^2(0.01) + \cdots + (3.00)^2(0.01) \\ &= 8.7067 \end{aligned}$$

For purposes of comparison, these are listed in order, together with $s(2000)$ and $S(2000)$ calculated for subdivisions of length 0.001:

$$\begin{aligned} s(10) &= 7.880 & S(10) &= 9.480 \\ s(20) &= 8.270 & S(20) &= 9.070 \\ s(200) &= 8.6267 & S(200) &= 8.7067 \\ s(2000) &= 8.662167 & S(2000) &= 8.670167 \end{aligned}$$

Note that every lower sum is less than any upper sum; this fact can be established in general. We shall define the *area* A bounded by this curve $y = x^2$, the x axis, the ordinate at $x = 1$, and the ordinate at $x = 3$ in the following way: It can be proved that, while $s(n) < S(n)$, these two sums approach the same limiting value as Δx is made to approach zero, while, at the same time, n is made larger and larger. *The area A is defined to be a quantity equal to this common limit.* Also, it can be proved that

$$s(n) < A < S(n)$$

for every n. These matters will now be discussed in more general form.

To this end, we consider a function $f(x)$ that is (1) continuous in the interval $a \leqq x \leqq b$, (2) increasing in the interval [that is, for $a \leqq x_1 < x_2 \leqq b$, $f(x_1) < f(x_2)$], and (3) positive or zero at $x = a$ (and hence posi-

* The formula

$$\sum_{\alpha=1}^{n} \alpha^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

can be used to make this particular calculation as follows:

$$\begin{aligned} s(20) &= (1.0^2 + 1.1^2 + 1.2^2 + \cdots + 2.9^2)(0.1) \\ &= (10^2 + 11^2 + 12^2 + \cdots + 29^2)(0.001) \\ &= \left(\sum_{\alpha=1}^{29} \alpha^2 - \sum_{\alpha=1}^{9} \alpha^2 \right)(0.001) \end{aligned}$$

tive throughout the remainder of the interval). Note that $f(x) = x^2$ has these properties in the interval $1 \leqq x \leqq 3$. We shall let $y = f(x)$ be the graph of this function (Fig. 113) and shall explain what is meant by the *area* bounded by the curve, the x axis, the ordinate at $x = a$, and the ordinate at $x = b$.

We first divide the interval from a to b into n subintervals by points whose abscissas are $x_1, x_2, \ldots, x_{n-1}$. Denote a by x_0 and b by x_n. Although, in the example already given, we made these intervals equal in length, there is no necessity for doing this. We denote the lengths of the subintervals as follows:

$$\begin{aligned} x_1 - x_0 = x_1 - a &= \Delta_1 x \\ x_2 - x_1 &= \Delta_2 x \\ x_3 - x_2 &= \Delta_3 x \\ &\cdots\cdots \\ x_n - x_{n-1} &= \Delta_n x \end{aligned}$$

Let ordinates be erected at each point of subdivision. Consider the *element of area* consisting of the rectangle $ACDE$ (Fig. 113) whose base is $\Delta_1 x$ and whose altitude is $f(a) = f(x_0)$, the ordinate at the left-hand end point of the subinterval. The area of this rectangle is $f(x_0)\ \Delta_1 x$. Similarly, the area of the rectangle whose base is $\Delta_2 x$ and whose altitude is $f(x_1)$ is $f(x_1)\ \Delta_2 x$. Continue until the rectangle with base $\Delta_n x$ and altitude $f(x_{n-1})$ is reached; its area is $f(x_{n-1})\ \Delta_n x$. Note that the tops of these rectangles all lie entirely below the curve since $f(x)$ is an increasing function in the interval $a \leqq x \leqq b$. Add these areas, obtaining a *lower sum* $s(n)$, thus:

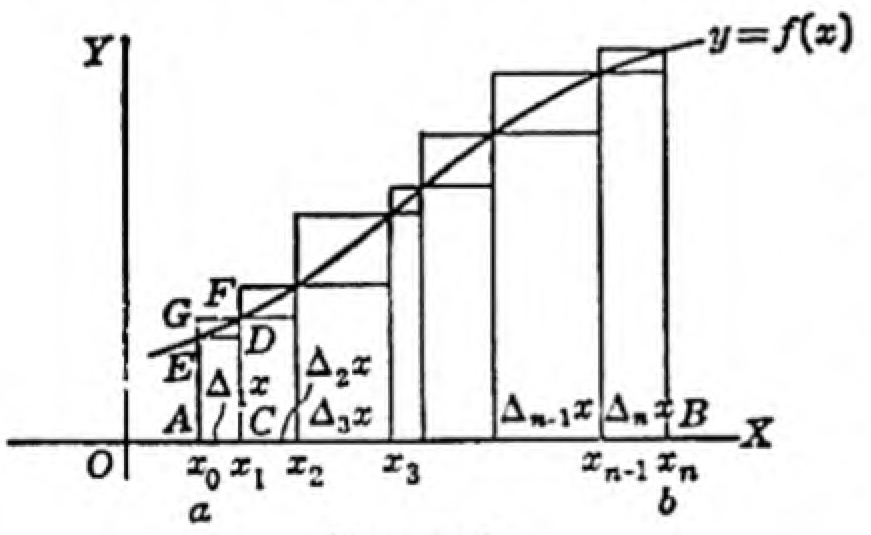


FIG. 113.

$$s(n) = f(x_0)\ \Delta_1 x + f(x_1)\ \Delta_2 x + \cdots + f(x_{n-1})\ \Delta_n x = \sum_{i=1}^{n} f(x_{i-1})\ \Delta_i x \tag{1}$$

In the same way, consider the rectangle $ACFG$ whose base is $\Delta_1 x$ and whose altitude is $f(x_1)$, the ordinate at the right-hand end of the subinterval. Its area is $f(x_1)\ \Delta_1 x$. Continue until the rectangle with base $\Delta_n x$, altitude $f(x_n)$, and area $f(x_n)\ \Delta_n x$ is reached. Note that the tops of these rectangles all lie *above* the curve. Add these areas, obtaining an *upper sum* $S(n)$, thus:

$$S(n) = f(x_1)\ \Delta_1 x + f(x_2)\ \Delta_2 x + \cdots + f(x_n)\ \Delta_n x = \sum_{i=1}^{n} f(x_i)\ \Delta_i x \tag{2}$$

It can be shown, although the proof will not be given here, that the lower sum for any subdivision is less than the upper sum for the same or any other subdivision. Let the number of subintervals, n, increase indefinitely, and at the same time so choose the points of subdivision that the length of the greatest subinterval approaches zero. It can be shown, although the proof will not be given here, that $s(n)$ and $S(n)$ approach a common limit. This limit is called *the area A bounded by the curve, the x axis, the ordinate at $x = a$, and the ordinate at $x = b$.* Again, it is possible to show that $s(n) < A < S(n)$ for every n. Thus, we may write

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta_i x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta_i x$$

the limits being taken for the maximum $\Delta_i x$ approaching zero and therefore for n increasing indefinitely.

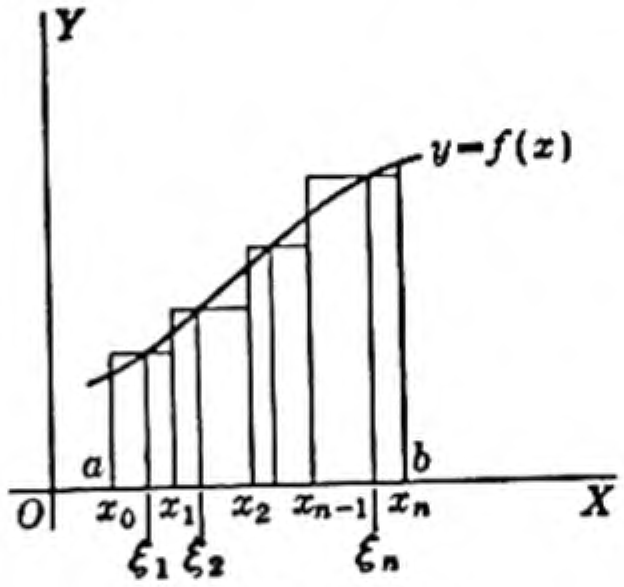


FIG. 114.

It should be remarked that we might choose points of subdivision of the interval from a to b and form a sum in the following way (Fig. 114):

Let ξ_1 be any point between $a = x_0$ and x_1; let ξ_2 be any point between x_1 and x_2; and so on. Form the sum

$$f(\xi_1)\,\Delta_1 x + f(\xi_2)\,\Delta_2 x + \cdots + f(\xi_n)\,\Delta_n x = \sum_{i=1}^{n} f(\xi_i)\,\Delta_i x \tag{3}$$

This is represented geometrically as the sum of the areas of the rectangles in Fig. 114. Evidently, this sum is between $s(n)$ and $S(n)$. In fact, it reduces to $s(n)$ if $\xi_1 = x_0$, $\xi_2 = x_1$, . . . , $\xi_n = x_{n-1}$ and to $S(n)$ if $\xi_1 = x_1$, $\xi_2 = x_2$, . . . , $\xi_n = x_n$. Hence, if the length of the greatest subinterval is made to approach zero and, therefore, n is made to increase indefinitely, we have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta_i x \tag{4}$$

We can remove the restriction that $f(x)$ be an increasing function. Suppose $f(x)$ to be *nondecreasing*, that is, for $x_1 < x_2, f(x_1) \leqq f(x_2)$. Here, the only modification needed in the above discussion is the remark that, if $f(x)$ is a constant throughout the interval $a \leqq x \leqq b$, then $s(n) = S(n)$.

Suppose $f(x)$ to be a *decreasing* function, that is, for $x_1 < x_2$, $f(x_1) > f(x_2)$. Here we note that the sum (1) gives a sum of rectangles whose tops lie above the curve $y = f(x)$. Hence, $s(n)$ is now an *upper* sum. Similarly, (2) gives a sum of rectangles whose tops are below the curve

$y = f(x)$. Hence, $S(n)$ is now a *lower* sum. We also have $s(n) > A > S(n)$. Otherwise, the discussion is the same as for an increasing function. If $f(x)$ is a *nonincreasing* function, that is, if for $x_1 < x_2$ we have $f(x_1) \geqq f(x_2)$, we need only observe that, for $f(x)$ a constant in $a \leqq x \leqq b$, $s(n) = S(n)$.

Finally, suppose $f(x)$ to have a finite number of maxima and minima in $a \leqq x \leqq b$. The interval can then be divided into a finite number of subintervals in each of which $f(x)$ is either nondecreasing or nonincreasing, and our argument applies in each subinterval separately.

We can remove the requirement that $f(x)$ be positive or zero in $a \leqq x \leqq b$. Suppose $f(x)$ is negative in this interval. Then, the curve $y = f(x)$ lies below the x axis. Form rectangles as indicated in Fig. 115. Here the measurement of area of the first rectangle is $|f(\xi_1)\, \Delta_1 x|$. Since $f(x) < 0$, we have $f(\xi_1)\, \Delta_1 x < 0$. Similarly for the remaining rectangles. Hence

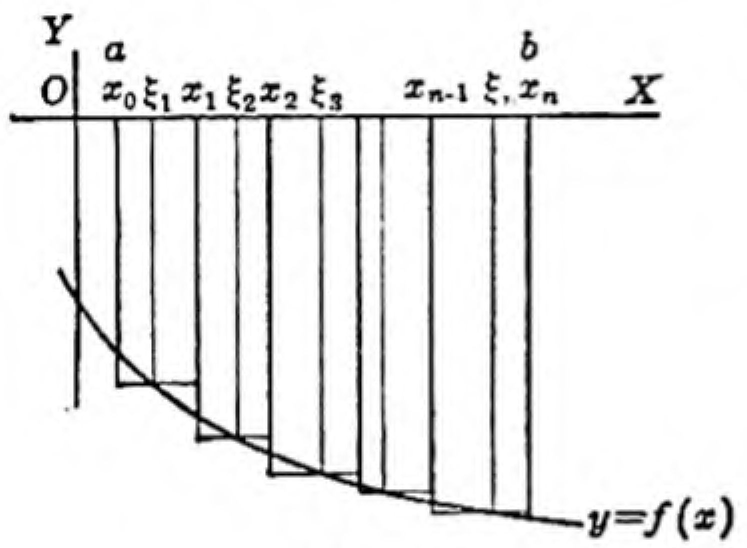


FIG. 115.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\, \Delta_i x$$

is negative, but its numerical value is the measure of the area bounded by the curve, the x axis, and the ordinates at $x = a$ and $x = b$. The negative result is due to the area's being *below* the x axis.

We have, therefore, ascribed a meaning to an "area" one of whose boundaries is a curved line, and we have seen that this area is the limit of a sum of suitably chosen rectangles.

EXERCISES

In Exercises 1 to 11, find approximately the areas described by considering in each case the sum of rectangles, as indicated. Draw a figure in each case.

1. Bounded by the line $y = 2x$, the x axis, and ordinates at $x = 1$ and $x = 2$. Compute lower and upper sums, using rectangles with base 0.2; compute the area by elementary geometry, and compare results.

2. Bounded by the line $y = 3x + 1$, the x axis, and ordinates at $x = 2$ and $x = 3$. Compute lower and upper sums, using rectangles with base 0.1; compute the area by elementary geometry, and compare results.

3. Bounded by the curve $y = 1/x$, the x axis, and the ordinates at $x = 1$ and $x = 3$. Compute a lower sum, using rectangles with base 0.1.

4. Bounded above by the curve $y = x^3$, below by the x axis, and on the right by the ordinate at $x = 2$. Compute a lower sum, using rectangles with base 0.1.

5. Bounded by the curve $y = \ln x$, the x axis, and the ordinates at $x = 2$ and $x = 3$. Compute a lower sum, using rectangles with base 0.1.

6. Bounded above by the curve $y = 8x - x^2$, below by the x axis, and on the right by the ordinate at $x = 4$. Compute a lower sum, using rectangles with base 0.2.

7. Bounded by the curve $y = e^{-x}$, the x axis, and the ordinates at $x = 0$ and $x = 2$. Compute a lower sum, using rectangles with base 0.1.

8. Bounded by the curve $y = \cosh x$, the x axis, and the ordinates at $x = 0$ and $x = 1$. Compute lower and upper sums, using rectangles with base 0.05.

9. Bounded by the curve $y = \sqrt{x}$, the x axis, and the ordinates at $x = 1$ and $x = 2$. Compute lower and upper sums, using rectangles with base 0.05.

10. Bounded by the x axis and the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$. Approximate the area, using rectangles with base $10° = \pi/18$ radian.

11. Bounded by the curve $y = 1/\sqrt{x^2 + 1}$, the x axis, and the ordinates at $x = 1$ and $x = 3$. Compute lower and upper sums, using rectangles with base 0.1.

12. Rewrite the discussion of the area under a curve given in Art. 104, assuming $f(x)$ to be a *decreasing* function.

105. The Definite Integral. We have a definition for *area under a curve* $y = f(x)$. We now proceed to find the connection between this area and a function whose derivative is $f(x)$, seemingly two quite different things. To this end, let $f(x)$ be a continuous function of x; for the sake of simplicity, we assume, in addition, that it is positive and nondecreasing in the interval $a \leqq x \leqq b$, that is, that $0 < f(x_1) \leqq f(x_2)$ for $a \leqq x_1 < x_2 \leqq b$. We wish to calculate the area $EFGH$ (Fig. 116) bounded by the graph of $y = f(x)$, the x axis, the ordinate at $x = a$, and the ordinate at $x = b$. We have an understanding of the term "area" in this case based on the discussion in Art. 104. Now, let P be any point whose abscissa x is between a and b. Draw the ordinate PS at the point P. The area $EPSH$ (shaded in the figure) is then a function of x which we shall denote by $A(x)$. We next find an expression for $\frac{dA}{dx}$. Let x be increased by an amount Δx. The area $A(x)$ is increased by an amount ΔA represented in the figure by the area $PQRS$. Since $f(x)$ is nondecreasing, a rectangle $PQTS$ whose height is $PS = f(x)$ and whose base is Δx is not greater than ΔA. Similarly, ΔA is not greater than a rectangle $PQRU$ whose height is $f(x + \Delta x)$ and whose base is Δx. That is,

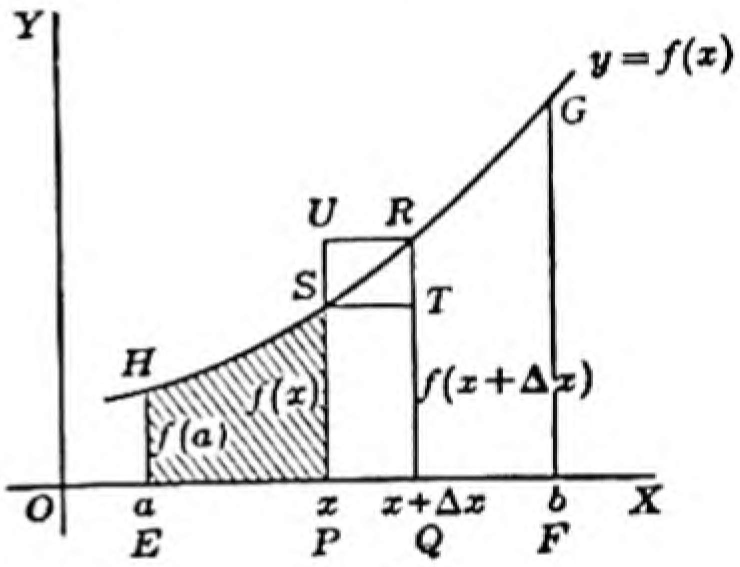


FIG. 116.

$$f(x)\,\Delta x \leqq \Delta A \leqq f(x + \Delta x)\,\Delta x$$

Dividing by Δx $(\Delta x > 0)$,

$$f(x) \leqq \frac{\Delta A}{\Delta x} \leqq f(x + \Delta x)$$

If we now let Δx approach zero, $f(x + \Delta x)$ will approach $f(x)$ since $f(x)$ is continuous. Hence,

$$f(x) \leqq \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \leqq \lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$$

and therefore
$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x)$$

Consequently, $A(x)$ turns out to be a function whose derivative is $f(x)$. That is, $A(x)$ is what we called in Art. 85 a *primitive function*, an *antiderivative*, or *an integral* of $f(x)$ and for which we had the notation $\int f(x)\,dx$. This proves the existence of such an integral, that is, of a function whose derivative is $f(x)$. As we have seen, this function is determined except for an additive constant. Let $F(x)$ be a function for which $F'(x) = f(x)$. Then $A(x) = F(x) + C$. We can determine C; for when $x = a$, the area $EPSH$ vanishes, that is, $A(a) = 0$. Hence

$$0 = A(a) = F(a) + C$$

so that $C = -F(a)$. Thus

$$A(x) = F(x) - F(a) \tag{5}$$

If we wish to calculate the area $EFGH$, we set $x = b$ in (5) obtaining $A(b) = F(b) - F(a)$. This is, of course, identical with the area defined before.

As in Art. 104, we can remove the condition that $f(x)$ be nondecreasing. In fact, although the proof will not be given here, it can be shown that *every continuous function $f(x)$ has an integral; that is, that there exists a function of which $f(x)$ is the derivative.*

We now observe that, since the area described here is given by the limit of the sum (4) of Art. 104, we may write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta_i x = F(b) - F(a)$$

Although we have appealed to the idea of area in forming the sum $\sum_{i=1}^{n} f(\xi_i)\,\Delta_i x$ and in showing that its limit is $F(b) - F(a)$, the sum can be set up without reference to area, and the proof that its limit is $F(b) - F(a)$ can be given in analytic form without referring to any geometrical interpretation. The *limit of the sum* is called *the definite integral* of $f(x)$ and is denoted by the symbol

$$\int_a^b f(x)\,dx$$

which we read, "the definite integral from a to b of $f(x)\,dx$."

We combine the results of this and the preceding section in the **fundamental theorem of the integral calculus**: *Let $f(x)$ be continuous in the interval $a \leqq x \leqq b$. Divide the interval into n subintervals $\Delta_1 x, \Delta_2 x, \ldots,$ $\Delta_n x$, such that the length of the maximum subinterval approaches zero as n*

increases indefinitely. Let ξ_i be any point of the subinterval $\Delta_i x$. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta_i x = \int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is a function whose derivative with respect to x is $f(x)$.

In the symbol $\int_a^b f(x)\,dx$ the numbers a and b are called, respectively, the *lower* and *upper limits of integration*. The interval $a \leqq x \leqq b$ is called the *interval of integration*. The function $f(x)$ is called the *integrand*, and x the *variable of integration*. It is convenient to make use of the symbol $F(x)\Big]_a^b$ in the following way:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Observe that, for constants a and b, $\int_a^b f(x)\,dx$ is a *number*, not a function of x. That is, $\int_a^b f(x)\,dx$ is a function of its upper and lower limits, and its value depends only upon a and b and the form of the function f.

Let the student note carefully that the argument of this and the previous sections holds whatever the physical or geometrical meaning of the function $f(x)$. For whatever this function may represent, it can always be interpreted as the ordinate of a point on a plane curve. Consequently, whenever a quantity can be expressed as the limit of a sum of *elements* of the type $f(\xi_i)\,\Delta_i x$, where $f(x)$ is continuous and where the maximum $\Delta_i x$ approaches zero as the number of elements in the sum increases indefinitely, that quantity can be found by calculating

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Here, $F(x)$ is a function whose derivative with respect to x is $f(x)$, and the numbers a and b are to be suitably chosen.

If we wish to express the area bounded by the curve, the x axis, the fixed ordinate at $x = a$, and the ordinate at a variable point x, we may write [from (5)]

$$A(x) = F(x) - F(a) = \int_a^x f(x)\,dx$$

The right-hand member could as well be written

$$\int_a^x f(t)\,dt = F(t)\Big]_a^x = F(x) - F(a)$$

since the variable appearing in the integrand is replaced in the final result by the limits of integration. The integral $\int_a^x f(t)\,dt$ is known as the

indefinite integral of $f(x)$. This is what was denoted in Art. 85 by $F(x) + C$ and called an *integral*, a *primitive function*, or an *antiderivative* of $f(x)$ and also written $\int f(x)\,dx$.

Note the difference between the definite integral and the function $F(x) + C$. The former is the *limit of a sum*, whereas the latter is simply a *function whose derivative is the integrand*, $f(x)$. The definite integral, which solves the problem of quadrature mentioned in Chap. 1, suggests the notation for integrals, the $\int$ being a conventionalized form of "S" for "sum."

We can now calculate the area under the parabola $y = x^2$ from $x = 1$ to $x = 3$, which we sought in the development of Art. 104, by using the connection between the area and the primitive function.

$$\int_1^3 x^2\,dx = \tfrac{1}{3}x^3\Big]_1^3 = \tfrac{1}{3}(27 - 1) = \tfrac{26}{3} = 8\tfrac{2}{3}$$

This is evidently much simpler than the previous calculation; furthermore, the result is the exact area.

Certain important facts about definite integrals follow. As a definition, we write

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{6}$$

This is in accordance with the previous rule; for if

$$\int f(x)\,dx = F(x) + C$$

(6) is simply the identity $F(b) - F(a) = -[F(a) - F(b)]$. It should be remarked that, in defining $\int_a^b f(x)\,dx$, we supposed a to be less than b; that is, we supposed the lower limit to be less than the upper limit. We therefore regard (6) as giving a definition for a definite integral whose lower limit is greater than its upper limit.

We also define

$$\int_a^a f(x)\,dx = 0 \tag{7}$$

Let the student demonstrate that, for $a < c < b$ and $f(x)$ continuous in $a \leqq x \leqq b$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{8}$$

Let m be the minimum and M the maximum of the continuous function $f(x)$ in the interval $a \leqq x \leqq b$, that is, $m \leqq f(x) \leqq M$. Then

$$m(b - a) \leqq \int_a^b f(x)\,dx \leqq M(b - a) \tag{9}$$

The proof follows evidently from geometrical considerations (Fig. 117).

For $m(b-a)$ is the area $ABCD$, and $M(b-a)$ is the area $ABEF$. These inequalities may be expressed by saying that $\int_a^b f(x)\,dx$ is *between* $m(b-a)$ and $M(b-a)$ or that it is *equal* to $\mu(b-a)$, μ being a suitable value between m and M. Since, as noted in Sec. 11, the continuous function $f(x)$ must assume the value μ for some value $x = \xi$ between a and b, we have the **mean value theorem**:

$$\int_a^b f(x)\,dx = (b-a)f(\xi) \qquad \text{where } a < \xi < b$$

As a special application of this theorem, we see that, if $f(x) \geqq 0$, then $\int_a^b f(x)\,dx \geqq 0$.

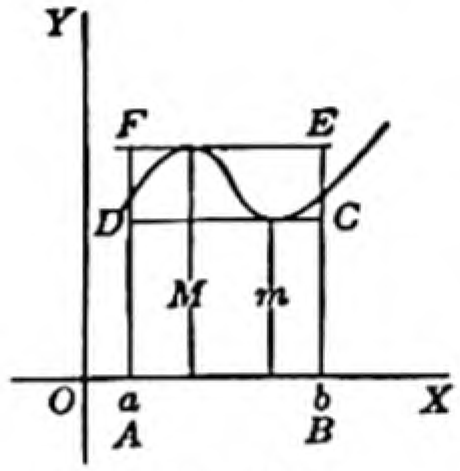


Fig. 117.

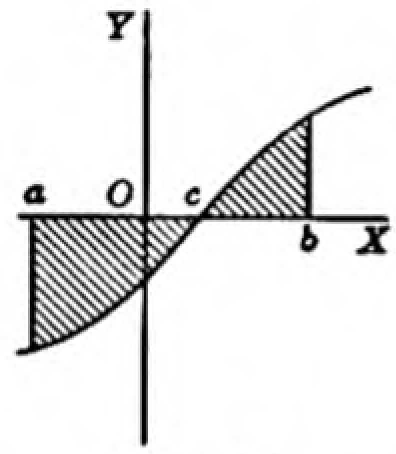


Fig. 118.

We next note that, if $f(x)$ has a graph such as is shown in Fig. 118, then $\int_a^c f(x)\,dx < 0$ while $\int_c^b f(x)\,dx > 0$. Since

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

if we calculate $\int_a^b f(x)\,dx$, we get the *algebraic* sum of the areas shaded in Fig. 118. If the number of *units of area* is required, we must calculate $\left|\int_a^c f(x)\,dx\right|$ and $\int_c^b f(x)\,dx$ and add.

Example 1. Find the value of $\int_{-1}^{2} x^3\,dx$. We have

$$\int_{-1}^{2} x^3\,dx = \tfrac{1}{4}x^4\Big]_{-1}^{2} = \tfrac{1}{4}(16-1) = \tfrac{15}{4}$$

This is the algebraic sum of the areas shaded in Fig. 119.

Example 2. Find the number of units of area shaded in Fig. 119. We note that $x^3 < 0$ for $x < 0$ and $x^3 > 0$ for $x > 0$. The area to the left of $x = 0$ is the numerical value of $\int_{-1}^{0} x^3\,dx = \tfrac{1}{4}x^4\Big]_{-1}^{0} = -\tfrac{1}{4}$, that is, $\tfrac{1}{4}$. The area to the right of $x = 0$ is $\int_0^2 x^3\,dx = \tfrac{1}{4}x^4\Big]_0^2 = 4$. The required result is, therefore, $4 + \tfrac{1}{4} = \tfrac{17}{4}$.

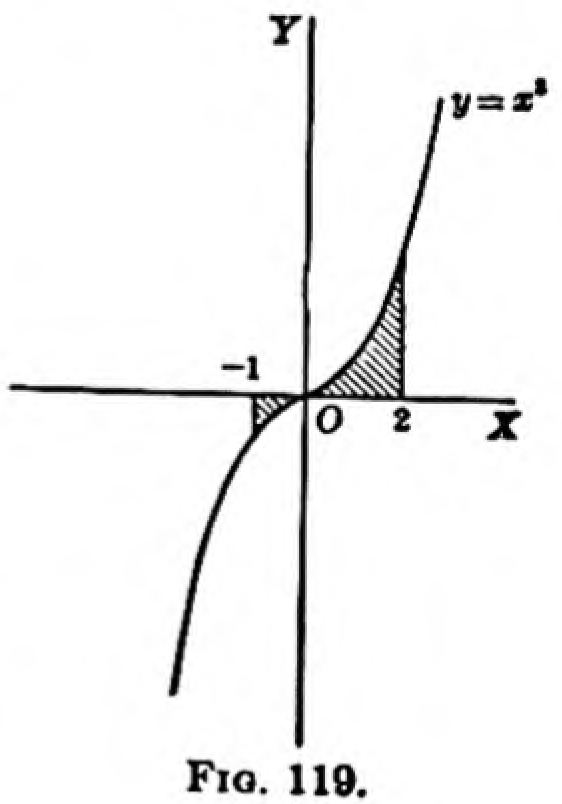


Fig. 119.

EXERCISES

Evaluate the following integrals, and compare results with those of correspondingly numbered exercises on pages 274 and 275. Interpret geometrically.

1. $\int_1^2 2x\,dx$
2. $\int_2^3 (3x + 1)\,dx$
3. $\int_1^3 \frac{dx}{x}$
4. $\int_0^2 x^3\,dx$
5. $\int_2^3 \ln x\,dx$
6. $\int_0^4 (8x - x^2)\,dx$
7. $\int_0^2 e^{-x}\,dx$
8. $\int_0^1 \cosh x\,dx$
9. $\int_1^2 \sqrt{x}\,dx$
10. $\int_0^\pi \sin x\,dx$
11. $\int_1^3 \frac{dx}{\sqrt{x^2 + 1}}$

Evaluate the following integrals (Ex. 12 to 33):

12. $\int_3^6 (x - 3)^2\,dx$
13. $\int_0^{\pi/3} \tan\theta\,d\theta$
14. $\int_0^{3\pi/4} \tan^2(\theta/3)\sec^2(\theta/3)\,d\theta$
15. $\int_0^{\pi/4} \sin^2 2\theta\,d\theta$
16. $\int_0^\pi \sin^3 x \cos^2 x\,dx$
17. $\int_2^4 \frac{dx}{x\sqrt{x^2 - 1}}$
18. $\int_0^2 \frac{dx}{4 + x^2}$
19. $\int_0^{4/3} \frac{dy}{\sqrt{16 - 9y^2}}$
20. $\int_0^1 xe^{x^2}\,dx$
21. $\int_0^{\ln 3} \frac{e^x\,dx}{1 + e^x}$
22. $\int_1^2 \frac{e^x + 1}{e^x - 1}\,dx$
23. $\int_0^1 xe^x\,dx$
24. $\int_0^\pi x\sin x\,dx$
25. $\int_a^{2a} x\sqrt{x^2 - a^2}\,dx$
26. $\int_a^{2a} \sqrt{x^2 - a^2}\,dx$
27. $\int_{-a}^{a} \sinh(x/a)\,dx$
28. $\int_{-a}^{a} a\cosh(x/a)\,dx$
29. $\int_1^e \frac{\ln x}{x}\,dx$
30. $\int_{-\pi/4}^{\pi/4} \tan^2\theta\,d\theta$
31. $\int_0^a \frac{x\,dx}{\sqrt{a^2 + x^2}}$
32. $\int_{-a}^{a} (x + a)^2\,dx$
33. $\int_0^a (x^2 + a^2)\,dx$

Evaluate and interpret geometrically the following integrals (Ex. 34 to 40). Make a sketch in each case.

34. $\int_{-\pi/2}^{\pi/2} \sin x \, dx$

35. $\int_{-\pi/2}^{\pi/2} \cos x \, dx$

36. $\int_{-a}^{a} \frac{a^3 \, dx}{a^2 + x^2}$

37. $\int_{-a}^{a} \sqrt{a^2 - x^2} \, dx$

38. $\int_{1}^{2} (1 - x) \, dx$

39. $\int_{0}^{3} x^2 \, dx$

40. $\int_{0}^{2\pi} \sin^2 x \, dx$

106. Evaluation of Definite Integrals. The student must note carefully that, in the symbol $\int_a^b f(x) \, dx$, a and b are the lower and upper limits for the *variable of integration* x, that is, for the *variable whose differential* dx *appears in the integrand.* If, in order to evaluate the integral, we change the variable by some substitution, $x = \varphi(z)$, it is essential either (1) to express the resulting primitive function in terms of the original variable before substituting the limits or (2) to change the limits to correspond to the change in the variable of integration. The required new limits can be found from the equation of substitution.

Example 1. Find $\int_{-a}^{a} \frac{dx}{(a^2 + x^2)^{3/2}}$.

Method 1. We first find the primitive function by the use of the substitution $x = a \tan \theta$, $dx = a \sec^2 \theta \, d\theta$, thus

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \int \frac{a \sec^2 \theta \, d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \int \cos \theta \, d\theta = \frac{1}{a^2} \sin \theta + C$$
$$= \frac{x}{a^2 \sqrt{a^2 + x^2}} + C$$

Substituting first $x = a$, then $x = -a$, and subtracting,

$$\int_{-a}^{a} \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{a}{a^2 \sqrt{2a^2}} - \frac{-a}{a^2 \sqrt{2a^2}} = \frac{2a}{a^3 \sqrt{2}} = \frac{\sqrt{2}}{a^2}$$

Method 2. Using the substitution $x = a \tan \theta$, we note, that when $x = -a$, $\theta = \arctan(-1) = -\pi/4$. As x increases continuously from $-a$ to a,

$$\theta = \arctan\left(\frac{x}{a}\right)$$

increases continuously from $-\pi/4$ to $\pi/4$. We therefore have $-\pi/4$ and $\pi/4$ as the

lower and upper limits, respectively, for the transformed integral. Thus,

$$\int_{-a}^{a} \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2} \int_{-\pi/4}^{\pi/4} \cos \theta \, d\theta = \frac{1}{a^2} \Big[\sin \theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{a^2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{a^2}$$

That care must be taken in making substitutions and changing the limits to correspond is indicated by the next example.

Example 2. Let us try to find $\displaystyle\int_{-a}^{a} \frac{dx}{(a^2 + x^2)^{3/2}}$ by use of the substitution $x = a/z$, $dx = -a/z^2\,dz$. Then, when $x = -a$, $z = -1$; and when $x = a$, $z = 1$. This would appear to give

$$\int_{-a}^{a} \frac{dx}{(a^2 + x^2)^{3/2}} = -\int_{-1}^{1} \frac{\frac{a}{z^2}\,dz}{\left(a^2 + \frac{a^2}{z^2}\right)^{3/2}} = -\frac{1}{a^2} \int_{-1}^{1} \frac{|z|\,dz}{(z^2 + 1)^{3/2}}$$

$$= -\frac{1}{a^2}(z^2 + 1)^{-1/2}\Big]_{-1}^{0} + \frac{1}{a^2}(z^2 + 1)^{-1/2}\Big]_{0}^{1} = \frac{\sqrt{2}}{a^2}(1 - \sqrt{2})$$

The reason why this substitution cannot be used is that, when x varies continuously from $-a$ to a, $z = a/x$ does *not* vary continuously from -1 to 1. In fact, z is not even defined for $x = 0$, one of the points within the interval of integration. Though a full discussion of the conditions that must be satisfied by the functions involved if such transformations are to be valid is beyond the scope of this book, it may be remarked that, if $f(x)$ is continuous in $a \leqq x \leqq b$ and if the inverse function of $x = \varphi(z)$, say $z = \psi(x)$, has a continuous derivative in $a \leqq x \leqq b$, then

$$\int_{a}^{b} f(x)\,dx = \int_{\alpha}^{\beta} f[\varphi(z)] \cdot \varphi'(z)\,dz$$

Here, $\alpha = \psi(a)$ and $\beta = \psi(b)$. Note that the function $\theta = \arctan(x/a) = \psi(x)$ satisfies this condition $\left[\text{since } \dfrac{d\theta}{dx} = \psi'(x) = \dfrac{a}{x^2 + a^2}\right]$ but that the function

$$z = \frac{a}{x} = \psi(x)$$

does *not* (since it is itself discontinuous at $x = 0$).

We shall frequently have occasion to evaluate an integral such as $\displaystyle\int_{a}^{b} f(x,y)\,dx$ where the variable y is expressed in terms of x by some equation. To do this, we need merely replace y by its value in terms of x and proceed. However, it is sometimes convenient to substitute for x and dx their values in terms of y and dy, then change the limits to correspond. The following examples illustrate these methods:

Example 3. Find $\int_a^{3a} y^2\,dx$ where $x^2 - y^2 = a^2$. Here we have x and y connected by an equation. Solving for y^2, we get

$$y^2 = x^2 - a^2$$

and therefore

$$\int_a^{3a} y^2\,dx = \int_a^{3a} (x^2 - a^2)\,dx = \left[\frac{x^3}{3} - a^2x\right]_a^{3a} = \frac{20}{3}a^3$$

Example 4. Find $\int^1 y\,dx$ where $x = \cos 2y$. Here we can solve for y in terms of x, obtaining $2y = \arccos x$, and $y = \frac{1}{2}\arccos x$. Thus

$$\int_0^1 y\,dx = \tfrac{1}{2}\int_0^1 \arccos x\,dx$$

This can be integrated by parts, as follows: Let

$$u = \arccos x \qquad dv = dx$$

$$du = -\frac{dx}{\sqrt{1 - x^2}} \qquad v = x$$

Therefore $$\tfrac{1}{2}\int \arccos x\,dx = \tfrac{1}{2}x\arccos x + \tfrac{1}{2}\int \frac{x\,dx}{\sqrt{1 - x^2}}$$

$$= \tfrac{1}{2}x\arccos x - \tfrac{1}{2}\sqrt{1 - x^2} + C$$

Hence $$\tfrac{1}{2}\int_0^1 \arccos x\,dx = \tfrac{1}{2}\left[x\arccos x - \sqrt{1 - x^2}\right]_0^1 = \tfrac{1}{2}$$

Alternative Method. Since $x = \cos 2y$, $dx = -2\sin 2y\,dy$. Also, when x varies continuously from 0 to 1, $2y$ varies continuously from $\pi/2$ to 0, and hence y from $\pi/4$ to 0. Therefore

$$\int_0^1 y\,dx = \int_{\pi/4}^0 y(-2\sin 2y)\,dy = -2\int_{\pi/4}^0 y\sin 2y\,dy$$

Integrating by parts, let

$$u = y \qquad dv = \sin 2y\,dy$$

$$du = dy \qquad v = -\tfrac{1}{2}\cos 2y$$

$$-2\int_{\pi/4}^0 y\sin 2y\,dy = -2\left[-\tfrac{1}{2}y\cos 2y\right]_{\frac{\pi}{4}}^0 - \int_{\pi/4}^0 \cos 2y\,dy$$

$$= 0 - \tfrac{1}{2}\sin 2y\Big]_{\frac{\pi}{4}}^0 = \tfrac{1}{2}$$

It is often necessary to evaluate integrals $\int_a^b f(x,y)\,dx$ where both x and y are given in terms of a third variable, that is, a parameter. We then express x, y, and dx in terms of the parameter and change the limits accordingly.

Example 5. Find $\int_0^a y^2\,dx$ where $x = a\cos\varphi$, $y = b\sin\varphi$. We have

$$dx = -a\sin\varphi\,d\varphi$$

Therefore, $$\int_0^a y^2\,dx = -ab^2\int_{\pi/2}^0 \sin^2\varphi\cdot\sin\varphi\,d\varphi$$

since φ varies continuously from $\pi/2$ to 0 when x varies continuously from 0 to a. Recalling that interchanging the upper and lower limits of integration changes the sign of the integral, we have

$$ab^2\int_0^{\pi/2}(1-\cos^2\varphi)\sin\varphi\,d\varphi = ab^2\Big[-\cos\varphi + \tfrac{1}{3}\cos^3\varphi\Big]_0^{\frac{\pi}{2}}$$
$$= ab^2(1-\tfrac{1}{3}) = \tfrac{2}{3}ab^2$$

EXERCISES

Evaluate the following integrals by use of a suitable substitution (Ex. 1 to 10):

1. $\int_{-a}^a \sqrt{a^2-x^2}\,dx$

2. $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

3. $\int_a^{2a} x^3\sqrt{x^2-a^2}\,dx$

4. $\int_0^4 \dfrac{dx}{(9+x^2)^{3/2}}$

5. $\int_0^4 \dfrac{dx}{2+\sqrt{x}}$

6. $\int_0^4 \dfrac{x\,dx}{2+\sqrt{x}}$

7. $\int_a^{2a} \dfrac{\sqrt{x^2-a^2}}{x}\,dx$

8. $\int_0^{16}\sqrt{1+\sqrt{x}}\,dx$

9. $\int_0^{\pi^2}\sin\sqrt{x}\,dx$

10. $\int_0^{\ln 5}\sqrt{e^x-1}\,dx$

Evaluate the following integrals (Ex. 11 to 30):

11. $\int_0^4 y^2\,dx$ where $y^2 = 16x$

12. $\int_0^1 xy\,dx$ where $y = x^2$

13. $\int_0^a x^2\,dy$ where $x^2+y^2 = a^2$

14. $\int_0^a xy\,dx$ where $x^2+y^2=a^2$ and $y \geqq 0$

15. $\int_{-a}^a x^2y\,dx$ where $x^2+y^2=a^2$ and $y \geqq 0$

16. $\int_0^a x^3y\,dx$ where $x^2+y^2=a^2$ and $y \geqq 0$

17. $\int_0^{\pi/2} x^2\,dy$ where $x = \sin y$

18. $\int_0^8 x^2\,dy$ where $y = x^3$

19. $\int_{-1}^1 x\,dy$ where $y = \sin x$

20. $\int_0^1 xy\,dy$ where $y = \cos x$

21. $\int_a^{2a} y^2\,dx$ where $x^2 - y^2 = a^2$

22. $\int_0^a xy\,dx$ where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $y \geqq 0$

23. $\int_{-a}^a y\,dx$ where $x = a\cos\varphi,\ y = a\sin\varphi$ and $y \geqq 0$

24. $\int_{-a}^a x^2y\,dx$ where $x = a\cos\varphi,\ y = a\sin\varphi$ and $y \geqq 0$

25. $\int_0^a xy\,dx$ where $x = a\cos\varphi,\ y = b\sin\varphi$ and $y \geqq 0$

26. $\int_0^a y\,dx$ where $x = a\cos^3\varphi,\ y = a\sin^3\varphi$ and $y \geqq 0$

27. $\int_0^b x^2\,dy$ where $x = a\sec\varphi,\ y = b\tan\varphi$ and $y \geqq 0$

28. $\int_0^a \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$ where $y = \sqrt{2ax - x^2}$

29. $\int_0^a \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$ where $x^{2/3} + y^{2/3} = a^{2/3}$ and $y \geqq 0$

30. $\int_0^a \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$ where $x = a\cos^3\varphi,\ y = a\sin^3\varphi$ and $y \geqq 0$

107. Improper Integrals; Integrand Infinite. So far we have considered only integrals of the form $\int_a^b f(x)\,dx$ where $f(x)$ is continuous over the interval of integration. We may well ask, "What happens if $f(x)$ becomes infinite either at a or b, or at one or more points between a and b?" We shall extend our definition of a definite integral to take into account such *improper integrals.*

Integrand infinite at an end point of the interval of integration. Suppose $f(x)$ is continuous in the interval $a \leqq x < b$ but that, when $x \to b^-$, $f(x)$ becomes infinite. If

$$\lim_{\beta\to b^-} \int_a^\beta f(x)\,dx = I$$

exists, then we *define* the symbol $\int_a^b f(x)\,dx$ to mean this limit.

Example 1. Find $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. Here the integrand becomes infinite at the upper limit. We calculate

$$\int_0^\beta \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big]_0^\beta = \arcsin \beta$$

This is represented geometrically by the shaded area in Fig. 120. We next take the limit as $\beta \to 1^-$.

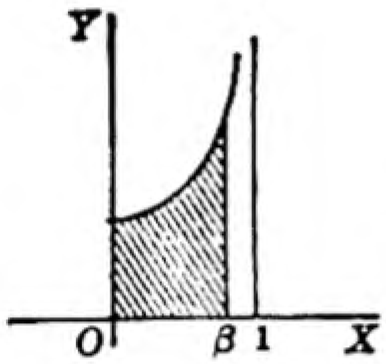


FIG. 120.

$$\lim_{\beta \to 1^-} \int_0^\beta \frac{dx}{\sqrt{1-x^2}} = \lim_{\beta \to 1^-} \arcsin \beta = \frac{\pi}{2}$$

Therefore, by definition, $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$. Geometrically, this means that the shaded area has a limiting value $\pi/2$ units as its right-hand boundary is made to approach the position of the line $x = 1$.

Now, suppose $f(x)$ is continuous in the interval $a < x \leqq b$ but that, when $x \to a^+$, $f(x)$ becomes infinite. If

$$\lim_{\alpha \to a^+} \int_\alpha^b f(x)\, dx = I$$

exists, we *define* the symbol $\int_a^b f(x)\, dx$ to mean this limit.

Example 2. Find $\int_0^4 \frac{dx}{\sqrt{x}}$. Here the integrand becomes infinite at the lower limit. We calculate $\int_\alpha^4 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_\alpha^4 = 4 - 2\sqrt{\alpha}$. This is represented geometrically by the shaded area in Fig. 121. We next take the limit as $\alpha \to 0^+$.

$$\lim_{\alpha \to 0^+} \int_\alpha^4 \frac{dx}{\sqrt{x}} = \lim_{\alpha \to 0^+} (4 - 2\sqrt{\alpha}) = 4$$

Therefore, by definition, $\int_0^4 \frac{dx}{\sqrt{x}} = 4$. Geometrically, this means that the shaded area has a limiting value as its left-hand boundary is made to approach the position of the line $x = 0$.

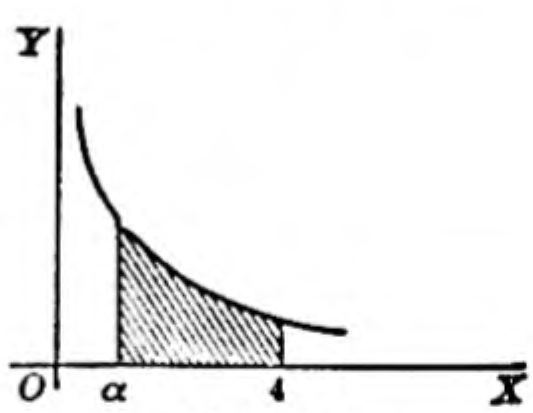


FIG. 121.

Finally, suppose $f(x)$ is continuous in the interval $a < x < b$ and that (x) becomes infinite when $x \to b^-$ and also when $x \to a^+$. If

$$\lim_{\substack{\beta \to b^- \\ \alpha \to a^+}} \int_\alpha^\beta f(x)\, dx = I$$

exists, we *define* the symbol $\int_a^b f(x)\, dx$ to mean this limit.

Example 3

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{\substack{\beta \to 1^- \\ \alpha \to -1^+}} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{1-x^2}} = \lim_{\substack{\beta \to 1^- \\ \alpha \to -1^+}} \arcsin x \Big]_{\alpha}^{\beta}$$

$$= \lim_{\substack{\beta \to 1^- \\ \alpha \to -1^+}} (\arcsin \beta - \arcsin \alpha) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Geometrically, this means that the shaded area in Fig. 122 has a limiting value as its left-hand boundary approaches the line $x = -1$ and its right-hand boundary approaches the line $x = 1$.

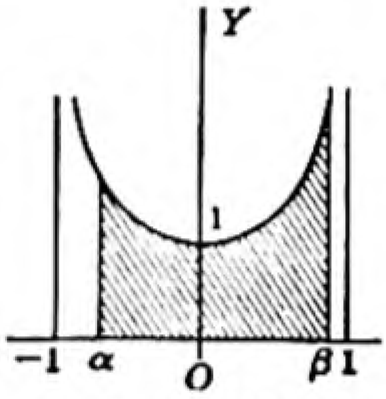


FIG. 122.

In each of the above cases, the integral $\int_a^b f(x)\,dx$ is called an *improper integral*. If the indicated limit I exists, the integral is said to *exist* or to *converge*, and we say that $f(x)$ is integrable in the interval $a \leqq x \leqq b$. If the limit I does not exist, we say that the integral *does not exist* and that *the symbol* $\int_a^b f(x)\,dx$ *is meaningless.* In this latter case the integral is said to diverge or to be *divergent.*

Example 4. $\int_0^1 \frac{dx}{x} = \lim_{\alpha \to 0^+} \int_{\alpha}^1 \frac{dx}{x} = \lim_{\alpha \to 0^+} (\ln 1 - \ln \alpha)$ which does not exist. Hence the symbol $\int_0^1 \frac{dx}{x}$ is meaningless.

Integrand infinite at a point within the interval of integration. Suppose $f(x)$ has a point of infinite discontinuity *between* a and b, say at $x = c (a < c < b)$. Then, if *both* the improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The integral $\int_a^b f(x)\,dx$ is again called an *improper integral*, and we have defined its meaning in terms of the improper integrals already considered. Note particularly that, if *either* of the two integrals on the right-hand side fails to converge, *then the symbol* $\int_a^b f(x)\,dx$ *is meaningless.*

Example 5. Find $\int_0^2 \frac{dx}{(x-1)^2}$. The careless student will say that this equals

$$-\frac{1}{(x-1)}\Big]_0^2 = -\frac{1}{1} + \frac{1}{-1} = -2$$

This is obviously absurd, since $\frac{1}{(x-1)^2} > 0$ and therefore the integral (if it exists)

is certainly positive. We must not fail to observe that, when $x = 1$, the integrand is discontinuous. In fact, when $x \to 1$, $1/(x-1)^2$ becomes infinite. Therefore

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

Now, consider the first of the two improper integrals on the right-hand side. We have

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{\beta\to 1^-} \int_0^\beta \frac{dx}{(x-1)^2} = \lim_{\beta\to 1^-} \left[-\frac{1}{x-1}\right]_0^\beta = \lim_{\beta\to 1^-} \left(\frac{1}{1-\beta} - 1\right) = +\infty$$

We may now say that $\int_0^2 \frac{dx}{(x-1)^2}$ does not exist, no matter what the second integral on the right-hand side may be (see Fig. 123 for a geometrical interpretation of this improper integral).

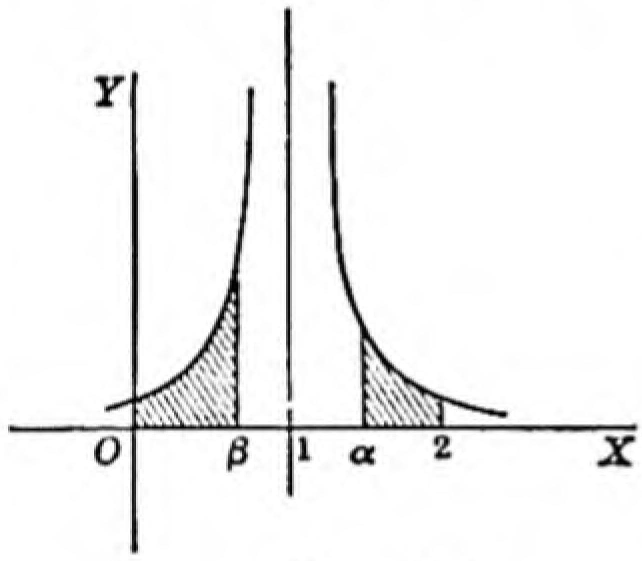


FIG. 123.

Example 6

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^{2/3}} &= \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} \\ &= \lim_{\beta\to 0^-} \int_{-1}^\beta \frac{dx}{x^{2/3}} + \lim_{\alpha\to 0^+} \int_\alpha^1 \frac{dx}{x^{2/3}} \\ &= \lim_{\beta\to 0^-} 3x^{1/3}\Big]_{-1}^\beta + \lim_{\alpha\to 0^+} 3x^{1/3}\Big]_\alpha^1 \\ &= \lim_{\beta\to 0^-} (3\beta^{1/3} + 3) + \lim_{\alpha\to 0^+} (3 - 3\alpha^{1/3}) \\ &= 3 + 3 = 6\end{aligned}$$

The integral therefore converges with value 6. Let the student interpret this geometrically.

Example 7. Find $\int_{-1}^1 \frac{dx}{x^3}$. Here we have

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^3} &= \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} \\ &= \lim_{\beta\to 0^-} \int_{-1}^\beta \frac{dx}{x^3} + \lim_{\alpha\to 0^+} \int_\alpha^1 \frac{dx}{x^3} \\ &= \lim_{\beta\to 0^-} \left(-\frac{1}{2\beta^2} + \frac{1}{2}\right) \\ &\quad + \lim_{\alpha\to 0^+} \left(-\frac{1}{2} + \frac{1}{2\alpha^2}\right)\end{aligned}$$

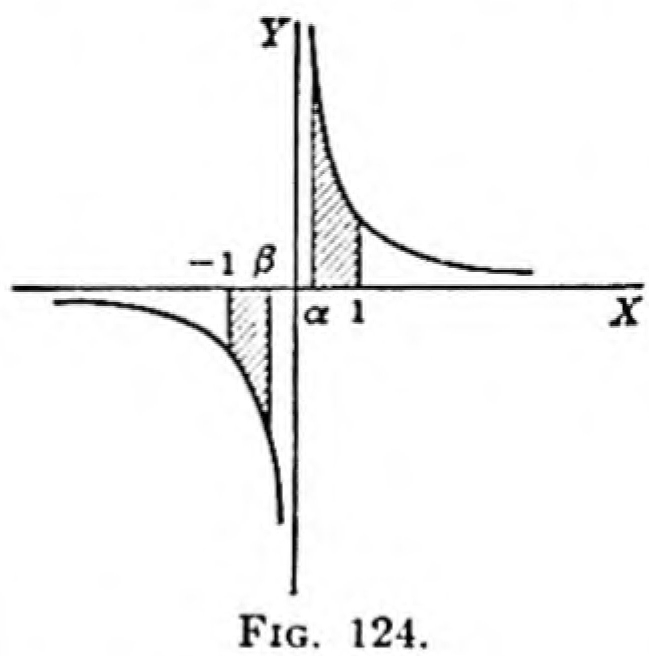


FIG. 124.

Since $-\frac{1}{2\beta^2}$ decreases toward $-\infty$ as $\beta \to 0^-$, the first of the integrals on the right-hand side does not exist. Consequently, the symbol $\int_{-1}^1 \frac{dx}{x^3}$ is meaningless. Similarly, since the second expression on the right-hand side increases to $+\infty$, the second integral does not exist. The student must not be misled by the fact that the first integral is $-\infty$ while the second is $+\infty$. We cannot say that the sum is zero—recall the discussion of indeterminate forms (Art. 81). Geometrically, the reason why the

integral does not converge is plain. In Fig. 124, neither the left-hand shaded area from -1 to β nor the right-hand shaded area from α to 1 has a limiting value. We cannot, therefore, speak of their sum.

It is now evident how we may extend our definition to cover the case where the integrand becomes infinite at a finite number of points in the interval $a \leqq x \leqq b$. Suppose that $f(x)$ becomes infinite for $x = c_1, c_2, \ldots, c_n$ where $a \leqq c_1 < c_2 \cdots < c_n \leqq b$. Then, *if each of the improper integrals on the right-hand side exists*, we define

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_n}^b f(x)\,dx \quad (10)$$

If any one of these integrals does not exist, the symbol $\int_a^b f(x)\,dx$ is meaningless.

As a first step in evaluating any definite integral, the student should search for values of the variable for which the integrand becomes infinite. If such values lie in the interval of integration, the integral is improper, and it must be handled as indicated in this section.

108. Improper Integrals; Interval of Integration Infinite. Consider the limit $\lim\limits_{b \to \infty} \int_a^b f(x)\,dx = I$ where we suppose that $f(x)$ is integrable in the interval $a \leqq x \leqq b$ for all $b > a$. If this limit exists, we define the symbol

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx = I$$

Similarly, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx = I$$

and

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{b \to +\infty \\ a \to -\infty}} \int_a^b f(x)\,dx = I$$

In these three cases, the integral is again called an *improper integral.* In the previous section, the integrals were improper because the *integrand* became infinite. In this section, the integrals are improper because the *interval of integration* becomes infinite. If the limit I exists, the integral is said to be *finite* or to *converge*, and we say that $f(x)$ is *integrable* in the infinite interval. If the limit I does not exist, we say that the integral *does not exist;* in this case the integral is called *divergent.*

Example 1. $\int_1^\infty \frac{dx}{x^2} = \lim\limits_{b \to +\infty} \int_1^b \frac{dx}{x^2} = \lim\limits_{b \to +\infty} \left(-\frac{1}{b} + \frac{1}{1}\right) = 1.$ Geometrically, this means that the shaded area from 1 to b, Fig. 125, has a limiting value 1 as the

right-hand boundary at $x = b$ moves indefinitely away to the right. We speak of this limiting value as the area bounded by the curve, its asymptote (the x axis), and the ordinate at $x = 1$.

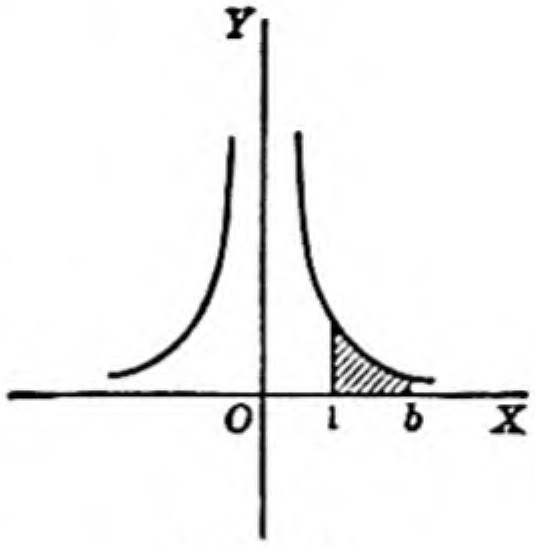


FIG. 125.

Example 2. $\displaystyle\int_1^{\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} (\ln b - \ln 1) = +\infty$. Therefore, the integral is divergent.

Example 3. $\displaystyle\int_0^{\infty} \sin x \, dx = \lim_{b \to +\infty} (-\cos b + 1)$, which does not exist. Therefore, the integral does not exist. Let the student interpret this geometrically.

Example 4. Find $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2}$. Notice that this integral is improper because the integrand becomes infinite as $x \to 0$ and also because the interval of integration is infinite. Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^2} = \int_{-\infty}^{0} \frac{dx}{x^2} + \int_0^{\infty} \frac{dx}{x^2}$$

Here

$$\int_{-\infty}^{0} \frac{dx}{x^2} = \lim_{\substack{\beta \to 0^- \\ \alpha \to -\infty}} \int_{\alpha}^{\beta} \frac{dx}{x^2} = \lim_{\substack{\beta \to 0^- \\ \alpha \to -\infty}} \left(-\frac{1}{\beta} + \frac{1}{\alpha} \right)$$
$$= \lim_{\beta \to 0^-} \left(-\frac{1}{\beta} \right)$$

since $\displaystyle\lim_{\alpha \to -\infty} \frac{1}{\alpha} = 0$. But this limit is $+\infty$. Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2}$ does not exist.

EXERCISES

Evaluate the following integrals if they exist, and interpret geometrically (Ex. 1 to 31):

1. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$

2. $\displaystyle\int_{-a}^a \frac{dx}{\sqrt{a^2 - x^2}}$

3. $\displaystyle\int_a^{2a} \frac{dx}{\sqrt{x - a}}$

4. $\displaystyle\int_a^{2a} \frac{dx}{(x - a)^{3/2}}$

5. $\int_a^b \frac{dx}{x-a}$

6. $\int_{-3}^{3} \frac{x\,dx}{\sqrt{9-x^2}}$

7. $\int_0^1 \ln x\,dx$

8. $\int_0^1 \frac{\ln x}{x}\,dx$

9. $\int_0^{\pi/2} \tan x\,dx$

10. $\int_{-1}^{1} \frac{dx}{x^2}$

11. $\int_{-a}^{a} \frac{dx}{(x-a)^{2/3}}$

12. $\int_{-1}^{1} \frac{dx}{x^5}$

13. $\int_{-1}^{1} \frac{dx}{x^{1/3}}$

14. $\int_{-\pi/2}^{\pi/2} \cot x\,dx$

15. $\int_0^\infty \frac{a^2\,dx}{x^2+a^2}$

16. $\int_0^\infty e^{-x}\,dx$

17. $\int_0^\infty xe^{-x^2}\,dx$

18. $\int_0^\infty xe^{-x}\,dx$

19. $\int_a^\infty \frac{x\,dx}{x^4+a^4}$

20. $\int_0^\infty \frac{dx}{(4+x^2)^{1/2}}$

21. $\int_{-1}^{1} \frac{dx}{x^2\sqrt{1-x^2}}$

22. $\int_{-1}^{1} \frac{dx}{x(1-x^2)}$

23. $\int_{-\infty}^{\infty} \frac{dx}{x^2-a^2}$

24. $\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2}$

25. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{a^2+x^2}}$

26. $\int_a^\infty \frac{dx}{x\sqrt{x^2-a^2}} \qquad a > 0$

27. $\int_a^\infty \frac{dx}{x^2\sqrt{x^2-a^2}} \qquad a > 0$

28. $\int_{-\infty}^{\infty} \tan x\,dx$

29. $\int_a^\infty \frac{dx}{x(x^2+a^2)} \qquad a > 0$

30. $\int_{-\infty}^{2} e^{2x}\,dx$

31. $\int_0^2 \frac{\sqrt{4+x^2}}{x}\,dx$

32. Show that $\int_a^b \frac{dx}{(x-a)^k}$ $(b > a)$ converges for all $k < 1$ and diverges for all $k \geqq 1$.

33. Show that $\int_a^\infty \frac{dx}{x^k}$ $(a > 0)$ converges for all $k > 1$ and diverges for all $k \leqq 1$.

34. Show that $\int_0^a x^k \ln x\,dx$ $(a > 0)$ converges for all $k > -1$ and diverges for all $k \leqq -1$.

35. Show that $\int_a^\infty \frac{(\ln x)^k}{x}\,dx$ $(a > 0$, but $\neq 1)$ converges for all $k < -1$ and diverges for all $k \geqq -1$.

MISCELLANEOUS EXERCISES

Evaluate the following integrals (Ex. 1 to 25):

1. $\displaystyle\int_0^2 \frac{x^3\,dx}{(x^4+16)^2}$

2. $\displaystyle\int_0^{2\sqrt{2}} \frac{x\,dx}{x^4+64}$

3. $\displaystyle\int_0^8 \frac{dt}{(t-4)^2}$

4. $\displaystyle\int_0^\infty \cos 3\theta\,d\theta$

5. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3\theta\,d\theta$

6. $\displaystyle\int_0^{\pi/9} \tan^4 3\theta \sec^2 3\theta\,d\theta$

7. $\displaystyle\int_0^{\pi/2} \cos^4\theta\,d\theta$

8. $\displaystyle\int_0^\infty x^2e^{-x^3}\,dx$

9. $\displaystyle\int_{-2}^1 \frac{dy}{y^2+4y+13}$

10. $\displaystyle\int_0^{5/2} \frac{dx}{\sqrt{25-4x^2}}$

11. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-\sqrt{x}}}$

12. $\displaystyle\int_5^6 \frac{dx}{(x^2-9)^{3/2}}$

13. $\displaystyle\int_2^3 \frac{dy}{\sqrt{4y-y^2-3}}$

14. $\displaystyle\int_0^\infty \cot^3 3\theta\,d\theta$

15. $\displaystyle\int_0^{3\ln 2} \frac{e^t\,dt}{\sqrt{1+e^t}}$

16. $\displaystyle\int_0^1 y^2e^y\,dy$

17. $\displaystyle\int_{-\infty}^1 t^2e^t\,dt$

18. $\displaystyle\int_0^\infty \frac{e^z\,dz}{3+e^{2z}}$

19. $\displaystyle\int_{-\infty}^\infty y^3e^{-y^2}\,dy$

20. $\displaystyle\int_{-\infty}^\infty \frac{y\,dy}{y^4-64}$

21. $\displaystyle\int_0^{\pi^2} \cos\sqrt{x}\,dx$

22. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x(x^2-4)}$

23. $\displaystyle\int_0^{\pi/3} \sec^3\theta\tan^5\theta\,d\theta$

24. $\displaystyle\int_0^{\operatorname{argcosh} 4} \frac{\sqrt{\cosh x}}{\operatorname{csch} x}\,dx$

25. $\displaystyle\int_0^2 x\sinh x\,dx$

26. Show that, if $f(x)$ is an *odd* function, then $\displaystyle\int_{-a}^a f(x)\,dx = 0$, provided the integral exists.

27. Show that, if $f(x)$ is an *even* function, then $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, provided the integral exists.

28. Show that, for m and n any positive integers, $\displaystyle\int_0^{2\pi} \cos mx \sin nx\,dx = 0$.

29. Show that, for m and n positive integers, $\displaystyle\int_0^{2\pi} \cos mx\cos nx\,dx$ and $\displaystyle\int_0^{2\pi} \sin mx\sin nx\,dx$ are both zero if $m \neq n$ and both equal to π if $m = n$.

30. Show that $\int_a^x f'(t)\,dt = f(x) - f(a)$ if $f'(t) = \frac{d}{dt}f(t)$.

31. Show that $\int_a^x (x - t)f''(t)\,dt = f(x) - f(a) - (x - a)f'(a)$ by using integration by parts.

32. Show that

$$\int_a^x \frac{(x - t)^2}{2!} f'''(t)\,dt = f(x) - f(a) - (x - a)f'(a) - \frac{(x - a)^2}{2!} f''(a)$$

33. Show that

$$\int_a^x \frac{(x - t)^n}{n!} f^{(n+1)}(t)\,dt = f(x) - f(a) - (x - a)f'(a) - \frac{(x - a)^2}{2!} f''(a) - \cdots - \frac{(x - a)^n}{n!} f^{(n)}(a)$$

and therefore that Taylor's theorem can be written

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \cdots + \frac{(x - a)^n}{n!} f^{(n)}(a) + \int_a^x \frac{(x - t)^n}{n!} f^{(n+1)}(t)\,dt \qquad (n \text{ a positive integer})$$

(Compare Art. 82.)

34. Show that $\int_0^{\pi/2} \sin^n \theta\,d\theta = \frac{n - 1}{n} \int_0^{\pi/2} \sin^{n-2} \theta\,d\theta$. (*Hint:* Let

$$\sin^n \theta = \sin^{n-1} \theta \sin \theta$$

and integrate by parts.)

35. Apply successively the result of Exercise 34 to show that

$$\begin{aligned}\int_0^{\pi/2} \cos^n \theta\,d\theta &= \int_0^{\pi/2} \sin^n \theta\,d\theta \\ &= \frac{(n - 1)(n - 3) \cdots 4 \cdot 2}{n(n - 2) \cdots 3 \cdot 1} \qquad \text{if } n \text{ is } odd \\ &= \frac{(n - 1)(n - 3) \cdots 3 \cdot 1}{n(n - 2) \cdots 4 \cdot 2} \cdot \frac{\pi}{2} \qquad \text{if } n \text{ is } even\end{aligned}$$

36. Show that $\int_0^{\pi/2} \sin^m \theta \cos^n \theta\,d\theta = \frac{m - 1}{m + n} \int_0^{\pi/2} \sin^{m-2} \theta \cos^n \theta\,d\theta$. (*Hint:* Let $\sin^m \theta = \sin^{m-1} \theta \sin \theta$, and integrate by parts.)

37. Apply successively the result of Exercise 36 to show that for m *odd* (n even or odd):

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta\,d\theta = \frac{(m - 1)(m - 3) \cdots 4 \cdot 2}{(m + n)(m + n - 2) \cdots (n + 3)(n + 1)}$$

and for m *even:*

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3)\cdots 3\cdot 1\cdot (n-1)(n-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots (n+2)n(n-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2} \qquad \text{for } n \textit{ even}$$

$$= \frac{(m-1)(m-3)\cdots 3\cdot 1\cdot (n-1)(n-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots (n+2)n(n-2)\cdots 3\cdot 1} \qquad \text{for } n \textit{ odd}$$

The formulas of Exercises 35 and 37 are called **Wallis's formulas.**

38. Use Wallis's formulas to verify that

(a) $\displaystyle\int_0^{\pi/2} \sin^4 \theta \, d\theta = \frac{3\pi}{16}$ (b) $\displaystyle\int_0^{\pi/2} \cos^9 \theta \, d\theta = \frac{128}{315}$

(c) $\displaystyle\int_0^{\pi/2} \sin^6 \theta \cos^4 \theta \, d\theta = \frac{3\pi}{512}$ (d) $\displaystyle\int_0^{\pi/2} \sin^5 \theta \cos^3 \theta \, d\theta = \frac{1}{24}$

(e) $\displaystyle\int_0^{\pi/2} \sin^5 \theta \cos^4 \theta \, d\theta = \frac{8}{315}$ (f) $\displaystyle\int_0^{\pi/2} \sin^4 \theta \cos^5 \theta \, d\theta = \frac{8}{315}$

CHAPTER 15

GEOMETRIC APPLICATIONS OF THE DEFINITE INTEGRAL

109. Plane Areas, Cartesian Coordinates. The simplest and most evident application of the fundamental theorem (Art. 105) is in finding areas bounded by plane curves whose equations are given in cartesian coordinates. Let us suppose the required area to be that shown in Fig. 126. Draw vertical lines at a distance Δw apart, and form n rectangular elements of area, as shown in the figure. Let the altitudes of these rectangles be $h_1, h_2, \ldots, h_n$. An approximation to the required area is then

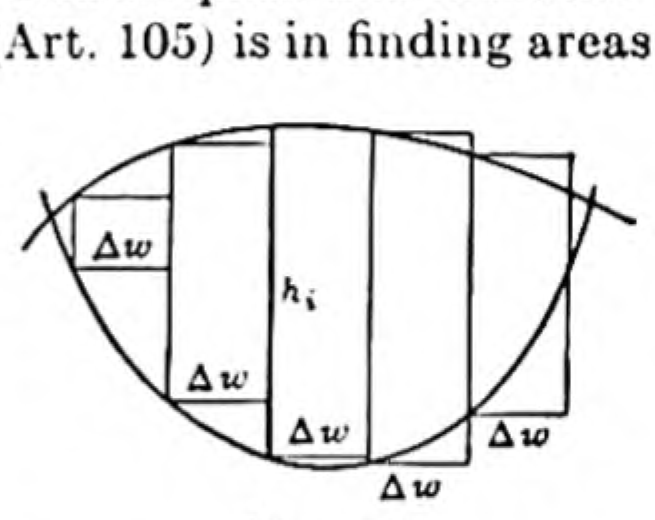


FIG. 126.

$$h_1\,\Delta w + h_2\,\Delta w + \cdots + h_n\,\Delta w = \sum_{i=1}^{n} h_i\,\Delta w$$

Furthermore, note that the approximation improves with an increase in n and a corresponding decrease in Δw. By the fundamental theorem, therefore, the area A is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} h_i\,\Delta w = \int_{w_1}^{w_2} h\,dw$$

the limit being taken as $\Delta w \to 0$ and $n \to \infty$, and with suitable limits of integration chosen to include the entire area. We may, of course, express h and dw in terms of the coordinates (x,y) of points on the curves involved. It would be possible to take rectangles with sides parallel to some fixed line in the plane other than the y axis. The limit of the sum of rectangles is also the measure of area of a certain square. Hence, finding an area is often referred to as *making a quadrature* (Chap. 1).

Example 1. Find the area bounded by the parabola $x^2 = 8y$ and the line

$$x - 2y + 8 = 0$$

We first find the points of intersection of the line and the parabola (Fig. 127). We have $y = x^2/8$, and substituting into the equation of the line,

$$x - \frac{x^2}{4} + 8 = 0 \qquad x^2 - 4x - 32 = 0$$
$$(x - 8)(x + 4) = 0$$

so that $x = 8$ or $x = -4$. From the equation of the line, we find that, when $x = -4$, $y = 2$; when $x = 8$, $y = 8$. The points of intersection are, therefore, $A(-4,2)$ and $B(8,8)$. Now, form rectangular elements of area by drawing vertical lines at a distance Δx apart, as shown in the figure. A typical element is heavily outlined. Note that in each element of area the upper extremity of the left-hand side is a point *of the line;* let its coordinates be (x_i, Y_i). The lower extremity is a point *of the parabola;* let its coordinates be (x_i, y_i). Note that these points have the same x coordinates since they lie on the same vertical line. The area of the typical element is, therefore, $h_i\,\Delta x = (Y_i - y_i)\,\Delta x$. The total area is approximately $\sum_{i=1}^{n} (Y_i - y_i)\,\Delta x$, and the exact area is the limit of this sum. Hence

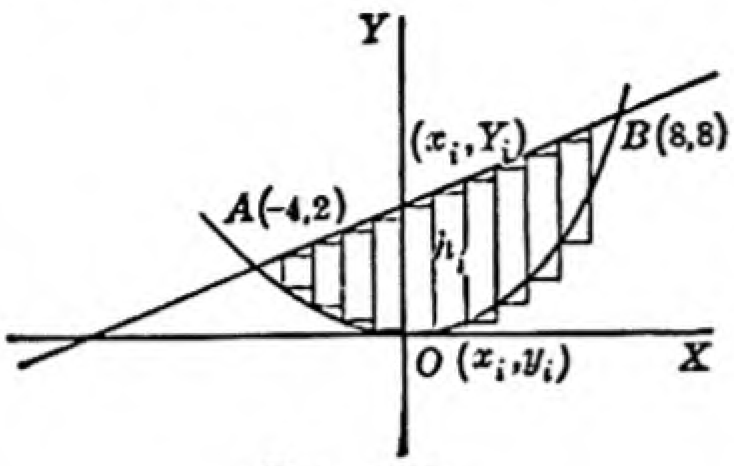


FIG. 127.

$$A = \int_{-4}^{8} (Y - y)\,dx$$

the limits being taken to include the entire area between A and B. We now express $Y - y$ in terms of x so that the integration can be effected. Since (x,Y) is a point of the line, $x - 2Y + 8 = 0$, and $Y = \dfrac{x + 8}{2}$. Since (x,y) is a point on the parabola, $x^2 = 8y$, and $y = x^2/8$. Remember that it is the same x in both cases. Hence

$$Y - y = \frac{x + 8}{2} - \frac{x^2}{8} = \frac{4x + 32 - x^2}{8}$$

Consequently

$$A = \int_{-4}^{8} (Y - y)\,dx = \tfrac{1}{8} \int_{-4}^{8} (4x + 32 - x^2)\,dx$$
$$= \frac{1}{8}\left[2x^2 + 32x - \frac{x^3}{3}\right]_{-4}^{8}$$
$$= 36 \text{ units of area}^*$$

Example 2. Find the area in the second quadrant bounded by the curve $y = x^3 + 8$ and the coordinate axes; choose elements of area in two different ways.

First choice of elements. Form n elements of area by drawing vertical lines at a distance Δx apart (Fig. 128). A typical element is heavily outlined. Its altitude is y_i, and its width Δx. Hence, its area is $y_i\,\Delta x$. The required area is approximately $\sum_{i=1}^{n} y_i\,\Delta x$. Taking the limit of this sum as n increases indefinitely and Δx approaches zero, we have

$$A = \int_{-2}^{0} y\,dx = \int_{-2}^{0} (x^3 + 8)\,dx = \frac{x^4}{4} + 8x\,\Big]_{-2}^{0}$$
$$= -4 + 16 = 12 \text{ units of area}$$

* These units will be square if the same scale is used on both x and y axes.

Second choice of elements. Form n elements of area by drawing horizontal lines a distance Δy apart (Fig. 129). A typical element is heavily outlined. Its length is the length PQ. Since P has coordinates (x_i,y_i) and Q has coordinates $(0,y_i)$, this

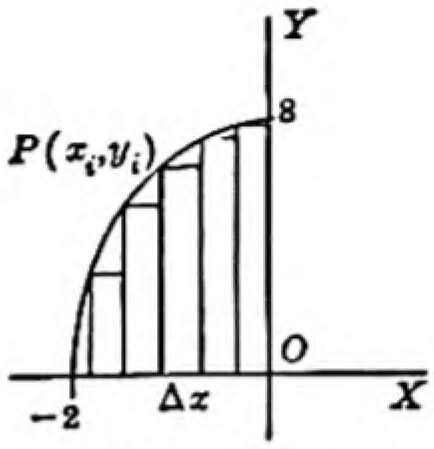


FIG. 128.

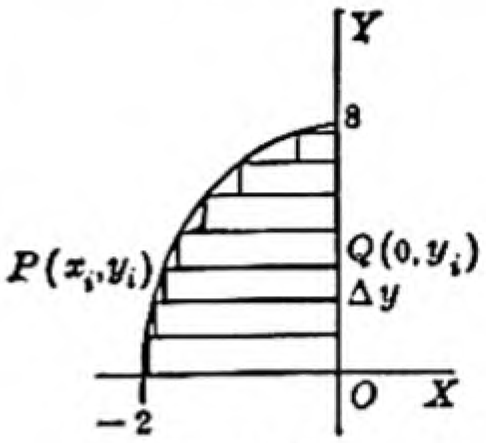


FIG. 129.

length is $0 - x_i = -x_i$. The altitude of the element is Δy, and its area is $(-x_i)\,\Delta y$. Hence, the total area is approximately $\sum_{i=1}^{n} (-x_i)\,\Delta y$, and the exact area is $\int_0^8 (-x)\,dy$. Evaluating this integral, we obtain

$$\begin{aligned}\int_0^8 (-x)\,dy &= -\int_0^8 x\,dy = -\int_0^8 (y-8)^{1/3}\,dy \\ &= -\tfrac{3}{4}(y-8)^{4/3}\Big]_0^8 = \tfrac{3}{4}(-8)^{4/3} = \tfrac{3}{4}(16) \\ &= 12 \text{ units of area}\end{aligned}$$

A quick method for setting up the integral in such area problems is as follows: A typical element of area is shown in Fig. 130. Its altitude is y, and its width is dx; hence, its area is $y\,dx$. Adding together all such elements of area and then taking the limit of the sum, we have for the required area the integral $\int y\,dx$. *In every case the student should make a suitable sketch, indicate a typical element, express its area in terms of the coordinates of the points involved, and then set up the required integral.*

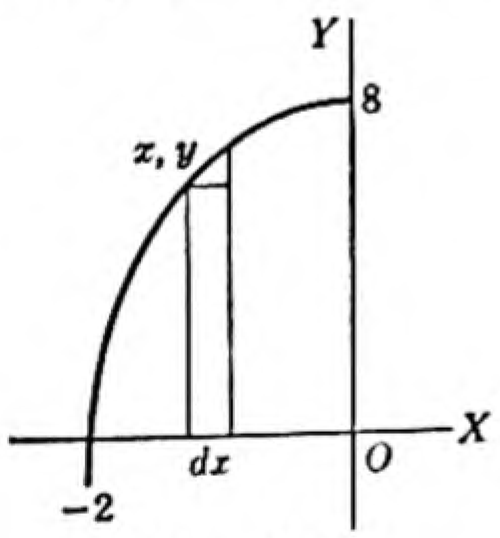


FIG. 130.

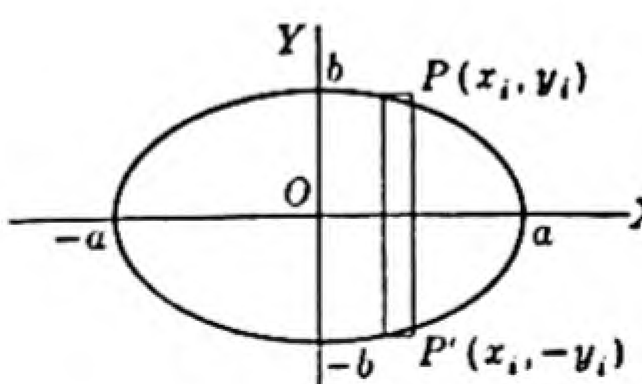


FIG. 131.

Example 3. Find the area bounded by the ellipse whose parametric equations are

$$x = a\cos\varphi \qquad y = b\sin\varphi$$

We form n rectangular elements of area by drawing vertical lines at a distance Δx apart (Fig. 131). Consider a typical element as indicated in the figure. If the point P has coordinates (x_i,y_i), then the point P' has coordinates $(x_i,-y_i)$, and the altitude of the element is $P'P = 2y_i$. Its area is then $2y_i\,\Delta x$, and the entire area is approxi-

mately $\sum_{i=1}^{n} 2y_i \, \Delta x$. It is clear from the figure that we may obtain as good an approximation to the area as is desired by taking n sufficiently large and Δx sufficiently small. Hence, the area is the limit of this sum,

$$A = \int_{-a}^{a} 2y \, dx = 2 \int_{-a}^{a} y \, dx$$

To evaluate this integral, notice that the coordinates of P must satisfy the parametric equations of the ellipse. Hence

$$\begin{aligned} A &= 2\int_{-a}^{a} y \, dx = 2 \int_{\pi}^{0} b \sin \varphi(-a \sin \varphi) \, d\varphi \\ &= 2ab \int_{0}^{\pi} \sin^2 \varphi \, d\varphi = ab \int_{0}^{\pi} (1 - \cos 2\varphi) \, d\varphi \\ &= ab \left[\varphi - \tfrac{1}{2} \sin 2\varphi \right]_{0}^{\pi} = \pi ab \end{aligned}$$

This result is so frequently useful that the student may well remember that *the area bounded by an ellipse is π multiplied by the product of the semiaxes.* Note that, when $b = a$, the ellipse becomes a circle; this formula gives πa^2 for the area, a fact already well-known to the reader.

Evidently, because of symmetry, we could have found the area in the first quadrant and then multiplied the result by 4. Thus

$$\begin{aligned} A &= 4 \int_{0}^{a} y \, dx = -4 \int_{\pi/2}^{0} ab \sin^2 \varphi \, d\varphi \\ &= 2ab \int_{0}^{\pi/2} (1 - \cos 2\varphi) \, d\varphi = 2ab \left[\varphi - \tfrac{1}{2} \sin 2\varphi \right]_{0}^{\frac{\pi}{2}} = \pi ab \end{aligned}$$

EXERCISES

1. Find the area bounded by the curve $y = 2x - x^2$ and the x axis.
2. Find the area bounded by the curve $y = 4 - x^2$ and the x axis. Choose elements of area in two different ways.
3. Find the area in the fourth quadrant bounded by the curve $y = x^3 - 8$. Choose elements of area in two different ways.
4. Find the area bounded by the parabola $y = x^2$ and the lines $y = 1$ and $y = 4$.
5. Find the area bounded by the parabola $y = x^2$ and the line $2x - y + 3 = 0$.
6. Find the area bounded by the curve $y = x^3$, the y axis, and the lines $y = 1$ and $y = 8$.
7. Find the area in the first quadrant bounded by the curve $y = x^3$ and the line $y = 4x$.
8. Find the area bounded by the parabolas $y^2 = ax$ and $x^2 = ay$.
9. Find the area bounded by $y = x^2 - 6x + 10$ and the lines $x = 6$ and $y = 2$.
10. Find by integration the area of the triangle bounded by the lines

$$x + 3y - 8 = 0$$

$5x - y - 8 = 0$, $x - y + 4 = 0$. Verify by elementary geometry.

11. Find the area between the parabolas $y^2 = a^2 + ax$ and $y^2 = a^2 - ax$.
12. Find by integration the area of the circle $x^2 + y^2 = a^2$.
13. Solve Exercise 12, using the equations $x = a \cos \varphi$, $y = a \sin \varphi$.

14. Find the area of the ellipse, using the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

15. Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$.
16. Find the area bounded by the curves $y = e^x$, $y = e^{-x}$, and the line $x = 4$.
17. Find the area enclosed by the curves $5y^2 = 16x$ and $y^2 = 8x - 24$.
18. Find the smaller area enclosed by the curves $3y^2 = 16x$ and $x^2 + y^2 = 25$.
19. Find the area common to the two circles $x^2 + y^2 = 25$ and

$$x^2 + y^2 - 16x + 39 = 0$$

20. Find the area bounded by the parabolas $x^2 - 2x - y - 3 = 0$ and

$$x^2 - 6x + y + 3 = 0$$

21. Find the area in the first quadrant bounded by the curve $y = e^{-x}$. Choose elements of area in two different ways.
22. Find the area in the fourth quadrant bounded by $y = \ln x$. Choose elements of area in two different ways.
23. Find the area bounded by $y = \cos x$ and $y = \sin x$ between two successive intersections.
24. Find the area bounded by $y = \sin x$ and $y = a \sin x$ between two successive intersections where $a > 1$; where $a < 1$.

25. Find the area bounded by the parabola $x^2 = 4ay$ and the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$.

26. Find the total area between the curve $y = \dfrac{x}{1 + x^2}$ and the line $y = \dfrac{x}{5}$.

27. Find the area between the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote.

28. Find the area enclosed by the curve $y^2 = x^3(4 - x)$.
29. Find the area bounded by the curve $(x/a)^2 + (y/b)^{2/3} = 1$.
30. Find the area bounded by the x axis and one arch of the cycloid

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

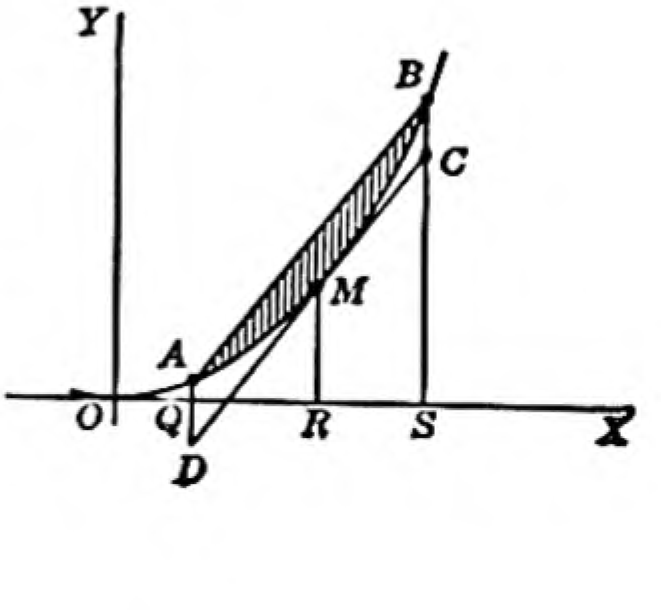
31. Find the area bounded by the four-cusped hypocycloid

$$x = a\cos^3\varphi \qquad y = a\sin^3\varphi$$

32. Solve Exercise 31, using the equation $x^{2/3} + y^{2/3} = a^{2/3}$.
33. Find the area bounded by the coordinate axes and the parabolic arc

$$x^{1/2} + y^{1/2} = a^{1/2}$$

34. Solve Exercise 33, using the parametric equations $x = a\cos^4\theta$, $y = a\sin^4\theta$.
35. Find the area of a circular sector of radius a and angle α.
36. Given the parabola $y = cx^2$. Let A and B be any two points upon it. Let M be the point at which the tangent is parallel to AB. If AQ, MR, BS are parallel to the axis of the parabola, MR is midway between AQ and BS (Exercise 20, page 98). Prove that the area of the shaded parabolic segment is two-thirds of the area of the circumscribing parallelogram $ABCD$. Note that this proposition is true for any segment of any parabola.



110. Plane Areas, Polar Coordinates. Suppose we wish to find the area enclosed by a curve $r = f(\theta)$ and two given radius vectors $\theta = \alpha$ and $\theta = \beta$ $(\alpha < \beta)$. We shall assume that $f(\theta)$ is a continuous function in the interval $\alpha \leqq \theta \leqq \beta$. Let the angle $\beta - \alpha$ be divided into n angles, each of measure $\Delta\theta$ (radian measure), by drawing radius vectors of lengths $r_1, r_2, \ldots, r_{n-1}$ (Fig. 132). Let the radius vector for $\theta = \alpha$ be r_0 so that $r_0 = f(\alpha)$, and let $r_n = f(\beta)$. Now, with O as center, describe arcs with radii $r_0, r_1, r_2, \ldots, r_{n-1}$ that form n circular sectors, each with central angle $\Delta\theta$, as indicated in the figure. These circular sectors are elements of area, and the area of a typical element is $\frac{1}{2}r_i^2\,\Delta\theta$.*

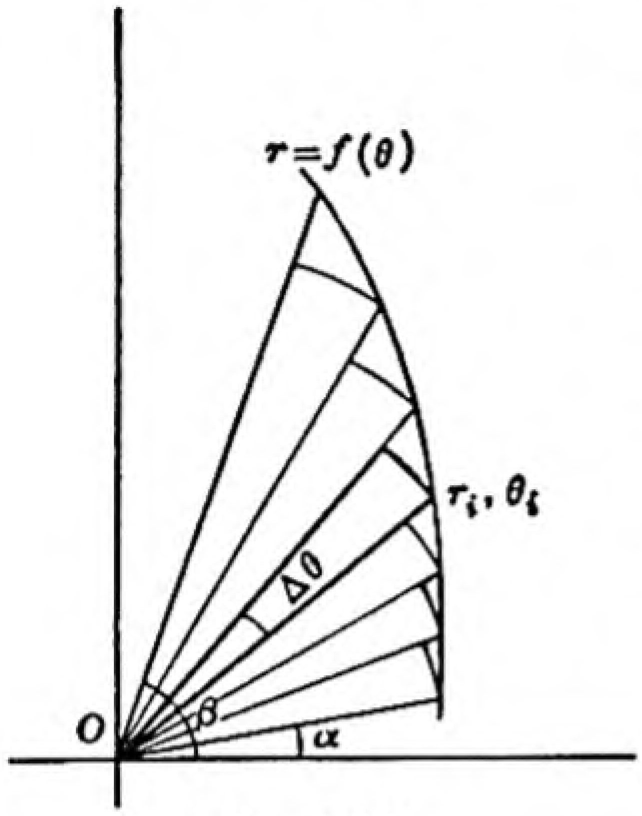


FIG. 132.

We shall assume it to be clear from the figure that the sum of these elements of area, $\sum_{i=0}^{n-1} \frac{1}{2}r_i^2\,\Delta\theta$, gives an arbitrarily close approximation to the required area if $\Delta\theta$ is sufficiently small and n correspondingly large. Hence, the required area is given by the limit of this sum taken for $\Delta\theta \to 0$ and, therefore, for $n \to \infty$. Since r is a function of θ, this limit is precisely the kind contemplated in the fundamental theorem. Therefore

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \tfrac{1}{2}r_i^2\,\Delta\theta = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta$$

and

$$A = \tfrac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

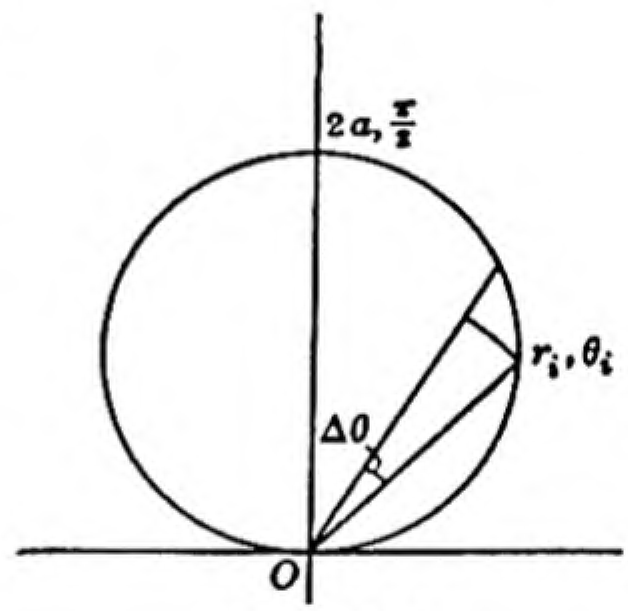


FIG. 133.

Example 1. Find the area bounded by the circle $r = 2a \sin\theta$ (Fig. 133). A typical element of area is shown. Since $r^2 = 4a^2 \sin^2\theta$, we have

$$A = \tfrac{1}{2}\int_0^\pi r^2\,d\theta = 2a^2\int_0^\pi \sin^2\theta\,d\theta$$
$$= a^2\int_0^\pi (1 - \cos 2\theta)\,d\theta = \pi a^2$$

In general, it is wise to take into account considerations of symmetry and find the smallest area of which the required area is a multiple. Following this principle, we

* From elementary geometry the area A_s of a circular sector of radius r and central angle φ is to the area A_c of the whole circle as φ (in radians) is to 2π. Therefore, $A_s/A_c = \varphi/2\pi$, that is, $A_s/\pi r^2 = \varphi/2\pi$, and $A_s = \frac{1}{2}r^2\varphi$ (see also Exercise 35, page 299).

find that the area would be $A = 2A_1$, where

$$A_1 = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = 2a^2\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{\pi}{2}a^2$$

Example 2. Find the area enclosed by the curve $r^2 = a^2 \sin\theta$. If we restrict r to positive values only, we have $r = a\sqrt{\sin\theta}$, and the curve is the heavy curve shown in Fig. 134. A typical element of area is shown. To find the area bounded by this heavy curve, let us use the fact that it is symmetrical to the line $\theta = \pi/2$. The area in the first quadrant is

$$A_1 = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{a^2}{2}\int_0^{\pi/2} \sin\theta\,d\theta$$

$$= -\frac{a^2}{2}\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{a^2}{2}$$

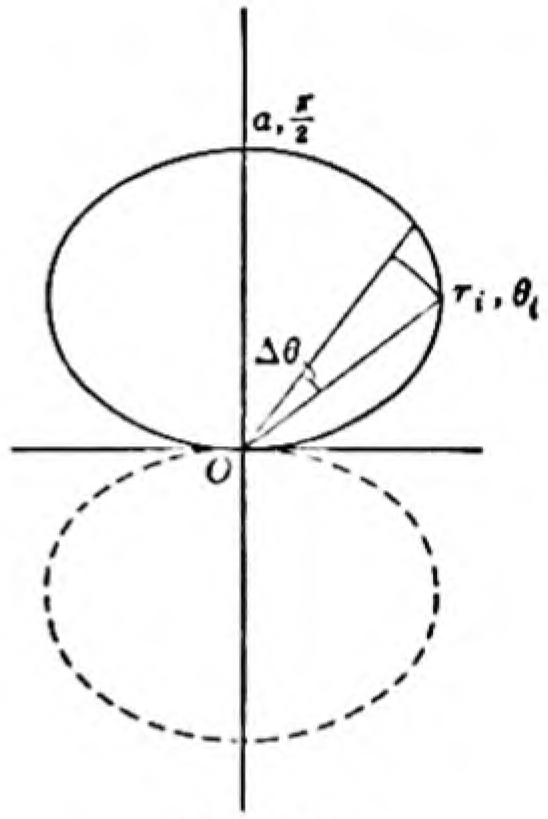


FIG. 134.

Therefore, the required area is $A = 2A_1 = a^2$.

Usually, polar coordinates are defined to admit negative values of r. In this case, the equation

$$r^2 = a^2 \sin\theta$$

becomes $r = \pm a\sqrt{\sin\theta}$, and the curve consists of two loops, the dotted curve as well as the heavy curve of Fig. 134 both belonging to the graph. The total area enclosed by the curve is then $A = 4A_1 = 2a^2$.

Example 3. Find the area, Fig. 135, bounded on the left by the vertical line through $(\frac{3}{4}a,0)$ and on the right by the cardioid $r = a(1 + \cos\theta)$. We first observe that the equation of this vertical line is $r = \frac{3}{4}a\sec\theta$. The intersections of the line and cardioid are found as follows:

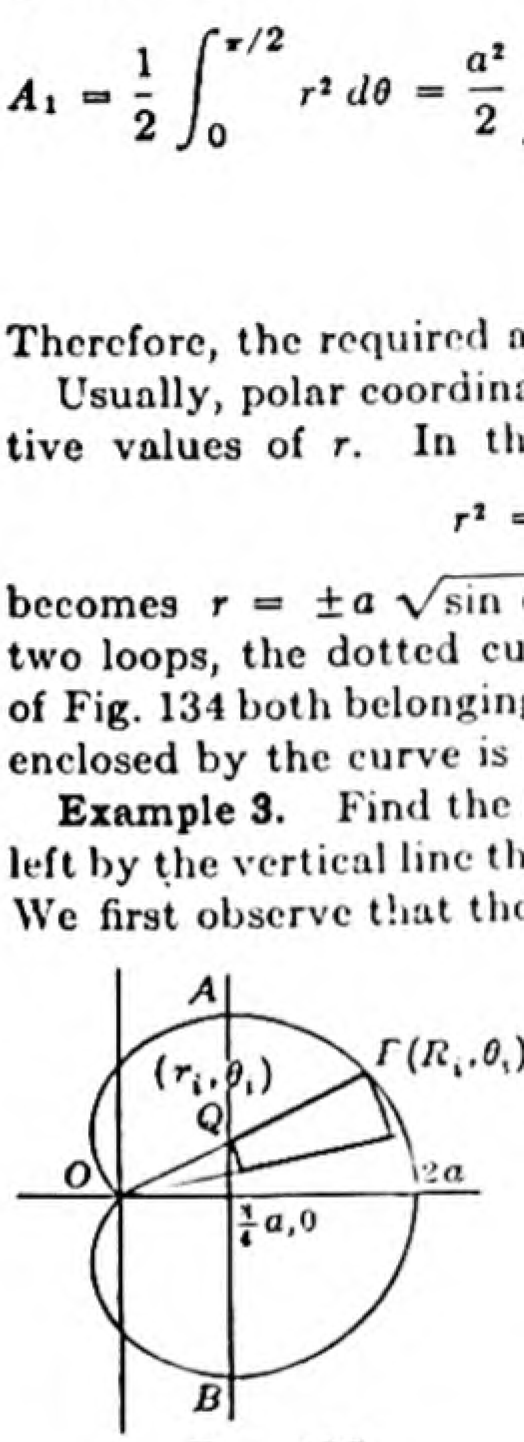


FIG. 135.

$$r = a(1 + \cos\theta) \qquad (1)$$
$$r = \tfrac{3}{4}a\sec\theta \qquad (2)$$

Substituting from (2) into (1),

$$\tfrac{3}{4}a\sec\theta = a(1 + \cos\theta)$$
$$3\sec\theta = 4 + 4\cos\theta$$
$$4\cos^2\theta + 4\cos\theta - 3 = 0$$
$$(2\cos\theta + 3)(2\cos\theta - 1) = 0$$

Evidently $2\cos\theta + 3 \neq 0$ since $\cos\theta$ is not less than $-$

Therefore

$$2\cos\theta - 1 = 0$$

and $\cos\theta = \frac{1}{2}$. Therefore $\theta = \pi/3$ or $\theta = -\pi/3$. Point A has coordinates $(\frac{3}{2}a,\pi/3)$, and B has coordinates $(\frac{3}{2}a, -\pi/3)$.

We next calculate the area of the element outlined in the figure. The sector with central angle $\Delta\theta$ and radius OP has area $\frac{1}{2}\overline{OP}^2\,\Delta\theta = \frac{1}{2}R_i^2\,\Delta\theta$ where $R_i = a(1 + \cos\theta_i)$ is the radius vector of a point on the *cardioid*. The sector with the same central angle and radius OQ has area $\frac{1}{2}\overline{OQ}^2\,\Delta\theta = \frac{1}{2}r_i^2\,\Delta\theta$ where $r_i = \frac{3}{4}a\sec\theta_i$ is the radius vector of a point on the *line*. Note that P and Q have the same vectorial angle θ_i. The element, therefore, has area $\frac{1}{2}R_i^2\,\Delta\theta - \frac{1}{2}r_i^2\,\Delta\theta = \frac{1}{2}(R_i^2 - r_i^2)\,\Delta\theta$. The required area is the limit of the sum of such elements. If we find the part of the area appearing in the first quadrant and then multiply by 2, we shall have the required area.

$$
\begin{aligned}
2\int_0^{\pi/3} \tfrac{1}{2}(R^2 - r^2)\,d\theta &= \int_0^{\pi/3} [a^2(1+\cos\theta)^2 - \tfrac{9}{16}a^2\sec^2\theta]\,d\theta \\
&= a^2\int_0^{\pi/3} (1 + 2\cos\theta + \cos^2\theta - \tfrac{9}{16}\sec^2\theta)\,d\theta \\
&= a^2\int_0^{\pi/3} (1 + 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{9}{16}\sec^2\theta)\,d\theta \\
&= a^2\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{9}{16}\tan\theta\right]_0^{\frac{\pi}{3}} \\
&= a^2\left(\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8} - \frac{9}{16}\sqrt{3}\right) \\
&= \frac{a^2}{16}(8\pi + 9\sqrt{3})
\end{aligned}
$$

EXERCISES

In these exercises, use the system of polar coordinates in which both positive and negative values of r are admitted.

1. Find the area of the circle $r = a$.

2. Find the area of the circle $r = 2a\cos\theta$.

3. Find the area of the cardioid $r = a(1 - \cos\theta)$.

4. Find the area of the cardioid $r = a(1 + \sin\theta)$.

5. Find the area cut off from the parabola $r = a\sec^2(\theta/2)$ by the latus rectum.

6. Find the area bounded by the curve $r^2 = a^2\cos\theta$.

7. Find the area bounded by the curve $r^2 = a^2\cos 2\theta$.

8. Find the area bounded by the curve $r = 2a\cos^2\theta$.

9. Find the total area of all the loops of the curve $r = a\sin 2\theta$.

10. Find the area bounded by the curve $r^2 = a^2\sin 2\theta$.

11. Find the area swept over by the radius vector of the curve $r = a\theta$ (spiral of Archimedes) when θ changes from 0 to 2π.

12. Show that the area bounded by the hyperbolic spiral $r\theta = a$ and any two of its radius vectors is proportional to the difference between the lengths of these vectors.

13. Show that the area of the three loops of the curve $r = a\cos 3\theta$ is one-fourth the area of the circumscribing circle.

14. Show that the total area of all the loops of the curve $r = a\sin n\theta$ is one-fourth or one-half the area of the circumscribing circle according as n is odd or even.

15. Find the area inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = 2a\cos\theta$.

16. Find the area inside the loops of $r = 2a\cos 2\theta$ and outside the circle $r = a$.

17. Find the area of the smaller loop of the limaçon $r = a(1 + 2\cos\theta)$.

18. Find the area common to the two circles $r = a$ and $r = 2a\cos\theta$.

19. Find the area common to the two circles $r = a$ and $r = 4a\cos\theta$.

20. Find the area inside $r = a\cos\theta$ and outside $r = a(1 - \cos\theta)$.

21. Find the area between the parabola $y = x^2$ and the line $y = 4x$ (change to polar coordinates).

22. Find the area between the line $y = x$ and the curve $y = x^3$.

23. Find the area outside the circle $x^2 + y^2 = a^2$ and bounded by the lines $y = x$ and $x = a$.

24. Find the area bounded by the curve $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

111. Volumes of Revolution. Suppose we have an area bounded by plane curves and a straight line, and suppose this area (the *generating area*) rotates about the straight-line boundary to form a solid of revolution. For simplicity, suppose the line to be vertical. From the generating area, let n elements of area be formed by drawing horizontal lines at a distance Δh apart, and let the length of a typical element be denoted by r_i. Each of these elements of area will generate an element of volume, namely, a right circular cylinder, or *disk*, of altitude Δh and radius r_i.

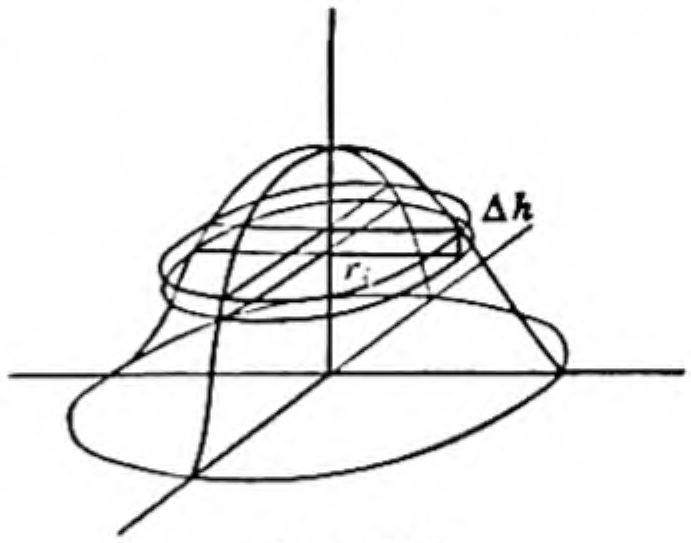


FIG. 136.

The volume of a typical disk (Fig. 136) is $\pi r_i^2\,\Delta h$. The volume of the solid is approximately equal to the sum of these n disks, $\sum_{i=1}^{n} \pi r_i^2\,\Delta h$. If Δh is made to approach zero and, at the same time, n is increased indefinitely, the limit of this sum is the exact volume. Since r is, in general, a function of h, this limit is of the kind contemplated in the fundamental theorem. Hence

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi r_i^2\,\Delta h = \pi \int_{h_1}^{h_2} r^2\,dh$$

with limits of integration chosen to include the entire volume.

Actually, we have given no definition for a *volume* bounded by curved surfaces. Such a definition will be given in Chap. 18. In the meantime, we shall assume that we have a correct intuitive idea of *volume*.

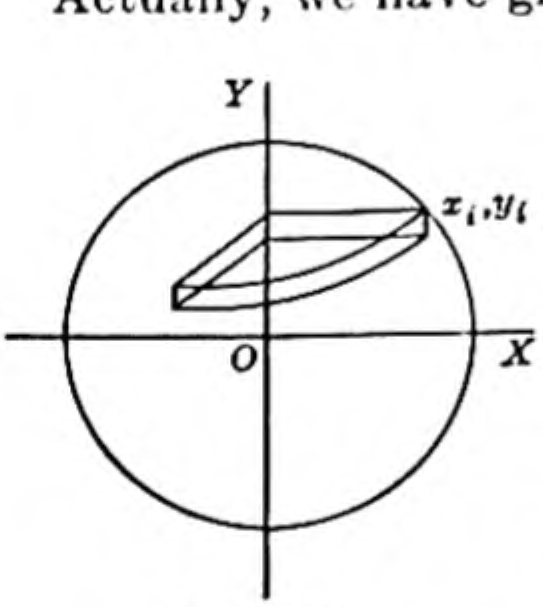


FIG. 137.

Example 1. Find the volume of a sphere. This volume is generated by rotating the area bounded by a semicircle about its diameter. Let the circle be $x^2 + y^2 = a^2$, and rotate the right-hand semicircle about the y axis. From this semicircular area, form n elements of area. A typical element is of width x_i and altitude Δy, where (x_i,y_i) is a point of the circle (Fig. 137). When rotated, this generates a circular disk of radius x_i, altitude Δy, and volume $\pi x_i^2\,\Delta y$. Adding together these elements of volume and taking the limit of the sum, we get $V = \pi \int_{-a}^{a} x^2\,dy$. Since (x,y) is a point of the circle, $x^2 = a^2 - y^2$, and

$$\begin{aligned} V &= \pi \int_{-a}^{a} (a^2 - y^2)\,dy = \pi \left[a^2y - \tfrac{1}{3}y^3 \right]_{-a}^{a} \\ &= \pi(a^3 - \tfrac{1}{3}a^3 + a^3 - \tfrac{1}{3}a^3) = \tfrac{4}{3}\pi a^3 \end{aligned}$$

Observe that, because of symmetry, we could rotate one quadrant of the circle

and multiply the result by 2, thus

$$V = 2\pi \int_0^a x^2\, dy$$

Example 2. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ rotates about the line $x = a$. Find the volume generated. Here (Fig. 138), a typical element of volume is obtained by rotating an element of area whose length is $a - x_i$ and height Δy about the line $x = a$. The volume of the element is, therefore, $\pi(a - x_i)^2\, \Delta y$. Adding all such elements and taking the limit of the sum, we obtain

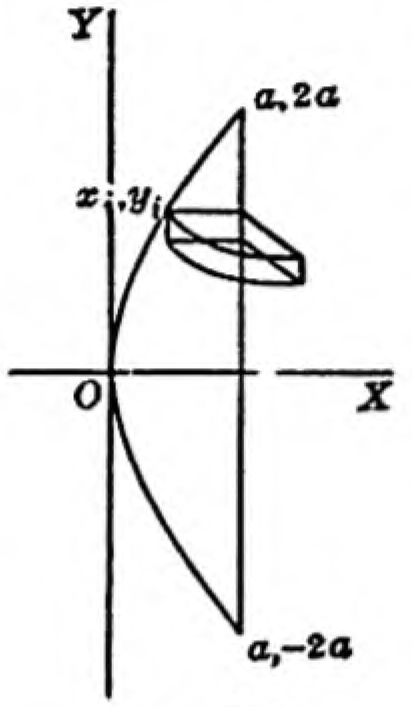


FIG. 138.

$$V = \pi \int_{-2a}^{2a} (a - x)^2\, dy = \pi \int_{-2a}^{2a} (a^2 - 2ax + x^2)\, dy$$
$$= \pi \int_{-2a}^{2a} \left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right)$$
$$dy = \pi \left[a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2}\right]_{-2a}^{2a} = \frac{32\pi a^3}{15}$$

Now, suppose the generating area is rotated about some line l (say a vertical line) entirely outside the area (Fig. 139). From the area, form horizontal elements of area as indicated in the figure. When rotated about line l, a typical element of area will generate a *washer-shaped* element of volume of outside radius R_i, inside radius r_i, and altitude Δh. The volume of this element is the volume of a disk of radius R_i minus the volume of a disk of radius r_i. Hence

$$\Delta V_i = \pi R_i^2\, \Delta h - \pi r_i^2\, \Delta h$$
$$= \pi(R_i^2 - r_i^2)\, \Delta h$$

The sum of all of these elements gives an approximation to the required volume, $\sum_{i=1}^{n} \pi(R_i^2 - r_i^2)\, \Delta h$.

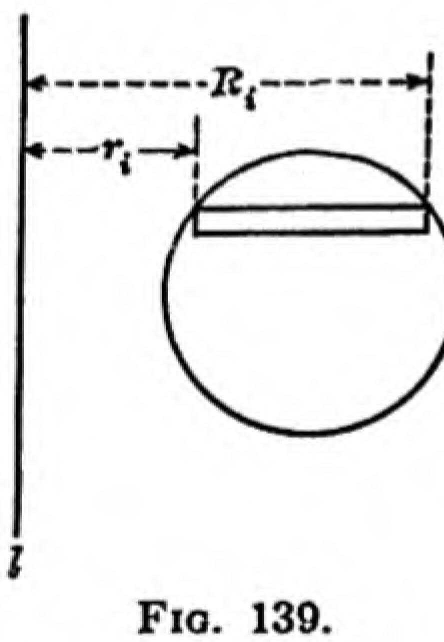


FIG. 139.

The limit of this sum is the required volume; and, by the fundamental theorem, we have

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(R_i^2 - r_i^2)\, \Delta h = \pi \int_{h_1}^{h_2} (R^2 - r^2)\, dh$$

with suitable limits of integration chosen to include the entire volume.

Example 3. Find the volume of the torus obtained by rotating the area bounded by the circle $x^2 + y^2 = a^2$ around the line $x = b$ $(b > a)$. From the circular area, form horizontal elements (Fig. 140). The typical element of area, when rotated, will generate a washer-shaped element of volume of thickness Δy, inside radius $(b - x_i)$, and outside radius $(b - x_i')$. The volume of this element is, therefore

$$\Delta V_i = \pi(b - x_i')^2\,\Delta y - \pi(b - x_i)^2\,\Delta y$$
$$= \pi[2b(x_i - x_i') + (x_i'^2 - x_i^2)]\,\Delta y$$

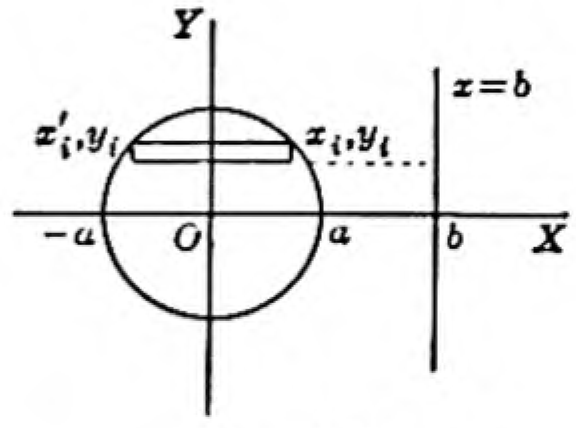


FIG. 140.

Now, $x^2 = a^2 - y^2$, and $x = \pm\sqrt{a^2 - y^2}$.

Hence,

$$x_i = \sqrt{a^2 - y_i^2} \qquad \text{and} \qquad x_i' = -\sqrt{a^2 - y_i^2}$$

Therefore

$$\Delta V_i = 4b\pi\sqrt{a^2 - y_i^2}\,\Delta y \qquad \text{and} \qquad V = 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\,dy$$

Setting $y = a \sin \theta$, we obtain

$$V = 4a^2b\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = 2a^2b\pi \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta = 2\pi^2a^2b$$

112. Volumes of Revolution, Alternative Method. Suppose, as in the last section, that an area rotates about a fixed line, say a vertical line. From this area, let us form n elements of area by drawing vertical lines at a distance Δr apart (Fig. 141). Let r_i be the distance from the fixed line to the midpoint P_i of the top of a typical element. Let the height h_i of the element be so chosen that P_i lies upon the curve bounding the area. When the area is rotated, each such element of area will generate an element of volume, namely, a *cylindrical shell* of altitude h_i, thickness Δr, inside radius $r_i - \frac{1}{2}\Delta r$, and outside radius $r_i + \frac{1}{2}\Delta r$.

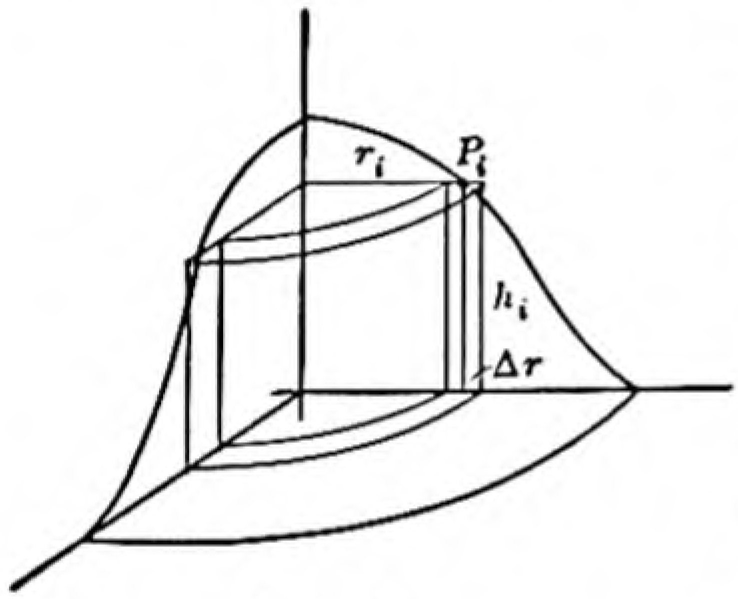


FIG. 141.

The volume of this shell is simply the volume of a right circular cylinder of radius $r_i + \frac{1}{2}\Delta r$ and altitude h_i minus the volume of a right circular cylinder of radius $r_i - \frac{1}{2}\Delta r$ and altitude h_i.

$$\Delta V_i = \pi(r_i + \tfrac{1}{2}\Delta r)^2h_i - \pi(r_i - \tfrac{1}{2}\Delta r)^2h_i = 2\pi r_ih_i\,\Delta r$$

If we add all such shells together, we get an approximation to the volume; thus, the required volume is approximately

$$\sum_{i=1}^{n} 2\pi r_ih_i\,\Delta r$$

The limit of this sum as n is indefinitely increased and Δr made to approach zero is the required volume. Hence, by the fundamental theorem

$$V = 2\pi \int_{r_1}^{r_2} rh\,dr$$

with limits of integration chosen to include the entire volume.

Example 1. This choice of an element of volume is particularly convenient if the line about which the generating area is rotated is not a part of the boundary of that area. Consider the torus of Example 3 of the preceding section. We take vertical elements of area of width Δx (Fig. 142). The height of a typical element is $2y_i$.

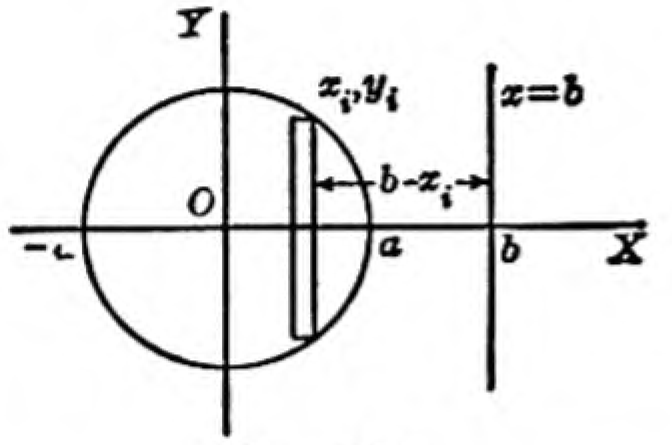


FIG. 142.

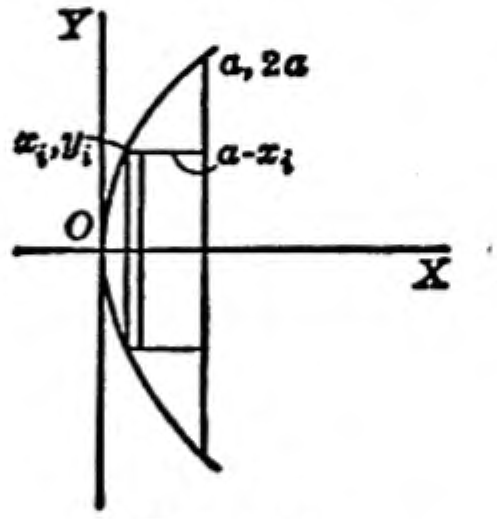


FIG. 143.

When this is rotated about the line $x = b$, it generates a cylindrical shell the volume of which is $2\pi r_i h_i \Delta r = 2\pi(b - x_i) \cdot 2y_i \Delta x$. The required volume is, therefore

$$
\begin{aligned}
V &= \int_{-a}^{a} 2\pi(b - x)\,2y\,dx = 4\pi \int_{-a}^{a} (b - x)y\,dx \\
&= 4\pi \int_{-a}^{a} (b - x)\sqrt{a^2 - x^2}\,dx \\
&= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx - 4\pi \int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx
\end{aligned}
$$

We observe that the second integral is zero since its integrand is an odd function (see Exercise 26, page 292). Therefore

$$
V = 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 2\pi^2 a^2 b
$$

Example 2. The area bounded by the parabola $y^2 = 4ax$ $(a > 0)$ and the line $x = a$ rotates about the line $x = a$ (see Example 2 of the preceding section). Find the volume generated. We consider vertical elements of area of width Δx (Fig. 143). The height of a typical element is $2y_i$. When this is rotated about the line $x = a$, it generates a cylindrical shell of volume

$$
\Delta V = 2\pi r_i h_i \Delta r = 2\pi(a - x_i)2y_i \Delta x
$$

The required volume is, therefore,

$$
\begin{aligned}
V &= 4\pi \int_{0}^{a} (a - x)y\,dx = 4\pi \int_{0}^{a} (a - x)\,2\sqrt{ax}\,dx \\
&= 8\pi\sqrt{a} \int_{0}^{a} (a\sqrt{x} - x^{3/2})\,dx = 8\pi\sqrt{a}\left[\tfrac{2}{3}ax^{3/2} - \tfrac{2}{5}x^{5/2}\right]_{0}^{a} = \tfrac{32}{15}\pi a^3
\end{aligned}
$$

EXERCISES

1. Find the volume of a sphere using cylindrical shells as elements of volume.

2. The area bounded by the parabola $y^2 = 4ax$, its axis, and the latus rectum is revolved about the x axis. Find the volume.

3. The area of Exercise 2 is revolved about the y axis. Find the volume by two methods.

4. Find the volume of a right circular cone of altitude h and radius of base a. Solve by two methods.

5. The area bounded by the hyperbola $x^2 - y^2 = a^2$, the x axis, and the line $x = 2a$ is revolved about the x axis. Find the volume.

6. The area bounded by the curve $y = \frac{1}{4}x^2$, the x axis, and the line $x = 2$ is revolved about the x axis. Find the volume.

7. The area of Exercise 6 is revolved about the y axis. Find the volume by two methods.

8. Find the volume of a paraboloid of revolution of altitude h and radius of base a.

9. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about its major axis. Find the volume (prolate spheroid).

10. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about its minor axis. Find the volume (oblate spheroid).

11. Find the volume of a prolate spheroid, using the equations $x = a \cos \varphi$,

$$y = b \sin \varphi$$

Solve by two methods.

12. The area in the first quadrant bounded by the parabola $y^2 = 2x - 4$ and the line $x = 4$ is revolved about the line $x = 4$. Find the volume.

13. The area bounded by the lines $x = 0$, $x = 2$, $x + y = 4$, and $x - y = 4$ is revolved about the line $x = 4$. Find the volume.

14. The area bounded by the parabolic arc $y = 2\sqrt{x}$, the line $x = 9$, and the line $y = 3$ is revolved about the x axis. Find the volume.

15. The area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$ is revolved about the y axis. Find the volume.

16. Solve Exercise 15, using the equations $x = a \sec \varphi$, $y = a \tan \varphi$.

17. The circle $x = a \cos \theta$, $y = a \sin \theta$ rotates about the line $x = b$ $(a < b)$. Find the volume by two methods, and check by comparing with Example 3, Art. 111.

18. The circle $x^2 + y^2 = 1$ is revolved about the line $x = 5$. Find the volume generated. Use two different methods, and check by comparing with the result of Exercise 17.

19. The area bounded by the x axis and the curve $y = \sin x$ from $x = 0$ to $x = \pi$ is rotated about the x axis. Find the volume.

20. The area of Exercise 19 is rotated about the y axis. Compare results.

21. Find the volume of a spherical segment of one base and altitude h cut from a sphere of radius a.

22. One arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is revolved about the x axis. Find the volume.

23. The right-hand half of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about the y axis. Find the volume.

24. Solve Exercise 23, using the equations $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

25. The area bounded by $y = e^x$, the x axis, and the lines $x = 1$ and $x = 3$ is revolved about the x axis. Find the volume.

26. The area of Exercise 25 is revolved about the y axis. Find the volume.

27. The area bounded by the curve $y = e^x$, the x axis, and the lines $x = -1$ and $x = 1$ is revolved about the line $x = 1$. Find the volume.

28. The area of Exercise 27 is revolved about $x = -1$. Find the volume.

29. That area in the first quadrant under the curve $y = e^{-x}$ is revolved about the x axis. Find the volume.

30. The area under the curve $y = \frac{8a^3}{x^2 + 4a^2}$ (witch) is revolved about the x axis. Find the volume.

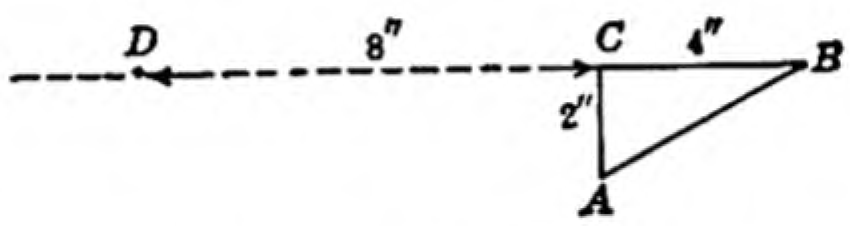
31. A ring of altitude $2h$ is generated by rotating about the y axis the area bounded by the circle $x^2 + y^2 = a^2$ and the chord of length $2h$ that is parallel to the y axis. Find the volume of this ring. Does it depend upon the radius of the circle?

32. That part of the curve $xy = 1$ from $x = 1$ to ∞ is revolved about the x axis. Find the volume generated.

33. A round hole of radius a is bored through the center of a solid sphere of radius $2a$. Find the volume cut out.

34. A round hole of radius a is bored through a solid paraboloid of revolution whose base has radius $2a$ and whose altitude is h. The axes of the hole and of the paraboloid coincide. Find the volume cut out.

35. A metal ring is formed by rotating the right triangle ABC horizontally about the point D as shown. Find the volume of this ring.



36. The area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about the line

$$x = c(c > a)$$

Find the volume generated.

37. Solve Exercise 36, using the equations $x = a \cos \varphi$, $y = b \sin \varphi$.

113. Miscellaneous Volumes. It is frequently necessary to calculate volumes whose boundaries are not surfaces of revolution and for which, therefore, the methods of the last two sections do not suffice. Such volumes can, in many cases, be found by use of simple integrals. Although a general method involving multiple integrals will be discussed in Chap. 18, we shall discuss a simple device for finding volumes of solids the areas of whose cross sections by parallel planes can be conveniently expressed. Suppose

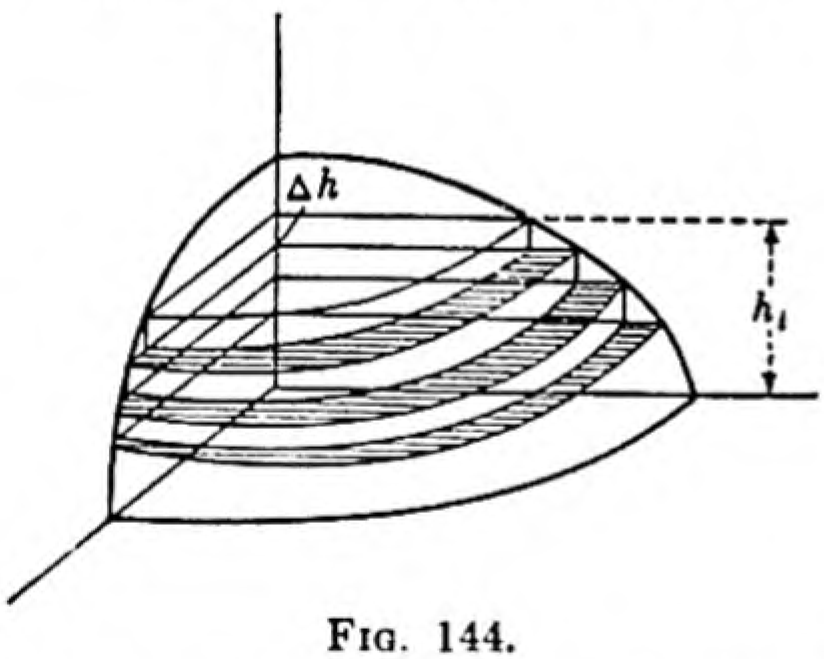


FIG. 144.

such a volume is cut by n parallel planes at a distance Δh apart (Fig. 144). Let the distance of the ith cutting plane from some fixed point (the origin, for example) be h_i. Suppose that the area of the cross section cut by this plane can be expressed as a function of h_i, say $A_i = f(h_i)$. Consider the volume of the right cylinder (or prism) of height Δh and base area A_i standing upon that area. This is called a *lamina*, and its volume is $A_i\,\Delta h$. Now, add together the n laminae, and their sum is an approximation to the required volume. By an already familiar argument, the limit of this sum can be expressed as a definite integral,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A_i \, \Delta h = \int_{h_1}^{h_2} A \, dh$$

the limit being taken as $\Delta h \to 0$ and $n \to \infty$, with proper limits of integration chosen to include the entire volume. Of course, A and dh should be expressed in terms of a single variable if the integral is to be evaluated.

Example 1. Find the volume cut from the paraboloid $x^2 + 4y^2 = z$ by the plane $z = 1$. Let the volume be cut into n laminae by planes parallel to the xy plane and at a distance Δz apart (Fig. 145). Each such plane makes an elliptical cross section whose area is π times the product of the semiaxes. The area of the ith lamina (one fourth of which is shown in the figure) is, therefore, $\pi CA \cdot CB$.

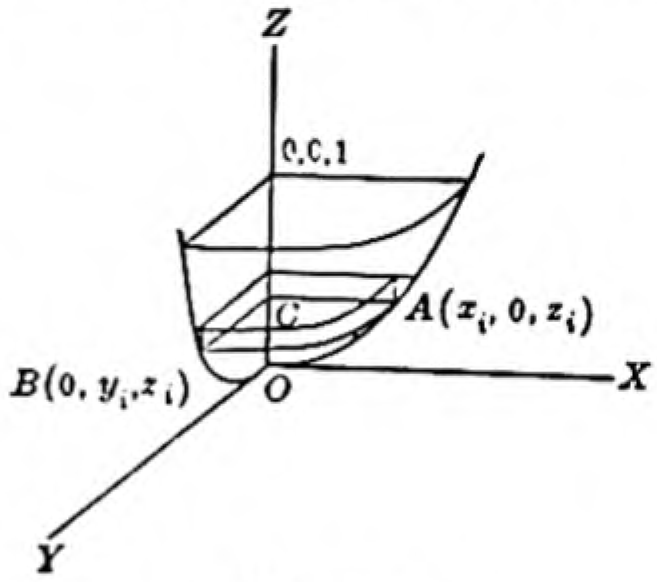


FIG. 145.

Now, points A and B have the same z coordinate, and so we shall express CA and CB in terms of z_i, the distance of one face of the lamina from the origin. Note that A is a point on the trace of the surface in the xz plane; and so, for this point, $x_i^2 = z_i$, or $x_i = \sqrt{z_i}$, the positive square root being taken since the x coordinate of A is positive. Similarly, B is a point on the trace in the yz plane, and therefore $4y_i^2 = z_i$, and $y_i = \frac{1}{2}\sqrt{z_i}$, the positive square root being taken since B has a positive y coordinate. Therefore

$$A_i = \pi x_i y_i = \pi \sqrt{z_i} \cdot \frac{1}{2} \sqrt{z_i} = \frac{\pi}{2} z_i$$

The volume of the lamina is

$$\Delta V_i = \pi x_i y_i \, \Delta z = \frac{\pi}{2} z_i \, \Delta z$$

Therefore, the required volume is

$$V = \int_0^1 \pi x y \, dz = \frac{\pi}{2} \int_0^1 z \, dz = \frac{\pi}{2} \left[\frac{z^2}{2} \right]_0^1 = \frac{\pi}{4}$$

Example 2. Find the volume in the first octant under the plane $z = y$ and inside the cylinder $y^2 = b(a - x)$.

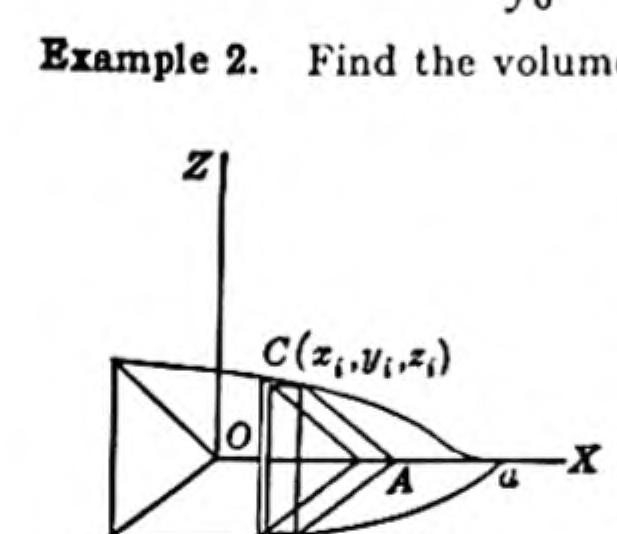


FIG. 146.

First choice of element. Let the volume be cut into n laminae by planes parallel to the yz plane (Fig. 146) and at a distance Δx apart. The ith plane will cut the xy plane in a line AB parallel to the y axis, the cylinder in a vertical element BC, and the plane $z = y$ in a line joining A and C. The lamina is, therefore, triangular in cross section, and its area can be expressed in terms of the coordinates of the point $C(x_i, y_i, z_i)$. Thus, $A_i = \frac{1}{2} y_i z_i$. The volume of the element of volume is, therefore,

$$\Delta V_i = \tfrac{1}{2} y_i z_i \, \Delta x$$

Hence, the required volume is $V = \frac{1}{2} \int_0^a yz \, dx$. Again, we have, for any point C on the curve of intersection of the cylinder and the plane $z = y$, $z = y$ and

$$y^2 = b(a - x)$$

Therefore
$$V = \frac{1}{2}\int_0^a yz\,dx = \frac{1}{2}\int_0^a y^2\,dx = \frac{b}{2}\int_0^a (a - x)\,dx$$
$$= \frac{b}{2}\left[ax - \frac{x^2}{2}\right]_0^a = \frac{b}{2}\left(a^2 - \frac{a^2}{2}\right) = \frac{1}{4}a^2b$$

Second choice of element. Let the volume be cut into n laminae by planes parallel to the xz plane and at a distance Δy apart. The ith plane (Fig. 147) will cut the xy plane in a straight line AB parallel to the x axis, the yz plane in a straight line AD parallel to the z axis, the plane $z = y$ in a straight line parallel to the x axis, and the cylinder in the vertical element BC. The lamina is therefore rectangular in cross section, and its area can be expressed in terms of the coordinates of the point $C(x_i,y_i,z_i)$. Thus, $A_i = x_iz_i$. The volume of the element of volume is, therefore, $\Delta V_i = x_iz_i\,\Delta y$. Hence, the required volume is

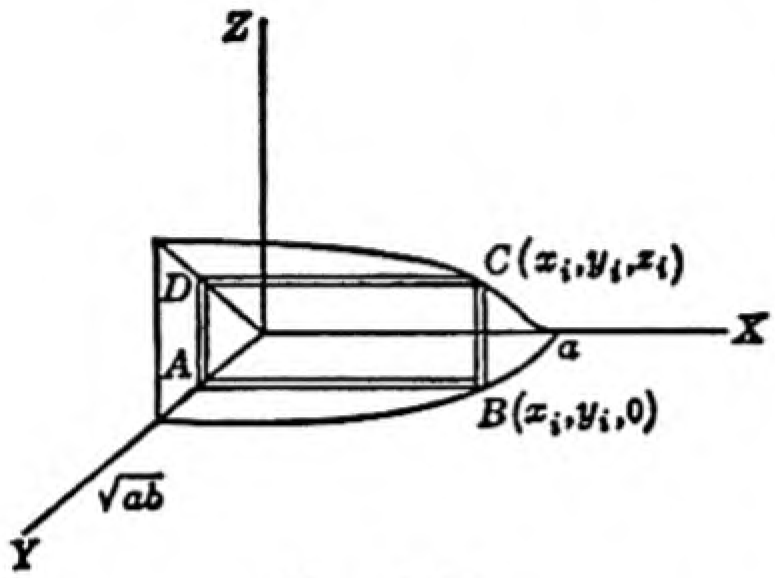


FIG. 147.

$$V = \int_0^{\sqrt{ab}} xz\,dy$$

For any point C on the curve of intersection of the plane $z = y$ and the cylinder, we have $z = y$ and $y^2 = b(a - x)$, that is, $x = a - (y^2/b)$. Therefore

$$V = \int_0^{\sqrt{ab}} \left(a - \frac{y^2}{b}\right) y\,dy = \int_0^{\sqrt{ab}} \left(ay - \frac{y^3}{b}\right) dy$$
$$= \left[\frac{ay^2}{2} - \frac{y^4}{4b}\right]_0^{\sqrt{ab}} = \frac{1}{4}a^2b$$

The student will recognize the method of finding a volume of revolution by use of disks or washer-shaped elements as a special case of this lamina method.

EXERCISES

1. A tetrahedron has three mutually perpendicular faces. The three mutually perpendicular edges are of lengths a, b, c, respectively. Find the volume.

2. Find the volume of a right elliptic cone if the semiaxes of the base are of lengths a and b and the altitude is h.

3. Find the volume of an ellipsoid whose three semiaxes are a, b, c, respectively.

4. A right pyramid of altitude h has a square base of side a. Find the volume.

5. A right pyramid of altitude h has a rectangular base with sides a and b. Find the volume.

6. Find the volume bounded by the surface $x^2 + 16y^2 = 16z$ and the plane $z = 4$.

7. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x$.

8. Find the volume in the first octant bounded by the cylinder $y^2 = 4x$ and the planes $y = 2x - 4$ and $z = 3x$.

9. Find the volume in the first octant bounded by the cylinders $y^2 = ax$, $y^2 = 4ax$ and the planes $z = x$ and $x = a$.

10. Find the volume bounded by the cylinders $y^2 = ax$, $y^2 + z^2 = a^2$ and the plane $x = a$.

11. Find the smaller volume bounded by the surfaces $x^2 + y^2 = 25$, $4y^2 = 9x$, $z = y$, $z = 0$.

12. Find the volume cut from the cylinder $y^2 = 4ax$ by the planes $x = a$, $x = 2a$, $z = 0$, $z = a$.

13. Find the volume cut from the cylinder $y^2 = 4ax$ by the planes $x + z = a$ and $z = 0$.

14. Find the volume in the first octant bounded by the cylinder $y^2 = 4ax$ and the planes $z = 0$, $y = x$, $x + z = 4a$.

15. A surface is generated by a circle of variable size moving with its plane always parallel to the yz plane and having the ends of a diameter always on the x axis and the line $z = x$, $y = 0$. Find the volume in the first and second octants bounded by this surface and the plane $x = a$.

16. A surface is generated by an ellipse of variable size moving with its plane always parallel to the yz plane and having the ends of its major axis on the x axis and the curve $z = \sqrt{x}$, $y = 0$. If the minor axis of the ellipse is always one-half the major axis, find the volume bounded by the surface and the plane $x = 4$.

17. The axes of two right circular cylinders, each of radius a, intersect at right angles. Find the volume common to both.

18. A hole of square cross section, a in. on the side, is cut through a right circular cylinder of diameter $2a$. The axis of the hole intersects the axis of the cylinder at right angles. Find the volume cut out.

19. Find the volume cut from the cylinder $x^2 - y^2 = a^2$ by the planes $z = 0$, $x = 2a$, and $z = x - a$.

20. A surface is generated by a circle of variable size moving parallel to the yz plane and having the ends of a diameter on the curves $x^2 + 2y - 4 = 0$, $z = 0$ and $x^2 + 8y - 4 = 0$, $z = 0$. Find the volume bounded by the surface and the plane $y = 0$.

21. A surface is generated by a square of variable size moving with its plane always parallel to the xz plane and having the ends of one side always on the line $y = x$, $z = 0$ and the parabola $y^2 = 4ax$, $z = 0$. Find the volume bounded by this surface.

22. Find the volume bounded by the cylinders $x^2 + y^2 = a^2$, $2y^2 = 2a^2 - az$, and the xy plane.

23. A surface (*conoid*) is generated by a straight line that moves always parallel to the xz plane and has points in common with the circle $x^2 + y^2 = a^2$, $z = 0$ and the line $z = c$, $x = 0$. Find the volume bounded by this surface and the plane $z = 0$.

24. Same as Exercise 23, except that the moving line always has points in common with the line $z = c$, $x = 0$ and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $z = 0$.

25. A surface (*hyperbolic paraboloid*) is generated by a line that moves always parallel to the xz plane and has points in common with the lines $y + z = a$, $x = 0$ and $x = b$, $z = 0$. Find the volume in the first octant bounded by this surface.

26. A surface is generated by a line moving always parallel to the xz plane and having points in common with the line $y + z = a$, $x = 0$ and the circle $x^2 + y^2 = a^2$, $z = 0$. Find the volume in the first octant bounded by this surface.

27. A square of variable size moves with its plane always parallel to the yz plane. The ends of one diagonal are points of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $z = 0$. Find the volume generated as the square moves from one vertex of the ellipse to the other.

28. The vertex of a cone is at the point $(0,0,a)$. Its base is the circle $x^2 + y^2 = 2bx$, $z = 0$. Find its volume.

114. Length of Arc. It has already been noted (Art. 59) that the length of a curvilinear arc is defined as the limit of the sum of lengths of chords. Thus, if the arc AB (Fig. 148) is divided into n short arcs by the points $P_1, P_2, \ldots, P_{n-1}$ and chords $AP_1, P_1P_2, P_2P_3, \ldots, P_{n-1}B$ are drawn, the length of arc AB is defined as the limit (as n increases indefinitely and the greatest arc approaches zero) of the sum $AP_1 + P_1P_2 + \cdots + P_{n-1}B$. We shall denote A by P_0 and B by P_n. We now discuss a method for evaluating such a limit. Suppose that the equation in rectangular coordinates of the curve of which AB is an arc is $y = f(x)$, and suppose $f(x)$ to have a continuous derivative in the interval $a \leqq x \leqq b$ where a and b are abscissas of A and B, respectively. We shall express the

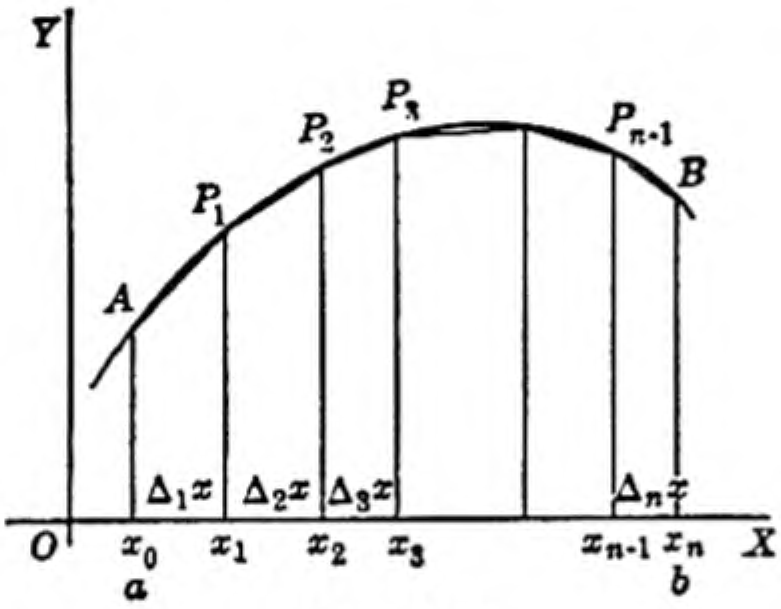


FIG. 148.

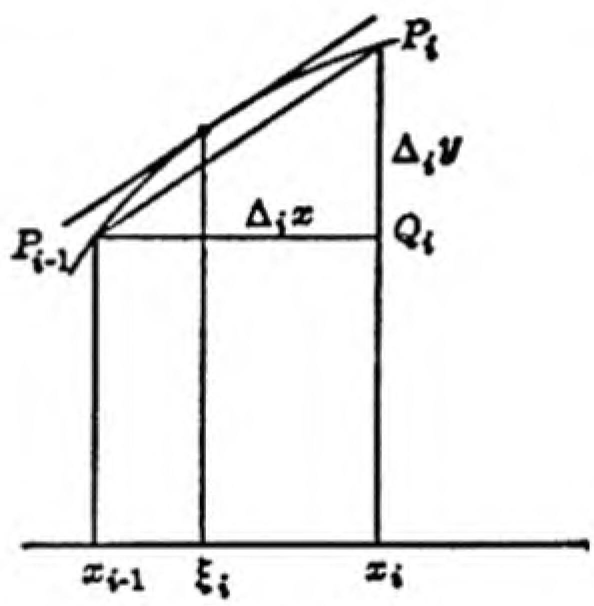


FIG. 149.

lengths of chords in terms of the coordinate system. To this end, consider a typical chord, say $P_{i-1}P_i$ (Fig. 149). Let the point P_i have abscissa x_i, and let $x_0 = a$ and $x_n = b$. Let the projection of the chord $P_{i-1}P_i$ upon the x axis be denoted by $\Delta_ix = x_i - x_{i-1}$, and let $Q_iP_i = \Delta_iy$. According to the theorem of the mean (Art. 77) of the differential calculus, there is a point on the arc $P_{i-1}P_i$ at which the tangent is parallel to the chord $P_{i-1}P_i$. Let the abscissa of this point be ξ_i. Then

$$\frac{\Delta_iy}{\Delta_ix} = f'(\xi_i) \qquad \text{or} \qquad \Delta_iy = f'(\xi_i)\,\Delta_ix$$

But

$$\begin{aligned} P_{i-1}P_i &= \sqrt{\Delta_ix^2 + \Delta_iy^2} \\ &= \sqrt{\Delta_ix^2 + [f'(\xi_i)]^2\,\Delta_ix^2} = \sqrt{1 + [f'(\xi_i)]^2}\,\Delta_ix \end{aligned} \tag{3}$$

The sum of chords is, therefore

$$\sum_{i=1}^{n} P_{i-1}P_i = \sum_{i=1}^{n} \sqrt{1 + [f'(\xi_i)]^2}\,\Delta_ix$$

Since $f(x)$ and its derivative are continuous, $\sqrt{1 + [f'(x)]^2}$ is a continuous function of x, and we may apply the fundamental theorem to evaluate the

limit of this sum when n increases indefinitely and the maximum $\Delta_i x$ approaches zero. Denoting the length of arc AB by s, we obtain

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Equivalent notations are

$$s = \int_a^b \sqrt{1 + y'^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Observe that the integrand is precisely the differential of arc length (Art. 60) so that $s = \int ds$, with limits chosen so that the integration includes the entire arc in question.

In case it is more convenient to express the equation of the curve as $x = \varphi(y)$, we simply use the equivalent expression for ds, obtaining

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The limits of integration, of course, refer to y. If the curve is given in parametric form $x = f_1(t)$, $y = f_2(t)$, then

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve is given in polar coordinates,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{r_1}^{r_2} \sqrt{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1}\, dr$$

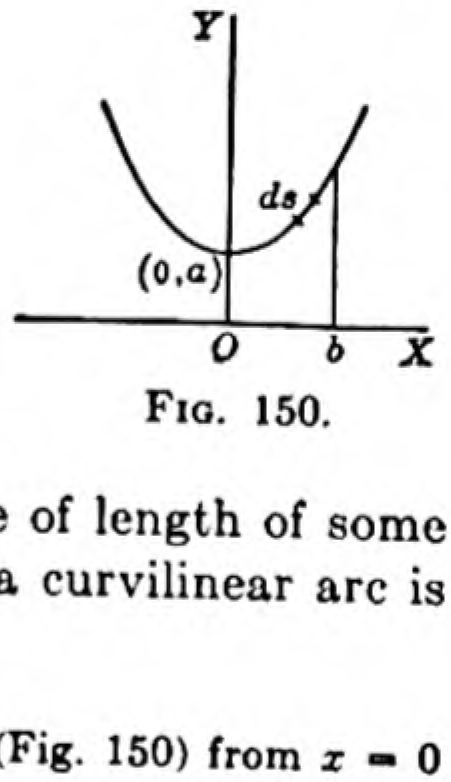


FIG. 150.

The limit of the sum of chords will be the measure of length of some straight-line segment. Hence, finding the length of a curvilinear arc is often called "rectifying" the arc.

Example 1. Find the length of the catenary $y = a \cosh \frac{x}{a}$ (Fig. 150) from $x = 0$ to $x = b$. We must calculate $s = \int_{x=0}^{x=b} ds$. We shall use $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, and so we first find $\frac{dy}{dx}$; thus, $\frac{dy}{dx} = \sinh \frac{x}{a}$. Therefore

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 \frac{x}{a} = \cosh^2 \frac{x}{a} \quad \text{and} \quad ds = \cosh \frac{x}{a}\, dx$$

Hence
$$s = \int_0^b \cosh \frac{x}{a}\, dx = a \sinh \frac{x}{a}\Big]_0^b = a \sinh \frac{b}{a}$$

Example 2. Find the length of the entire cardioid $r = a(1 + \sin\theta)$ (Fig. 151). Here, we use $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$. We have

$$\frac{dr}{d\theta} = a\cos\theta$$

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= a^2(1 + \sin\theta)^2 + a^2\cos^2\theta \\ &= a^2(1 + 2\sin\theta + \sin^2\theta + \cos^2\theta) \\ &= 2a^2(1 + \sin\theta) \end{aligned}$$

If we let θ vary from $-\pi/2$ to $\pi/2$, the point P will generate half the entire curve. Therefore

$$s = 2\int_{-\pi/2}^{\pi/2} \sqrt{2}\,a\,\sqrt{1 + \sin\theta}\,d\theta$$

In order to effect the integration, we make the substitution

$$\theta = \frac{\pi}{2} - \varphi \qquad d\theta = -d\varphi$$

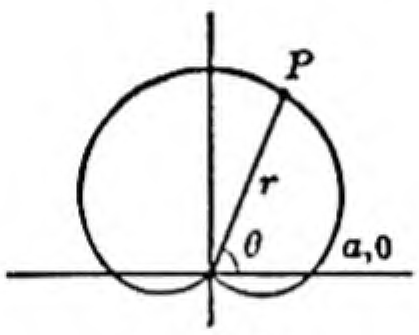


FIG. 151.

Consequently

$$\begin{aligned} s &= -2\sqrt{2}\,a\int_{\pi}^{0}\sqrt{1 + \cos\varphi}\,d\varphi \\ &= 4a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi = 8a\sin\frac{\varphi}{2}\Big]_0^{\pi} = 8a \end{aligned}$$

Example 3. If s is a short arc of the curve $y = f(x)$ of our preceding discussion and c its chord, we can verify that $\lim_{s\to 0}\frac{s}{c} = 1$. Let (x_1,y_1) be the coordinates of one end point of s and $(x_1 + \Delta x, y_1 + \Delta y)$ the coordinates of the other end point. Then

$$s = \int_{x_1}^{x_1+\Delta x}\sqrt{1 + [f'(x)]^2}\,dx$$

By the mean value theorem for definite integrals (Art. 105) this is equal to

$$s = \Delta x\sqrt{1 + [f'(\eta)]^2}$$

where η is between x_1 and $x_1 + \Delta x$. We have already seen in (3) that

$$c = \Delta x\sqrt{1 + [f'(\xi)]^2}$$

where ξ is between x_1 and $x_1 + \Delta x$. Consequently

$$\frac{s}{c} = \frac{\Delta x\sqrt{1 + [f'(\eta)]^2}}{\Delta x\sqrt{1 + [f'(\xi)]^2}}$$

When $s \to 0$, $\Delta x \to 0$ so that ξ and η both approach x_1. Hence $\lim_{s\to 0} s/c = 1$ (see Art. 59).

EXERCISES

Find the length of arc as indicated (Ex. 1 to 18).

1. $y = x^{3/2}$ from $x = 0$ to $x = 5$
2. $y = x^{2/3}$ from $x = 0$ to $x = 8$
3. $y = 3 \ln x$ from $x = 1$ to $x = 4$
4. $y = \ln \sec x$ from $x = 0$ to $x = \pi/4$
5. $y = \ln \cos x$ from $x = 0$ to $x = \pi/3$
6. $y = x^2$ from $x = 0$ to $x = 1$
7. $y^2 = 4ax$ from $x = 0$ to $x = a$ on the upper branch
8. Same as Exercise 7, using the equations $x = 4at^2$, $y = 4at$
9. The entire length of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$
10. Same as Exercise 9, using the equations $x = a \cos^3 \varphi$, $y = a \sin^3 \varphi$
11. One arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$
12. $y = \ln \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 3$
13. $18y^2 = x(x - 6)^2$ from $x = 0$ to $x = 6$
14. The entire length of $r = a(1 + \cos \theta)$
15. The entire length of $r = a \sin^3 (\theta/3)$
16. The curve $r = a \sin^4 (\theta/4)$ from $\theta = 0$ to $\theta = 2\pi$
17. The entire length of the parabolic arc $x = a \cos^4 \varphi$, $y = a \sin^4 \varphi$
18. $\begin{cases} x = e^{-t} \cos t \\ y = e^{-t} \sin t \end{cases}$ from $t = 0$ to $t = \pi$

19. A point moves in the xy plane with law of motion $x = e^{-2t} \cos 3t$, $y = e^{-2t} \sin 3t$. Find the distance traveled from $t = 0$ to $t = \pi$.

20. A point moves in the xy plane with law of motion $x = t^2$, $y = 2t$. Find the distance traveled from $t = 0$ to $t = 2$.

115. Area of a Surface of Revolution. A surface of revolution is generated when an arc AB (Fig. 152) of a plane curve is rotated about some line in its plane. If the arc AB is divided into n small arcs by points $P_0 = A, P_1, P_2, \ldots, P_{n-1}, B = P_n$, then each of the n chords P_0P, $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$ will generate a frustum of a right circular cone. The limit of the sum of the areas of these frustums, when n is increased indefinitely and the greatest arc is made to approach zero, is defined to be the area of the surface. Let the length of the ith chord be $P_{i-1}P_i = \Delta_i's$ (a prime mark is used since $\Delta_i s$ means the ith *arc*), and let R_i be the distance from the axis of revolution to the mid point of the chord. The area of the ith frustum is, therefore, $2\pi R_i \Delta_i's$ since the area of a conical frustum is the circumference of the middle section times the slant height. The limit of the sum of such areas is

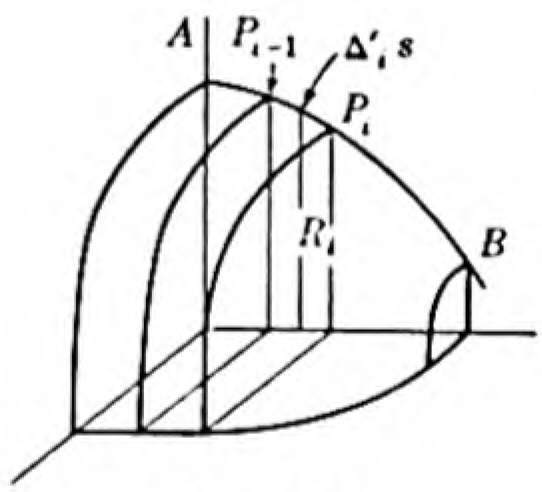


FIG. 152.

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi R_i \, \Delta_i's = 2\pi \int_a^b R \, ds$$

with limits of integration a, b chosen to include the whole surface.

That the limit of this sum is actually this integral is not obvious. A proof based upon geometrical reasoning follows: We shall choose the axis of revolution as the x axis and rotate an arc of the curve $y = f(x)$ about this axis. Consider the ith subdivision of the arc AB (Fig. 153). Let K be the mid-point of chord $P_{i-1}P_i$. Then HK is the radius of the middle section of the ith frustum. The area of the frustum is

$$\Delta_i A = 2\pi \overline{HK} \cdot \overline{P_{i-1}P_i}$$

This must be expressed as a function of the abscissa of some point in the interval $\Delta_i x$. By using the theorem of mean value as in Art. 114 [and supposing $f(x)$ to have a continuous derivative in the interval $a \leqq x \leqq b$], the chord

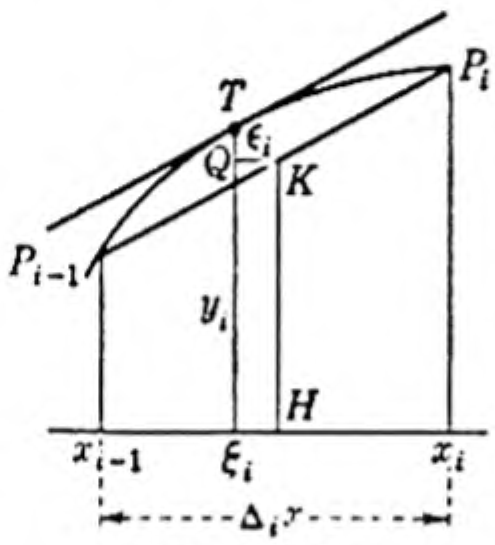


FIG. 153.

$$P_{i-1}P_i = \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x$$

where ξ_i is the abscissa of the point T at which the tangent is parallel to the chord. We designate the ordinate of T by $y_i = f(\xi_i)$. Now let Q be the point of intersection of a horizontal line through K with the ordinate of T. Denote QT by ϵ_i. Note particularly that, when $\Delta_i x \to 0$, $\epsilon_i \to 0$ on account of the continuity of $f(x)$. We have $HK = y_i - \epsilon_i$; therefore

$$\Delta_i A = 2\pi(y_i - \epsilon_i) \cdot \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x$$

Consequently, the sum of the conical frustums is

$$\sum_{i=1}^{n} \Delta_i A = 2\pi \sum_{i=1}^{n} y_i \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x - 2\pi \sum_{i=1}^{n} \epsilon_i \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x$$

The limit of the first sum in the right-hand member as $n \to \infty$ and the greatest $\Delta_i x \to 0$ is given by the fundamental theorem, since $y_i = f(\xi_i)$. This limit is

$$2\pi \int_a^b y \sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_{x=a}^{x=b} y\, ds$$

The second sum has a limit zero. To show this, we note that, because $f(x)$ was supposed continuous in the interval $a \leqq x \leqq b$, the greatest of the positive numbers $|\epsilon_1|, |\epsilon_2|, \ldots, |\epsilon_n|$ must approach zero. Call this number ϵ. Hence

$$\begin{aligned} \left| 2\pi \sum_{i=1}^{n} \epsilon_i \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x \right| &\leqq 2\pi \sum_{i=1}^{n} |\epsilon_i| \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x \\ &\leqq 2\pi \cdot \epsilon \cdot \sum_{i=1}^{n} \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x \\ &\leqq 2\pi\epsilon \cdot (b - a) \cdot M \end{aligned}$$

where M is the greatest of the $\sqrt{1 + [f'(\xi_i)]^2}$. This approaches zero since $\lim \epsilon = 0$. Therefore $S = 2\pi \int_{x=a}^{x=b} y\, ds$, and our original formula for S is established.

Example 1. Find the surface area of a sphere. We revolve the right-hand semicircle $x^2 + y^2 = a^2$ around the y axis (Fig. 154). Let this semicircle be divided into n small arcs each of length Δs, and let (x_i, y_i) be a point of the ith arc. Upon rotation, this generates a band on the surface of the sphere, the radius R_i of the circle described by (x_i, y_i) being x_i. The area of this band is approximately $2\pi x_i \Delta s$. Adding together all such bands and taking the limit of the sum, we have

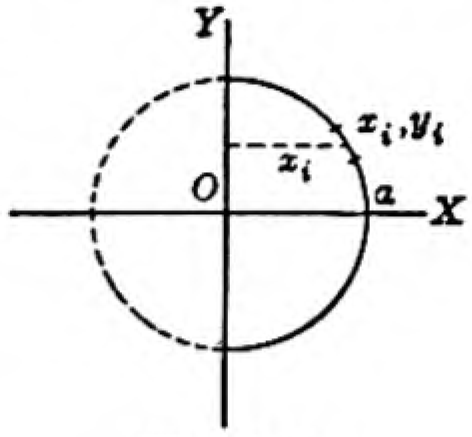


FIG. 154.

$$S = 2\pi \int_{y=-a}^{y=a} x\, ds$$

the limits being chosen to include the right-hand semicircle. We must express $x\, ds$ in terms of a single variable. Let us use

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

We have $\qquad x^2 + y^2 = a^2 \qquad 2x\frac{dx}{dy} + 2y = 0 \qquad \frac{dx}{dy} = -\frac{y}{x}$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2} = \frac{a^2}{x^2}$$

Therefore $\qquad ds = \frac{a}{x}\, dy$

and $\qquad S = 2\pi \int_{-a}^{a} x \cdot \frac{a}{x}\, dy = 2\pi a \int_{-a}^{a} dy = 4\pi a^2$

Example 2. Find the surface area generated by rotating the upper half of the cardioid $r = a(1 + \cos\theta)$ around the polar axis (Fig. 155). Divide the arc into n small arcs of length Δs. Let (r_i, θ_i) be a point in the ith arc. Upon rotation about the polar axis, this point will describe a circle of radius $R_i = r_i \sin\theta_i$. The area of the band described by the ith arc is approximately $\Delta_i S = 2\pi r_i \sin\theta_i \Delta s$. Hence, by the usual argument,

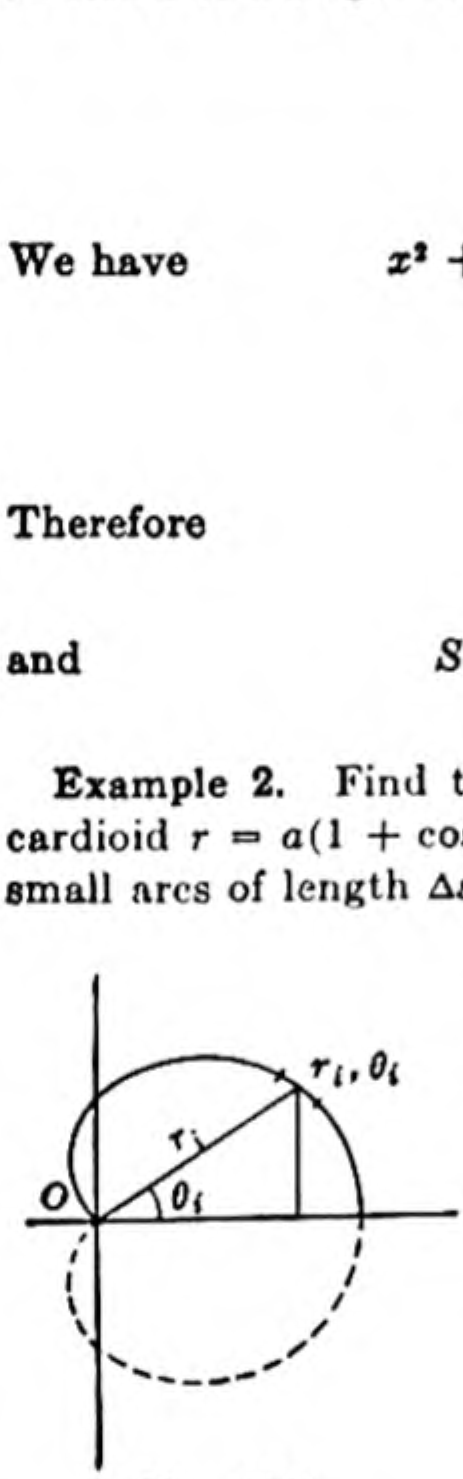


FIG. 155.

$$S = 2\pi \int_{\theta=0}^{\theta=\pi} r \sin\theta\, ds$$

To find $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$, we have

$$\frac{dr}{d\theta} = -a \sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta = a^2(2 + 2\cos\theta)$$

Therefore
$$\begin{aligned} S &= 2\pi \int_0^{\pi} a(1 + \cos\theta)\sin\theta \cdot a\sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\pi a^2 \sqrt{2} \int_0^{\pi} (1 + \cos\theta)^{3/2} \sin\theta\, d\theta \\ &= -2\pi a^2 \sqrt{2} \cdot \tfrac{2}{5}(1 + \cos\theta)^{5/2}\Big]_0^{\pi} = \tfrac{32}{5}\pi a^2 \end{aligned}$$

116. Area of a Cylindrical Surface. Suppose that the arc $\widehat{AB}$ of a curve in the plane HK (Fig. 156) is the directrix of a cylinder whose elements are perpendicular to the plane HK. Let us calculate the area of the cylindrical surface bounded by $\widehat{AB}$ and the arc $\widehat{CD}$ of another bounding curve. Note that $\widehat{CD}$ need not be a plane curve. Divide $\widehat{AB}$ into n short arc lengths $\Delta_1 s, \Delta_2 s, \ldots, \Delta_n s$. Let the chords subtended by these short arcs be $\Delta_1' s, \Delta_2' s, \ldots, \Delta_n' s$. Draw the elements of the cylinder at each point of subdivision, and let the lengths cut from them by $\widehat{CD}$ be $h_1, h_2, \ldots, h_n$. Consider the ith chord $\Delta_i' s$ and a rectangular plane area standing upon it as base, with left-hand side h_i and right-hand side and top as indicated in Fig. 156. The area of this rectangle is $h_i \Delta_i' s$.

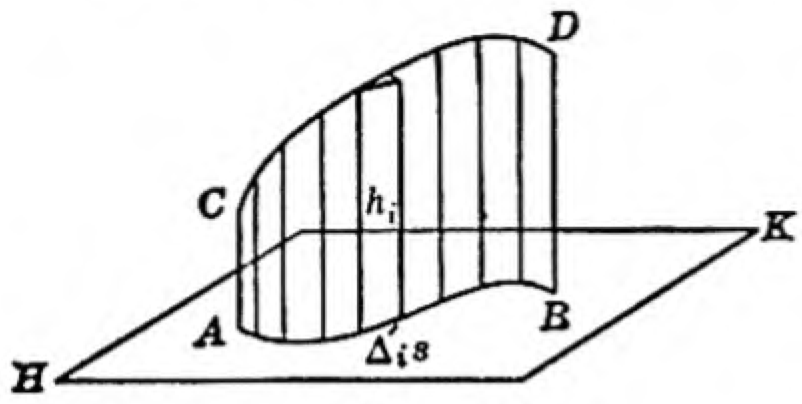


FIG. 156.

The sum of all such rectangular areas is $\sum_{i=1}^{n} h_i \Delta_i' s$. It is clear that the required area of the cylindrical surface is the limit of this sum taken as the greatest $\Delta_i s$, and therefore the greatest $\Delta_i' s$ is made to approach zero while n increases indefinitely. It can be shown that

$$\lim_{n \to \infty} \sum_{i=1}^{n} h_i \Delta_i' s = \int_{s_1}^{s_2} h \, ds$$

with suitable limits of integration chosen.

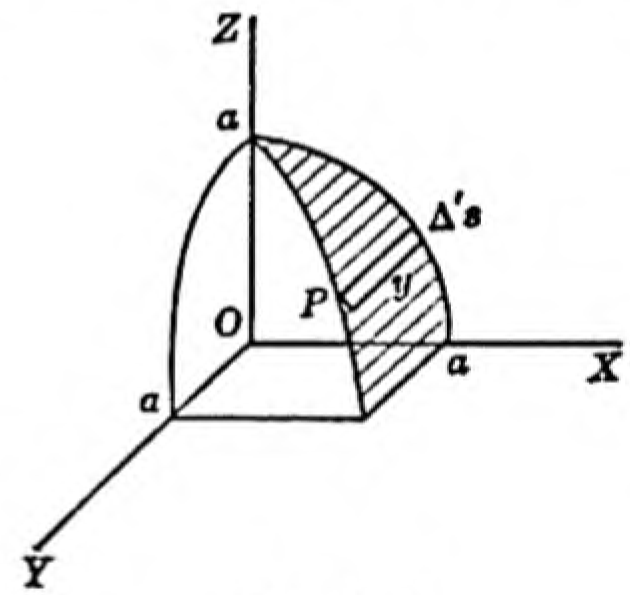


FIG. 157.

Example. Find the area in the first octant cut from the cylinder $x^2 + z^2 = a^2$ by the cylinder $y^2 + z^2 = a^2$. The area is shaded in Fig. 157, and a typical element is heavily outlined. Let $P(x,y,z)$ be the point where this element touches the curve of intersection of the two cylinders. The height h of the element is the y coordinate of P. Since P is a point of the cylinder $y^2 + z^2 = a^2$, we have $h = y = \sqrt{a^2 - z^2}$. The differential of arc length, ds, is found from the equation of the curve

$$x^2 + z^2 = a^2 \qquad y = 0$$

Thus

$$2x \frac{dx}{dz} + 2z = 0 \qquad \frac{dx}{dz} = -\frac{z}{x}$$

$$1 + \left(\frac{dx}{dz}\right)^2 = 1 + \frac{z^2}{x^2} = \frac{x^2 + z^2}{x^2} = \frac{a^2}{x^2}$$

$$ds = \frac{a}{x} dz = \frac{a}{\sqrt{a^2 - z^2}} dz$$

Hence

$$\int_{s_1}^{s_2} h \, ds = \int_0^a \sqrt{a^2 - z^2} \cdot \frac{a}{\sqrt{a^2 - z^2}} dz = a^2$$

EXERCISES

1. Find the surface of a sphere of radius a, using the equations $x = a \cos \varphi$, $y = a \sin \varphi$.

2. Find the surface of the sphere generated by rotating the circle $r = a$ around the polar axis.

3. The arc of the parabola $x^2 = 4ay$ from $x = 0$ to $x = 2a$ is revolved about the y axis. Find the area of the surface generated.

4. A sphere has radius a. A *zone* is cut from the surface of this sphere by two parallel planes at a distance h apart. Find the area of the zone.

5. The upper half of the cardioid $r = a(1 - \cos \theta)$ is revolved about the line $\theta = \pi$. Find the area of the surface generated.

6. Find the area of the surface of a right circular cone of altitude h and radius of base a.

7. The reflector of a searchlight is a paraboloid of revolution. If it is 6 in. deep and 18 in. wide, find the area of the surface.

8. Find the surface of the torus generated by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$ $(b > a)$.

9. Solve Exercise 8, using the equations $x = a \cos \varphi$, $y = a \sin \varphi$.

10. One arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is revolved about the x axis. Find the surface generated.

11. Find the surface generated by revolving the right half of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about the y axis.

12. Solve Exercise 11, using the equations $x = a \cos^3 \varphi$, $y = a \sin^3 \varphi$.

13. The lemniscate $r^2 = a^2 \cos 2\theta$ is revolved about the horizontal axis. Find the surface generated.

14. The right-hand loop of the lemniscate $r^2 = a^2 \cos 2\theta$ is revolved about the vertical axis. Find the surface generated.

15. The arc of the catenary $y = a \cosh (x/a)$ from $x = 0$ to $x = a$ is revolved about the y axis. Find the surface generated.

16. The upper branch of the curve $y^2 = x^3$ from $x = 0$ to $x = \frac{4}{3}$ is revolved about the y axis. Find the surface generated.

17. The part of the curve $y = \ln x$ in the fourth quadrant is revolved about the y axis. Find the surface generated.

18. One arch of the curve $y = \cos x$ is revolved about the x axis. Find the area of the surface generated.

19. The arc of the logarithmic spiral $r = e^{a\theta}$ from $\theta = 0$ to $\theta = \pi/2$ is revolved about the line $\theta = \pi/2$. Find the surface generated.

20. The arc of the hyperbola $x^2 - y^2 = a^2$ from $x = a$ to $x = 2a$ is revolved about the x axis. Find the surface generated.

21. The upper half of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is revolved around the x axis. Find the surface of the resulting prolate spheroid.

22. Solve Exercise 21, using the equations $x = a \cos \varphi$, $y = b \sin \varphi$.

23. Find the area in the first octant cut from the cylindrical surface $x^2 + z^2 = a^2$ by the plane $y = x$.

24. Find the area cut from the cylindrical surface $x^{2/3} + y^{2/3} = a^{2/3}$ by the cylinder $x^{2/3} + z^{2/3} = a^{2/3}$.

25. Find the area in the first octant cut from the cylindrical surface $y^2 = a^2 - ax$ by the plane $z = y$.

26. Find the area cut from the cylindrical surface $x^2 + y^2 - 2ax = 0$ by the sphere $x^2 + y^2 + z^2 = 4a^2$.

117. Mean Value of a Function. Given $y = f(x)$, a continuous function of x in the interval $a \leqq x \leqq b$. Divide the interval into n equal subintervals of length Δx by points $x_1, x_2, \ldots, x_{n-1}$. Let $x_0 = a$ and $x_n = b$. Designate the ordinates at these points (Fig. 158) by $y_0, y_1, \ldots, y_n$. Form the expression

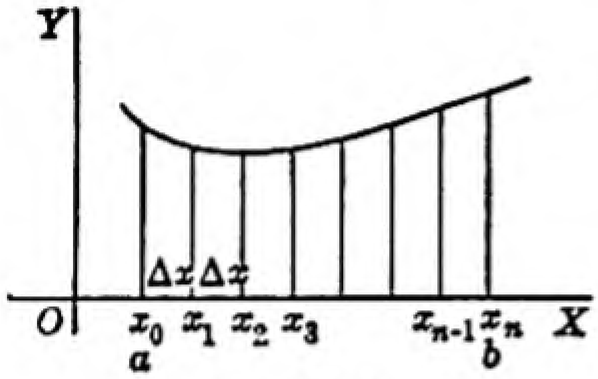


FIG. 158.

$$M_n = \frac{y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x}{n\,\Delta x}$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} y_i\,\Delta x = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

since $n\,\Delta x = b - a$. Observe that, if we divide numerator and denominator by Δx, we get the familiar expression for the *arithmetic mean* of the n ordinates $y_1, y_2, \ldots, y_n$. Now take the limit of this sum as Δx is made to approach zero and n to increase indefinitely. The result is called the *mean value of the function* $f(x)$ over the interval a to b on the x axis.

$$M = \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

We may take the mean value of the function over an interval of a line or curve other than the x axis. This will be illustrated in Example 2.

Example 1. Find the mean value of the ordinates of the curve $y = \sin x$ over the interval along the x axis from 0 to $\pi/2$ (Fig. 159). Here we have $a = 0$, $b = \pi/2$, and

$$M = \frac{1}{\pi/2}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{2}{\pi}\Big[-\cos x\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi} \text{ units}$$

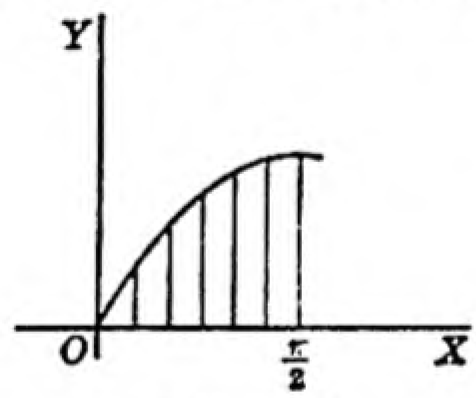


FIG. 159.

Example 2. Find the mean value of ordinates of a semicircle

$$y = \sqrt{a^2 - x^2}$$

over the arc of the semicircle, Fig. 160. Here, we think of the ordinates as equally spaced along the arc of the semicircle. If the semicircle is divided into n arcs each of length Δs and the corresponding ordinates are $y_1, y_2, \ldots, y_n$, then

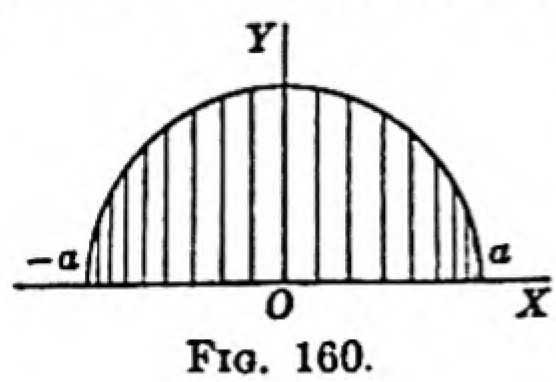


FIG. 160.

$$M_n = \frac{y_1\,\Delta s + y_2\,\Delta s + \cdots + y_n\,\Delta s}{n\,\Delta s} = \frac{1}{\pi a}\sum_{i=1}^{n} y_i\,\Delta s$$

since $n\,\Delta s = \pi a$. Hence

$$M = \frac{1}{\pi a}\int_{s_1}^{s_2} y\,ds$$

with suitable limits chosen. To find ds, we have

$$x^2 + y^2 = a^2 \qquad 2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{a^2}{y^2}$$

and therefore

$$ds = \frac{a}{y}\,dx$$

Hence

$$M = \frac{1}{\pi a}\int_{-a}^{a} y\cdot\frac{a}{y}\,dx = \frac{1}{\pi}\int_{-a}^{a} dx = \frac{2a}{\pi}$$

EXERCISES

1. Find the mean value of the ordinates of the curve $y = \cos x$ along the x axis from $x = -\pi/2$ to $x = \pi/2$.

2. Find the mean value of the ordinates of the upper branch of $y^2 = 4ax$ along the x axis from $x = 0$ to $x = a$.

3. Find the mean value of the ordinates of the semicircle $y = \sqrt{a^2 - x^2}$ along the x axis from $x = -a$ to $x = a$.

4. Find the mean value of the ordinates of the curve $y = \ln x$ along the x axis from $x = 1$ to $x = 4$.

5. Squares whose planes are perpendicular to the xy plane are constructed on the double ordinates of the circle $x^2 + y^2 = a^2$. Find the mean value of the areas of the squares if they are equally spaced along the x axis from $x = -a$ to $x = a$.

6. Solve Exercise 5 if the squares are equally spaced along the circumference of the circle.

7. Rectangles are inscribed in the circle $x^2 + y^2 = a^2$. Find the mean value of their areas if their vertical sides are equally spaced along the x axis.

8. Solve Exercise 7 if the vertical sides of the rectangles are equally spaced along the circumference.

9. Rectangles are inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the mean value of their areas if their vertical sides are equally spaced along the x axis.

10. Solve Exercise 9 if the horizontal sides of the rectangles are equally spaced along the y axis (same result as in Exercise 9).

11. Isosceles triangles with a common vertex are inscribed in a circle of radius a. Successive bases are equally spaced along a diameter. Find the mean value of their areas.

12. Solve Exercise 11 if successive bases are equally spaced along the circumference of the circle.

13. Find the mean length of the radius vector of the circle $r = 2a\cos\theta$ if successive vectors make equal angles with one another.

14. Solve Exercise 13 if the vectors are drawn to points equally spaced along the circumference (same result as in Exercise 13).

15. Find the mean length of the radius vector of one loop of the rose-leaf curve $r = a\cos n\theta$ if successive vectors make equal angles with one another.

16. Find the mean length of the radius vector of the spiral of Archimedes $r = a\theta$ from $\theta = \theta_1$ to $\theta = \theta_2$ if successive vectors make equal angles with one another.

17. Find the mean length of the radius vector of the cardioid $r = a(1 + \cos\theta)$ if successive vectors make equal angles with one another.

18. If a body falls from rest and is acted upon only by gravity, show that the mean value of its velocity with respect to time of fall is one-half the velocity at the end of the fall.

19. A body falls from rest through a distance S. If the only force acting is gravity, show that the mean value of the velocity over this distance is $\frac{2}{3}\sqrt{2gS}$, that is, two-thirds the velocity at the end of the fall.

20. The kinetic energy of a particle in uniform motion is $\frac{1}{2}mv^2$ where m is the mass and v the velocity. If a particle moves in a straight line with a law of motion

$$x = r\cos\alpha t$$

(simple harmonic motion) where r and α are constants, calculate the mean value, with respect to time, of the kinetic energy during one complete vibration. Show that this is equal to half the maximum kinetic energy.

21. In Exercise 20, if the mean value is taken with respect to the distance traveled during one complete vibration, show that the result is two-thirds the maximum kinetic energy.

MISCELLANEOUS EXERCISES

1. Find the area bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, $y = -1$.

2. Find the area enclosed by the curve $y^2 = x^2(9 - x)$.

3. Find the area bounded by one arch of the curve $y = \sin^2 x$ and the x axis.

4. Find the area bounded by the curve $y = e^{-\frac{x}{2}}$ and the lines $y = 0$, $x = -1$, $x = 3$.

5. Find the area bounded by the curves $a^2y = x^3 - ax^2$ and $ay = x^2$.

6. A parabolic segment is bounded by the parabola $y = kx^2$ and a line perpendicular to the axis of the parabola. Prove without reference to Exercise 36, page 299, that the area of such a segment is two-thirds the area of the rectangle which circumscribes the segment.

7. A cow is tethered on the outside of a circular fence of radius r ft. by a rope of length πr ft. Find the area over which the cow can graze.

8. Find the area enclosed by the curve $r = 2a\sin^2\theta$.

9. Find the area bounded by the limaçon $r = a(2 + \cos\theta)$.

10. Find the area inside the loops of the curve $r = 2a\cos 3\theta$ and outside the circle $r = a$.

11. Find the area inside the cardioid $r = a(1 + \sin\theta)$ and outside the circle $r = 2a\sin\theta$.

12. The area bounded by $y^2 = 4x$, the line $x = 1$, and the line $x = 4$ is revolved about the y axis. Find the volume generated.

13. A circle of radius a is rotated about a tangent line. Find the volume generated.

14. The area bounded by the coordinate axes and the parabolic arc $x^{1/2} + y^{1/2} = a^{1/2}$ is revolved about the x axis. Find the volume generated.

15. Solve Exercise 14, using the equations $x = a\cos^4\theta$, $y = a\sin^4\theta$.

16. The area bounded by $y = \ln x$, the x axis, and the line $x = 4$ is revolved about the x axis. Find the volume generated.

17. The area bounded by the catenary $y = a\cosh(x/a)$, the y axis, and the line $y = 2a$ rotates about the y axis. Find the volume.

18. The cissoid $y^2 = \dfrac{x^3}{2a - x}$ is revolved about its asymptote. Find the volume generated.

19. Find the volume in the first octant under the plane $z = y$ and inside the cylinder $y^2 = a^2 - ax$.

20. A right circular cone has altitude h and radius of base a. Two planes through the axis of the cone making an angle of φ radians with one another cut out a wedge-shaped volume. Find this volume.

21. A solid right circular cylinder has radius a. A piece is cut off by a plane tangent to the edge of the circular base and making an angle α with the plane of the base. Find the volume of this piece.

22. A surface is generated by a circle moving with its plane always parallel to the yz plane and having the ends of a diameter always on the curves $z = 2\sqrt{ax}$, $y = 0$ and $z = \sqrt{ax}$, $y = 0$. Find the volume bounded by the surface and the plane $x = a$.

23. Find the length of the curve $y = \ln \sin x$ from $x = \pi/4$ to $x = \pi/2$.

24. Find the length of the curve $y = e^x$ from $x = 0$ to $x = 1$.

25. Find the length of the curve $y = \ln (1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$.

26. Find the length of the curve $r = a \cos^3 (\theta/3)$.

27. Express as a definite integral the length of the curve $r = a \sin^n (\theta/n)$ from $\theta = 0$ to $\theta = 2\pi$ (n a positive integer).

28. Find the entire length of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$.

29. Solve Exercise 28, using the equations $x = a \cos^3 \alpha$, $y = b \sin^3 \alpha$.

30. Find the length of the spiral of Archimedes $r = a\theta$ from $\theta = 0$ to $\theta = 2\pi$.

31. Find the length of the logarithmic spiral $r = e^{a\theta}$ from (r_1,θ_1) to (r_2,θ_2).

32. Find the surface of the sphere generated by rotating the circle $r = 2a \sin \theta$ about the line $\theta = \pi/2$.

33. Find the surface of the torus generated by revolving the circle $r = a$ about the line $r = b \sec \theta$ (compare Exercise 8, page 319).

34. The arc of the catenary $y = \cosh x$ from $x = -1$ to $x = 1$ is revolved about the x axis. Find the area of the surface generated.

35. Find the area cut from the cylindrical surface $x^2 + z^2 = a^2$ by the planes $y = x$ and $y = 2x$.

36. Find the area in the first octant cut from the cylindrical surface $y^2 = ax$ by the planes $z = y$ and $x = a$.

37. Right circular cones with a common vertex are inscribed in a sphere of radius a. Their bases are equally spaced along a diameter of the sphere. Find the mean value of their volumes.

38. Find the mean length of the radius vectors of the cardioid $r = a(1 + \cos \theta)$ if these vectors are drawn from the pole to points equally spaced along the cardioid (see Exercise 14, page 321).

39. When a liquid flows through a pipe of radius a ft., the velocity v of a particle of the liquid at a distance r ft. from the axis of the pipe is $v_0\left(1 - \dfrac{r^2}{a^2}\right)$, where v_0 ft. per second is the velocity along the axis. What volume of liquid will flow past a given point per second?

40. In Exercise 39, find the mean value of the velocity v along a diameter of the pipe.

CHAPTER 16

PHYSICAL APPLICATIONS OF THE DEFINITE INTEGRAL

118. Work. If an object is moved in a straight line against the action of a constant force, the *work done* in moving the object is defined as the product of the distance by the force. If the distance is one foot and the force is one pound, the unit of work is one foot-pound. For example, the work done in lifting a 16-lb. shot a vertical distance of 3 ft. is $3 \times 16 = 48$ ft.-lb. Note that the *weight* of the shot is simply the *force* exerted upon it by the earth.

If, however, the force is not constant but varies with the position of the object, the calculation of work requires something more complicated than a simple multiplication. Suppose a body moves against a force whose point of application on the body is P and that P moves along a straight line. For simplicity, let this line be the x axis. Suppose that the component of the force along the x axis is the continuous function $f(x)$ where x is the abscissa of P. We wish to calculate the work done in moving P from $x = a$ to $x = b$. Divide the interval from a to b into n subintervals of length Δx. We shall express the work $\Delta_i W$ (termed an *element of work*) done in moving P across the ith subinterval, namely, from x_i to x_{i+1} (Fig. 161). Let m_i be the minimum and M_i the maximum values of $f(x)$ in this subinterval. Then $\Delta_i W$ is between $m_i\,\Delta x$ and $M_i\,\Delta x$, say $\Delta_i W = \mu_i\,\Delta x$ where μ_i is between m_i and M_i. Since $f(x)$ is continuous, $\mu_i = f(\xi_i)$ where $x_i \leqq \xi_i \leqq x_{i+1}$. That is, $\Delta_i W = f(\xi_i)\,\Delta x$ where ξ_i is some point in the ith subinterval. Hence, the work done in moving P from a to b is the sum of such elements, namely,

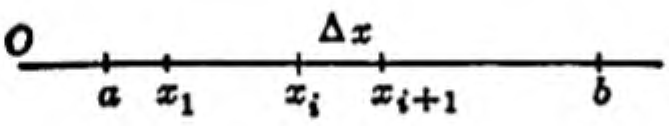


FIG. 161.

$$W = \sum_{i=1}^{n} f(\xi_i)\,\Delta x^*$$

By the usual argument, if Δx is made to approach zero and, consequently,

* Note that we do not know what values the ξ_i assume since each of them is simply *some* value in the appropriate subinterval. They depend not only upon $f(x)$ but also upon the particular subdivision chosen. We therefore take the limit of the sum as Δx is made to approach zero in order to calculate the work.

n to increase indefinitely, we obtain

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta x = \int_a^b f(x)\,dx$$

Example 1. According to Hooke's law, the force required to stretch a helical spring is proportional to the distance stretched. The natural length of a given spring is 10 in. A force of 3 lb. will stretch it to a total length of 12 in. Find the work done in stretching it from its natural length to a total length of 18 in.

Let the spring be attached at A (Fig. 162), and take the origin at the other end of the spring. Let the force required to produce an elongation of x in. be $f(x)$. We then have $f(x) = kx$. To find k, the *spring constant* for this particular spring, we observe that the force required to stretch the spring to a total length of 12 in. is 3 lb. Note that this is an elongation of 2 in. Consequently

FIG. 162.

$$f(2) = k(2 \text{ in.}) = 3 \text{ lb.}$$

Hence $$k = \tfrac{3}{2} \text{ lb./in.}$$

Therefore $$f(x) = \tfrac{3}{2}x \text{ lb.}$$

When the spring is stretched from a total length of 10 to 18 in., x varies from 0 to 8. Therefore,

$$W = \int_0^8 \tfrac{3}{2}x\,dx = \tfrac{3}{4}x^2\Big]_0^8 = 48 \text{ in.-lb.} = 4 \text{ ft.-lb.}$$

Example 2. A circular cylindrical tank (axis horizontal) of radius 2 ft. and length 15 ft. is full of water. Find the work done in pumping all the water to a delivery pipe 10 ft. above the center of the tank. Choose coordinate axes with the origin at the center of one of the circular ends of the tank, as shown in Fig. 163. Divide the circle into horizontal elements of area, and imagine a horizontal "slice" of water of which a typical element of area forms one end. The volume of this slice is $15(2x_i\,\Delta y)$, as is evident from the figure. If water weighs w lb. per cubic foot, the weight of the slice is $30wx_i\,\Delta y$. The approximate value of the work done in raising all the water in this slice to a point 10 ft. above O is the weight (force) multiplied by $10 - y_i$, that is,

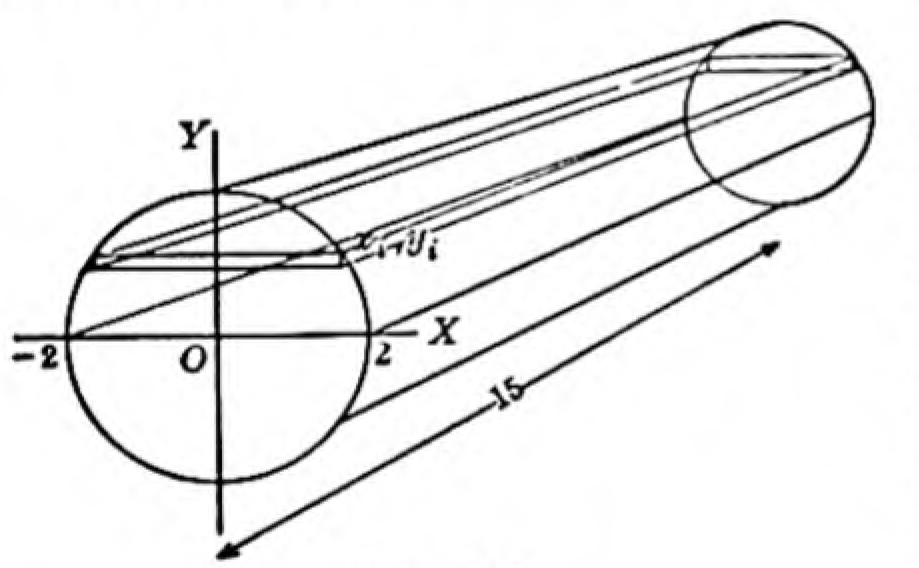


FIG. 163.

$$\Delta_i W = 30w(10 - y_i)x_i\,\Delta y$$

The total work done in raising all the water is, therefore

$$W = \int_{-2}^{2} 30w(10 - y)x\,dy$$

Since $x^2 + y^2 = 4$, we have

$$W = 30w \int_{-2}^{2} (10 - y)\sqrt{4 - y^2}\,dy = 600\pi w \text{ ft.-lb.}$$

Using $w = 62.4$ lb. per cubic foot, we obtain

$$W = 117{,}600 \text{ ft.-lb. approximately}$$
$$= 58.8 \text{ ft.-tons}$$

EXERCISES

1. If a force of 10 lb. will stretch a certain spring 8 in., find the work done in stretching it 2 ft.

2. If a force of 75 g. will stretch a spring 5 cm., find the work done in stretching it 10 cm.

3. If a force of w lb. will stretch a spring h ft., find the work done in stretching it nh ft.

4. A particle is moved in a straight line with law of motion $x = t^2 + 2t + 2$ (x in feet, t in seconds). The resisting force (in pounds) is numerically equal to the square of the velocity. Find the work done in moving the particle against this resistance from $t = 0$ to $t = 4$.

5. Solve Exercise 4 if $x = 5 \cos t$ and the work done in the time interval $t = 0$ to $t = \pi/2$ is required.

6. The base of a vertical cylindrical tank is a circle of radius 10 ft., and the altitude is 25 ft. If the tank is full of water, find the work done in pumping the contents to the top of the tank.

7. Find the work done in pumping all the water in the tank of Exercise 6 to a point 5 ft. above the top of the tank.

8. A trough full of water ($w = 62.4$ lb. per cubic foot) is 10 ft. long and has isosceles triangular cross sections 2 ft. across the top and 2 ft. deep. Find the work done in pumping the water to the top of the trough.

9. A circular cylindrical tank (axis horizontal) of radius 3 ft. and length 20 ft. is full of water. Find the work done in pumping the water to the top of the tank.

10. Solve Exercise 9 if the tank is half full of water.

11. The top of a conical cistern is a circle of radius 3 ft. The cistern is 9 ft. deep and is full of water. Find the work done in pumping the water to a point 5 ft. above the top of the cistern.

12. Solve Exercise 11 if the water in the cistern is 6 ft. deep.

13. The top of an elliptical conical reservoir is an ellipse with major axis 6 ft. and minor axis 4 ft. It is 6 ft. deep and full of water ($w = 62.4$ lb. per cubic foot). Find the work done in pumping the water to an outlet at the top of the reservoir.

14. Find the work done in pumping the water from a full hemispherical tank bowl of radius 12 ft. to a point 4 ft. above the top of the tank.

15. The top of a tank in the form of a paraboloid of revolution is a circle of radius 2 ft. The tank is 6 ft. deep at the center. Find the work done in pumping the contents to the top of the tank.

16. A tank has the form of an elliptic paraboloid. The top is an ellipse with major axis 8 ft. and minor axis 4 ft. It is 6 ft. deep at the center and full of water. Find the work done in pumping the contents to the top of the tank.

17. A full water tank consists of a hemisphere of radius 4 ft., surmounted by a circular cylinder of the same radius and of altitude 8 ft. Find the work done in pumping the water to an outlet at the top of the tank.

18. An oil tank full of oil ($w = 50$ lb. per cubic foot) is a cylinder 10 ft. long with horizontal axis. The cross section is an ellipse with vertical minor axis 4 ft. long and major axis 6 ft. Find the work done in pumping the contents to a point 2 ft. above the top of the tank.

119. Density. Consider a solid body of mass M and volume V. Let P be some point within or on the surface of the body, and let ΔV be a small portion of the solid of which P is a point. Let ΔM be the mass of this piece. We define the *average density* of the piece to be the ratio $\frac{\Delta M}{\Delta V}$ units of mass per unit of volume. Now, imagine ΔV to decrease so that its greatest linear dimension approaches zero, the point P being always in ΔV. If, under these circumstances, the ratio $\frac{\Delta M}{\Delta V}$ approaches a limit, we call this limit the *density* δ of the solid *at the point* P. Thus, $\delta = \frac{dM}{dV}$. When δ is a constant and is, therefore, independent of the position of P, we call the mass M *homogeneous*. In this case, with limits chosen to include the whole volume, $M = \int \delta\, dV = \delta \int dV = \delta V$. If, on the other hand, δ varies with the position of the point P, we call the mass M *heterogeneous*. In this case, the integral $M = \int \delta\, dV$ gives the total mass; but its evaluation will, in general, be best effected by the use of a so-called *iterated integral* which will be discussed in Chap. 18. For the present, we shall restrict ourselves to a consideration of homogeneous masses.

The notion of a *mass point*, or of a mass concentrated at a certain point, is useful and of frequent application. Imagine a body of mass m whose maximum linear dimension approaches zero but whose density increases so that the mass remains constant. The limiting form of this body may be thought of as a point endowed with the mass m or at which the mass m is concentrated. Such a mass point is often called a *particle* of mass m.

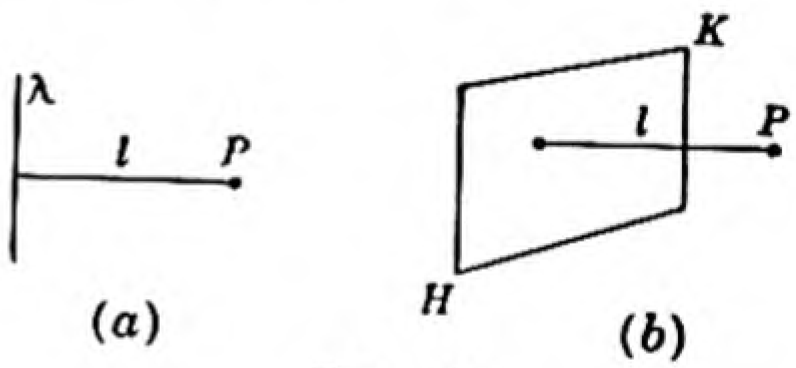


FIG. 164.

120. Centroid of a System of Particles. Consider a mass m concentrated at a point P. If P is at a distance l from a line λ (Fig. 164*a*) or at a distance l from a plane HK (Fig. 164*b*), then lm is called the *mass moment* or *moment of first order* with respect to the line or plane.

Next, consider a system of n masses $m_1, m_2, \ldots, m_n$ concentrated, respectively, at points in space whose coordinates are $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2), \ldots, P_n(x_n,y_n,z_n)$. The total mass of the system is

$$M = m_1 + m_2 + \cdots + m_n$$

The moments of the masses of the n particles with respect to the yz plane are $x_1m_1, x_2m_2, \ldots, x_nm_n$. We define the moment G_{yz} of the system with respect to the yz plane as the sum of the moments of the separate particles. Thus

$$G_{yz} = x_1m_1 + x_2m_2 + \cdots + x_nm_n = \sum_{i=1}^{n} x_im_i$$

Similarly, we define the moments of the system with respect to the zx and xy planes as, respectively,

$$G_{zx} = \sum_{i=1}^{n} y_i m_i \qquad G_{xy} = \sum_{i=1}^{n} z_i m_i$$

Let us now imagine the total mass $M = \sum_{i=1}^{n} m_i$ of the system to be concentrated at a single point C, and let us find coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of C so that the moments with respect to the coordinate planes of M concentrated at C will be G_{yz}, G_{zx}, G_{xy}, respectively. Evidently, we must have

$$\begin{aligned} M\bar{x} &= G_{yz} = \sum_{i=1}^{n} x_i m_i \\ M\bar{y} &= G_{zx} = \sum_{i=1}^{n} y_i m_i \\ M\bar{z} &= G_{xy} = \sum_{i=1}^{n} z_i m_i \end{aligned} \tag{1}$$

Therefore

★ $$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i} \qquad \bar{z} = \frac{\sum_{i=1}^{n} z_i m_i}{\sum_{i=1}^{n} m_i} \tag{2}$$

The point $C(\bar{x},\bar{y},\bar{z})$ is called the *centroid*, or *center of mass*, of the system of n particles. It can easily be shown that the centroid coincides with the *center of gravity*. If the n particles could be thought of as connected by rigid wires having no mass, the system would balance if supported at C.

In the special case in which the points $P_1, P_2, \ldots, P_n$ all lie in one of the coordinate planes, the moments with respect to coordinate *planes* reduce to moments with respect to coordinate *axes*, and the centroid lies in this plane. For example, if the points lie in the xy plane, G_{yz} reduces to $G_y = \sum_{i=1}^{n} x_i m_i$, the moment of the system with respect to the y axis, and G_{zx} reduces to $G_x = \sum_{i=1}^{n} y_i m_i$, the moment of the system with respect to the x axis. The centroid C is the point of the xy plane whose coordinates $\bar{x}$, $\bar{y}$ are given by the equations (2).

Although we have been dealing with the centroid of a system of mass

particles, we can find the centroid of certain continuous masses by use of the following considerations.

Homogeneous masses:

1. Any axis or plane of symmetry must pass through the centroid.

2. Consequently, if a body has a geometrical center, that point is the centroid.

Any mass:

3. If the body consists of two or more portions for each of which the centroid can be found, each portion may be thought of as concentrated at its centroid, and the centroid of this system of mass points will be the centroid of the body.

The centroid of a mass has the following property of major importance in mechanics: *The moment of the mass with respect to any plane is the same as if the whole mass were concentrated at the centroid.* An evident corollary is that *the moment of a mass with respect to any plane containing the centroid is zero.*

Example 1. Five particles are located as follows: mass proportional to 3 at $(5, -1, 2)$; mass proportional to 4 at $(1, 2, -3)$; mass proportional to 1 at $(4, 3, 1)$; mass proportional to 2 at $(-6, -2, 4)$; mass proportional to 6 at $(-2, 1, 4)$. Find the centroid of the system of particles. We have $m_1 = 3k$, $m_2 = 4k$, $m_3 = k$, $m_4 = 2k$, $m_5 = 6k$. Hence

$$M = \sum_{i=1}^{5} m_i = 16k$$

Applying equations (1), we have

$$\begin{aligned}
16k\bar{x} &= 5 \cdot 3k + 1 \cdot 4k + 4 \cdot k + (-6) \cdot 2k + (-2) \cdot 6k \\
\bar{x} &= -\tfrac{1}{16} \\
16k\bar{y} &= (-1) \cdot 3k + 2 \cdot 4k + 3 \cdot k + (-2) \cdot 2k + 1 \cdot 6k \\
\bar{y} &= \tfrac{5}{8} \\
16k\bar{z} &= 2 \cdot 3k + (-3) \cdot 4k + 1 \cdot k + 4 \cdot 2k + 4 \cdot 6k \\
\bar{z} &= \tfrac{27}{16}
\end{aligned}$$

Example 2. A homogeneous sheet of metal of uniform thickness τ is cut into the shape shown in Fig. 165. Find the centroid. We first choose convenient coordinate axes as shown in the figure and then subdivide into rectangular sheets I, II, III. Since these have geometrical centers, we can at once write down the coordinates of their centroids, namely, $C_1(4, \frac{15}{2}, \frac{1}{2}\tau)$, $C_2(11, 4, \frac{1}{2}\tau)$, $C_3(16, \frac{5}{2}, \frac{1}{2}\tau)$. Now consider the mass of each rectangular sheet as concentrated at its centroid. If the constant density is δ, the mass of I is the area times the thickness times the density, thus: $m_1 = 8 \cdot 15 \cdot \tau \cdot \delta = 120\tau\delta$. Similarly, the masses of the other two rectangular sheets are $m_2 = 48\tau\delta$ and $m_3 = 20\tau\delta$. The total mass is $M = m_1 + m_2 + m_3 = 188\tau\delta$. The centroid of the system of particles C_1, C_2, C_3 has coordinates obtained by use of

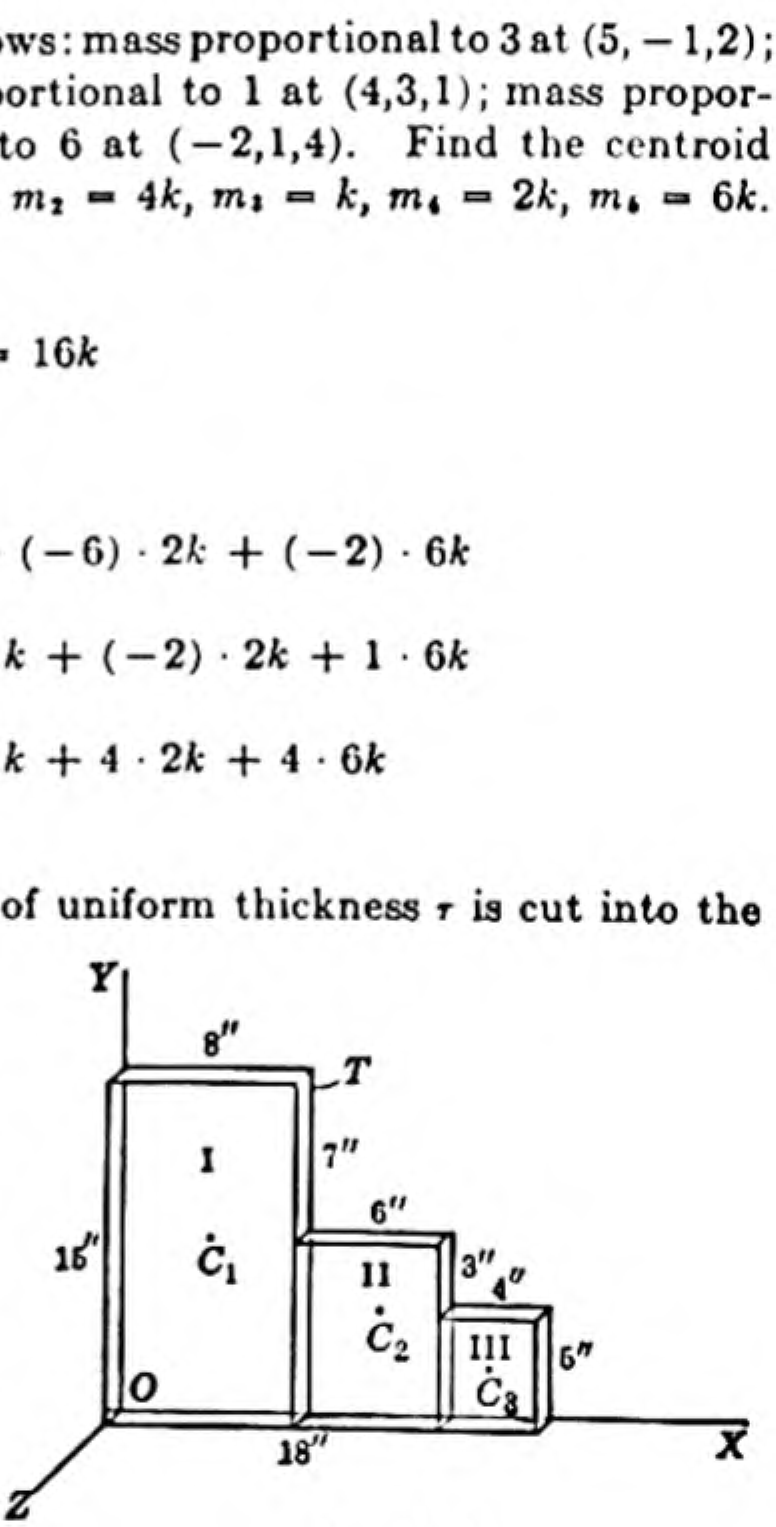


FIG. 165.

equations (1),

$$188\tau\delta\bar{x} = 4(120\tau\delta) + 11(48\tau\delta) + 16(20\tau\delta) = 1328\tau\delta$$
$$\bar{x} = \tfrac{332}{47}$$

Similarly, $\bar{y} = \frac{571}{94}$. Evidently, $\bar{z} = \frac{1}{2}\tau$. The centroid of the metal plate, therefore, has coordinates $\frac{332}{47}$, $\frac{571}{94}$, $\frac{1}{2}\tau$.

EXERCISES

Find the centroid of the following systems of particles (Ex. 1 and 2):

1. Masses of 2, 3, 5, 6 units at points $(1,-2)$, $(3,4)$, $(-3,7)$, $(-1,-2)$, respectively

2. Masses of 2, 4, 5, 9 at points $(1,3,2)$, $(-2,4,4)$, $(0,-1,-3)$, $(6,1,-3)$, respectively

3. Show that the centroid of a system of two particles divides the line joining them into segments inversely proportional to the masses of the particles.

4. Show that the centroid of a system of three particles of equal masses not in a straight line is at the intersection of the medians of the triangle at whose vertices they lie.

5. A piece of wire 10 in. long is bent to form a right angle of which one side is 6 in. and the other 4 in. long. Let the origin be at the vertex and the x axis along the 6-in. side. Find the centroid.

6. A thin rod 24 in. long is cut into two segments each 12 in. long. One of them is joined to the mid-point of the other (let this point be the origin) to form a T-shaped article. Find the centroid.

7. The ends of two slender rods, one 12 in. long and the other 4 in. long and twice as heavy as the first, are joined to form a right angle. Find the centroid if the vertex is chosen for origin and the x axis is the shorter side.

8. The bottom of a tin box is 8 by 10 in. The box is 4 in. deep. If there is no top, find the centroid.

9. Solve Exercise 8 if the box has a cover weighing twice as much per unit area as the sides and bottom.

10. A solid cube of edge 4 in. is placed on top of a cube of the same material of edge 10 in., the centers being in the same vertical line. Find the centroid.

11. A solid cube of edge 10 in. is surmounted by a sphere of the same material and radius 4 in., the centers being in the same vertical line. Find the centroid.

12. Solve Exercise 11 if the cube and sphere are hollow and made of thin sheet metal of density w lb. per square inch.

13. Find the centroid of an open cylindrical tin can of radius 2 in. and altitude 6 in.

14. From a circular plate of radius 8 in., a circular hole of radius 4 in. is cut. If the smaller circle is tangent (internally) to the larger circle, find the centroid. (*Hint:* Consider the portion cut out as having negative mass.)

121. Centroids Found by Integration. Suppose we have a volume V occupied by a mass M, and let it be divided into n elements of volume $\Delta_1 V$, $\Delta_2 V$, . . . , $\Delta_n V$. We shall call the masses $\Delta_1 M$, $\Delta_2 M$, . . . , $\Delta_n M$, which occupy, respectively, these n elements of volume, *elements of mass*. Let $C_1(X_1,Y_1,Z_1)$, $C_2(X_2,Y_2,Z_2)$, . . . , $C_n(X_n, Y_n, Z_n)$ be the centroids of these elements of mass, and let δ_1, δ_2, . . . , δ_n be the densities at points C_1, C_2, . . . , C_n. Since we may regard the mass of each element of mass as concentrated at its centroid, the centroid of the system of

such mass particles has coordinates given by

$$\bar{x}_n = \frac{X_1\,\Delta_1 M + X_2\,\Delta_2 M + \cdots + X_n\,\Delta_n M}{\Delta_1 M + \Delta_2 M + \cdots + \Delta_n M} = \frac{\sum_{i=1}^{n} X_i\,\Delta_i M}{\sum_{i=1}^{n} \Delta_i M}$$

$$\bar{y}_n = \frac{\sum_{i=1}^{n} Y_i\,\Delta_i M}{\sum_{i=1}^{n} \Delta_i M} \qquad \bar{z}_n = \frac{\sum_{i=1}^{n} Z_i\,\Delta_i M}{\sum_{i=1}^{n} \Delta_i M}$$

If we define the centroid of the original mass M as the point whose coordinates $(\bar{x},\bar{y},\bar{z})$ are the limits approached by $\bar{x}_n$, $\bar{y}_n$, $\bar{z}_n$ as the greatest $\Delta_i M$ approaches zero and n increases indefinitely, we have

$$\bar{x} = \frac{\int X\,dM}{\int dM} \qquad \bar{y} = \frac{\int Y\,dM}{\int dM} \qquad \bar{z} = \frac{\int Z\,dM}{\int dM} \tag{3}$$

limits being chosen to include the whole mass. Since we have supposed the density at the point C_i to be δ_i, we have $\lim\limits_{\Delta_i V \to 0} \dfrac{\Delta_i M}{\Delta_i V} = \delta_i$. In other words, δ_i is the value of $\delta = \dfrac{dM}{dV}$ at point C_i. That is, $dM = \delta\,dV$ where δ is the density at the centroid of an element of mass. Hence, equations (3) may be written

$$\bar{x} = \frac{\int X\delta\,dV}{\int \delta\,dV} \qquad \bar{y} = \frac{\int Y\delta\,dV}{\int \delta\,dV} \qquad \bar{z} = \frac{\int Z\delta\,dV}{\int \delta\,dV}$$

limits being chosen to include the entire volume. In the most general case, these integrals are so-called *triple integrals* which will be discussed in Chap. 18. However, certain relatively simple cases can be handled by methods already developed. We note that if the mass is homogeneous, δ is a constant and divides out, and the physical attribute of density plays no part in the determination of $\bar{x}$, $\bar{y}$, $\bar{z}$. We are thus led to speak of the centroid of a *volume*, or of a plane *area*, a curved *surface*, or an *arc* length.

122. Centroid of a Plane Area. To find the centroid of a plane area, we proceed as in the following example.

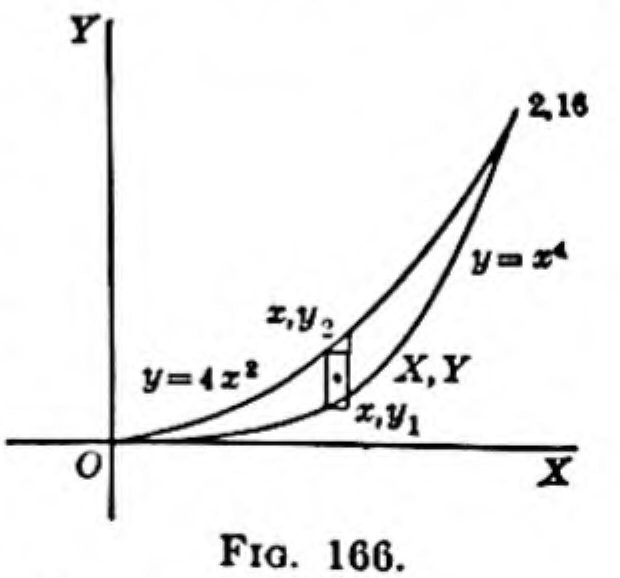


FIG. 166.

Example. Find the centroid of the area in the first quadrant bounded by the curves $y = 4x^2$ and $y = x^4$.

Let us use vertical rectangular elements of area (Fig. 166). We have

$$A = \int_0^2 (y_2 - y_1)\,dx = \int_0^2 (4x^2 - x^4)\,dx = \tfrac{64}{15}$$

To find $\bar{x}$, we note that we may take* $X = x$ and $dM = dA = (y_2 - y_1)\,dx$. Therefore

$$\int X\,dM = \int_0^2 x(y_2 - y_1)\,dx = \int_0^2 x(4x^2 - x^4)\,dx$$
$$= \int_0^2 (4x^3 - x^5)\,dx = \tfrac{16}{3}$$

Hence $$\bar{x} = \tfrac{16}{3} \div \tfrac{64}{15} = \tfrac{5}{4}$$

To find $\bar{y}$, we note that $Y = \frac{1}{2}(y_2 + y_1)$, and $dM = dA = (y_2 - y_1)\,dx$. Therefore

$$\int Y\,dM = \tfrac{1}{2}\int_0^2 (y_2 + y_1)(y_2 - y_1)\,dx = \tfrac{1}{2}\int_0^2 (y_2^2 - y_1^2)\,dx$$
$$= \tfrac{1}{2}\int_0^2 (16x^4 - x^8)\,dx = \tfrac{1024}{45}$$

Hence $$\bar{y} = \tfrac{1024}{45} \div \tfrac{64}{15} = \tfrac{16}{3}$$

In statistical analysis, a frequency distribution is often represented by a *frequency function*, $y = f(x)$. The arithmetic mean of the distribution is simply the abscissa of the centroid of the area between the x axis and the graph of $y = f(x)$.

EXERCISES

Find the centroid of each of the areas described; as a rough check, make a sketch of the area and estimate the position of the centroid (Ex. 1 to 17).

1. The area in the first quadrant bounded by the parabola $y^2 = 8x$ and the line $x = 2$
2. The area bounded by the parabola $y^2 = 4ax$ and its latus rectum
3. The area bounded by the parabola $ay = a^2 - x^2$ and the x axis
4. The area bounded by the parabolas $y^2 = x$ and $x^2 = y$
5. The area in the first quadrant bounded by the line $y = x$ and the curve $y = x^3$
6. The area of a semicircle of radius a
7. The area of one quadrant of a circle of radius a
8. The area in the first quadrant bounded by the ellipse $x = a\cos\varphi$, $y = b\sin\varphi$
9. The area of a right triangle
10. The area of any triangle
11. The area bounded by the parabola $y^2 = 4x$, and the line $2x + y = 4$
12. The area in the first quadrant under the curve $y = e^{-x}$
13. The area in the first quadrant under the curve $y = 1/x^3$ and to the right of the line $x = 1$
14. The area under the curve $y = \cos x$ from $x = 0$ to $x = \pi/2$
15. The area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$
16. The area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$
17. The area in the first quadrant bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$

18. Show that the centroid of the area of a circular sector of radius a and half angle α is at a distance $\dfrac{2}{3}a\dfrac{\sin\alpha}{\alpha}$ from the center. Show that, as α approaches zero, this distance approaches $\frac{2}{3}a$.

* We may take for X any value from x to $x + \Delta x$ (see the fundamental theorem, Art. 105).

19. In polar coordinates, the distance from the pole to the centroid of an element of area of radius r and angle $\Delta\theta$ is $\frac{2}{3}r$, apart from infinitesimals of higher order (see Exercise 18). Show that, therefore, the coordinates of the centroid of the area bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$ can be obtained from the equations

$$A\bar{x} = \tfrac{1}{3}\int_\alpha^\beta r^3 \cos\theta\, d\theta \qquad A\bar{y} = \tfrac{1}{3}\int_\alpha^\beta r^3 \sin\theta\, d\theta$$

20. Find the centroid of the upper half of the circular area bounded by $r = a$.

21. Find the centroid of the area bounded by the cardioid $r = a(1 + \cos\theta)$.

22. Find the centroid of the area consisting of the upper halves of the two loops of the curve $r^2 = a^2\cos\theta$.

123. Centroid of a Volume.

To find the centroid of a volume, we proceed as in the following examples.

Example 1. Find the centroid of a hemispherical volume of radius a. This may be regarded as the volume of revolution obtained by rotating the area in the first

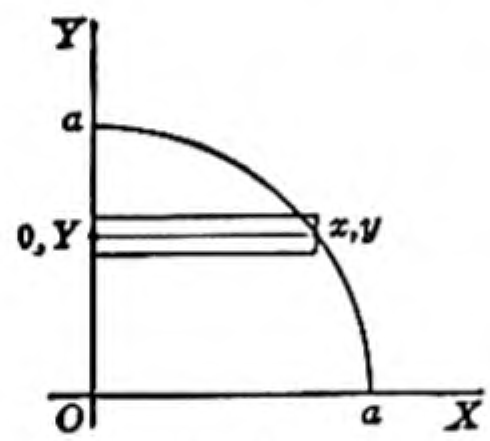


FIG. 167a.

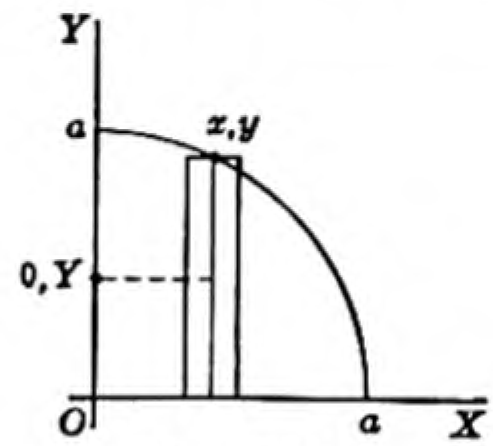


FIG. 167b.

quadrant of the circle $x^2 + y^2 = a^2$ about the y axis. Since the y axis is a line of symmetry, $\bar{x} = 0$. To find $\bar{y}$, we may choose elements of volume in either of two ways.

1. *Disk Method.* Let the horizontal element of area shown in Fig. 167a rotate about the y axis. It generates a circular disk whose centroid is at the point $(0,y)$. We then have $Y = y$, $dM = dV = \pi x^2\, dy$. Hence

$$\bar{y} = \frac{\int_0^a y\cdot \pi x^2\, dy}{\int_0^a \pi x^2\, dy} = \frac{\pi\int_0^a y(a^2 - y^2)\, dy}{\frac{2}{3}\pi a^3} = \frac{\pi}{4}a^4\cdot\frac{3}{2\pi a^3} = \frac{3}{8}a$$

2. *Cylindrical-shell Method.* Let the vertical element of area shown in Fig. 167b rotate about the y axis. It generates a cylindrical shell with centroid at the point $(0,\frac{1}{2}y)$. We have, therefore

$$Y = \tfrac{1}{2}y \qquad dM = dV = 2\pi xy\, dx$$

Hence $$\bar{y} = \frac{\int_0^a \frac{1}{2}y\cdot 2\pi xy\, dx}{\int_0^a 2\pi xy\, dx} = \frac{\pi\int_0^a xy^2\, dx}{\frac{2}{3}\pi a^3} = \frac{\pi\int_0^a x(a^2 - x^2)\, dx}{\frac{2}{3}\pi a^3} = \frac{3}{8}a$$

Example 2. Find the centroid of the volume of an elliptic cone of altitude h and with semiaxes of the base equal to a and b. Let the base of the cone be placed

with major axis on OX and minor axis on OY, and let the vertex be on OZ (Fig. 168). Slice the cone by planes parallel to the xy plane, forming elements of volume (one-fourth of a typical element is shown in the figure). Since OZ is a line of symmetry, $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we evidently may take $Z = z$, $dM = dV = \pi xy\,dz$. Hence

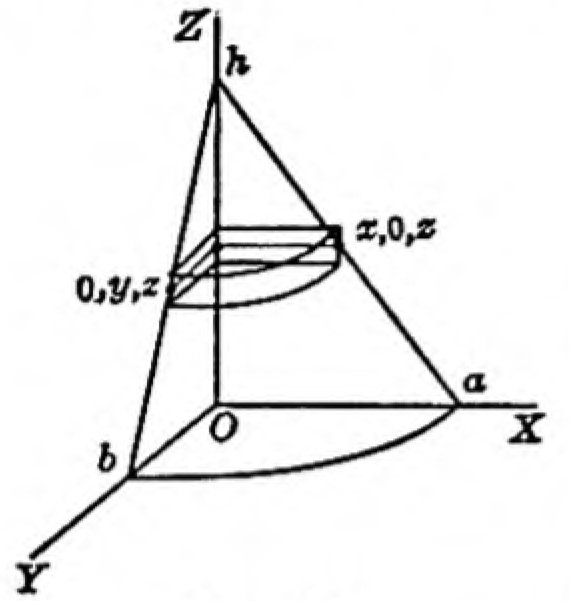


FIG. 168.

$$\bar{z} = \frac{\int_0^h z \cdot \pi xy\,dz}{\int_0^h \pi xy\,dz}$$

We have the following equations to determine x and y in terms of z:

$$\frac{x}{a} + \frac{z}{h} = 1 \qquad y = 0$$

so that

$$x = \frac{a}{h}(h - z)$$

$$\frac{y}{b} + \frac{z}{h} = 1 \qquad x = 0$$

so that

$$y = \frac{b}{h}(h - z)$$

Hence

$$\pi \int_0^h zxy\,dz = \pi \frac{ab}{h^2} \int_0^h z(h - z)^2\,dz = \frac{1}{12}\pi abh^2$$

and

$$V = \pi \int_0^h xy\,dz = \frac{\pi ab}{h^2} \int_0^h (h - z)^2\,dz = \frac{1}{3}\pi abh$$

Therefore

$$\bar{z} = \frac{1}{12}\pi abh^2 \cdot \frac{3}{\pi abh} = \frac{1}{4}h$$

EXERCISES

In the following, as a check on your calculations, estimate the result from a figure.

1. The area bounded by the parabola $y^2 = 4ax$, the x axis, and the line $x = a$ rotates about the x axis. Find the centroid of the volume.

2. The area of Exercise 1 rotates about the y axis. Find the centroid of the volume. Solve by two methods.

3. The area of Exercise 1 rotates about the line $x = a$. Find the centroid of the volume. Solve by two methods.

4. Find the centroid of the volume of a right circular cone of altitude h and radius of base a.

5. The area in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ rotates about the x axis (half of a prolate spheroid). Find the centroid of the volume.

6. The area in the first quadrant bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$ rotates about the x axis. Find the centroid of the volume.

7. The area in Exercise 6 rotates about the y axis. Find the centroid of the volume.

8. The upper half of the area bounded by the circle $x^2 + y^2 = a^2$ rotates about the

line $x = b$ $(b > a)$. Find the centroid of the volume generated (upper half of a torus).

9. The upper half of the area bounded by the curves $y^2 = 8x$ and $y^2 - 4x - 8 = 0$ rotates about the x axis. Find the centroid of the volume generated.

10. Find the centroid of the volume inside the cylinder $x^2 + y^2 = 2a^2$, outside the hyperboloid $x^2 + y^2 - z^2 = a^2$, and above the xy plane.

11. Find the centroid of the volume bounded by the paraboloid $x^2 + y^2 = 4az$ and the cone $x^2 + y^2 = z^2$.

12. A tetrahedron has three mutually perpendicular faces. The three mutually perpendicular edges are of lengths a, b, c. Find the centroid of the volume.

13. An ellipsoid has semiaxes of lengths a, b, c. Find the centroid of the volume cut off by a plane through the center perpendicular to an axis.

14. A right pyramid of altitude h has a square base of side a. Find the centroid of the volume.

15. Find the centroid of the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x$.

16. Find the centroid of the volume in the first octant under the plane $z = y$ and inside the cylinder $y^2 = 4 - x$.

17. A wooden hemisphere has radius $2a$. A cylindrical hole of radius a is bored through, the axis of the cylinder coinciding with the axis of the hemisphere. Find the centroid of the remaining solid.

18. A solid wooden buoy is made of two right circular cones with common radius 1 ft. The upper cone has altitude 6 in., and the bottom cone has altitude 3 ft. Find the centroid.

19. A plumb bob consists of a cone of radius $\frac{1}{2}$ in. and altitude 2 in., surmounted by a cylinder of radius $\frac{1}{4}$ in. and altitude $\frac{1}{2}$ in. Find the centroid.

20. A weight consists of a cylinder of radius 1 in. and altitude 2 in. surmounted by a hemisphere of radius 2 in. The center of the base of the hemisphere is at the center of the top of the cylinder. Find the centroid.

124. Centroid of an Arc; Centroid of a Surface of Revolution.

To find the centroid of an arc or of a surface of revolution, we proceed as in the following examples.

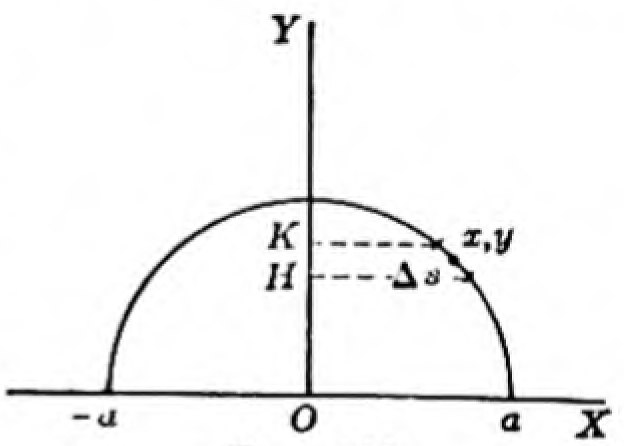


FIG. 169.

Example 1. Find the centroid of a semicircular arc of radius a. We choose the upper semicircle of $x^2 + y^2 = a^2$, that is, $y = \sqrt{a^2 - x^2}$ (Fig. 169). Since the y axis is a line of symmetry, $\bar{x} = 0$. To find $\bar{y}$, we choose an element of arc length Δs. Let (x,y) be the middle point of this element of arc. Now, $\int dM = \int ds$ with suitable limits chosen is simply the length of the semicircle, namely, πa. To find

$$\int Y\,dM = \int Y\,ds$$

we have $\qquad x^2 + y^2 = a^2 \qquad$ and therefore $\qquad 2x + 2y\dfrac{dy}{dx} = 0$

Hence

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$ds = \sqrt{1 + \frac{x^2}{y^2}}\,dx = \sqrt{\frac{y^2 + x^2}{y^2}}\,dx = \frac{a}{y}\,dx$$

We may take $Y = y$.* This gives

$$\int Y\,ds = \int_{-a}^{a} y \cdot \frac{a}{y}\,dx = a\int_{-a}^{a} dx = 2a^2$$

Therefore
$$\bar{y} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

Example 2. Find the centroid of a hemispherical surface of radius a. Let the first quadrant of the circle $x^2 + y^2 = a^2$ rotate about the y axis. Since the axis of revolution is a line of symmetry, $\bar{x} = 0$. To find $\bar{y}$, divide the arc into elements of length Δs. A typical element is shown in Fig. 170; let its middle point be (x,y). When the arc rotates about the y axis, this element generates a circular band whose centroid is at the point $(0,Y)$. If the area of this band is ΔA, then

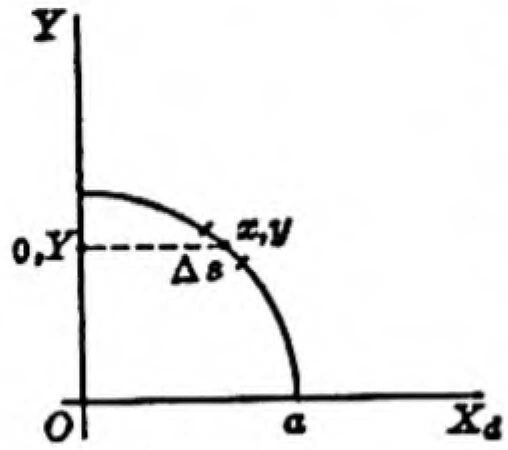


FIG. 170.

$$\bar{y} = \frac{\int Y\,dA}{\int dA}$$

with suitable limits chosen. To evaluate these integrals, we first note that $\int dA$ is the area of the hemisphere, namely, $2\pi a^2$. We may take $Y = y$. Also, since

$$ds = \left(\frac{a}{y}\right) dx$$

(see Example 1),

$$\int Y\,dM = \int y\,dA = \int y \cdot 2\pi x\,ds = \int_0^a y \cdot 2\pi x \cdot \frac{a}{y}\,dx$$
$$= \pi a \int_0^a 2x\,dx = \pi a^3$$

Therefore
$$\bar{y} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}$$

125. Theorems of Pappus. There are two useful theorems named after the Greek mathematician Pappus (*ca.* 340 A.D.).

First proposition of Pappus. Let a surface of revolution be generated by an arc lying entirely on one side of the axis of revolution; the area of the surface is equal to the length of the generating arc, multiplied by the circumference of the circle described by the centroid of the arc. Thus, if the y axis is the axis of revolution, $A = 2\pi\bar{x}s$.

Second proposition of Pappus. Let a volume of revolution be generated by an area lying entirely on one side of the axis of revolution; the volume is equal to the generating area multiplied by the circumference of the circle described by the centroid of that area. Thus, if the y axis is the axis of revolution, $V = 2\pi\bar{x}A$.

* Although Y is not equal to y, since the centroid of Δs is not at (x,y), still it is between the horizontal lines through H and K. The fundamental theorem of integral calculus allows us to take any value for Y between the lengths OH and OK in finding $\int Y\,ds$; y is such a value.

The student should formulate proofs of these propositions as an exercise.

EXERCISES

1. Find the centroid of one quadrant of the circumference of a circle of radius a.
2. Solve Exercise 1, using the equation $r = a$ (polar coordinates).
3. Find the centroid of the first quadrant arc of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.
4. Solve Exercise 3, using the parametric equations $x = a\cos^3\varphi$, $y = a\sin^3\varphi$.
5. Find the centroid of the arc of the circle $r = a$ between $\theta = -\alpha$ and $\theta = \alpha$.
6. Find the centroid of the arc of the first arch of the cycloid

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$$

7. Find the centroid of the entire arc of the cardioid $r = a(1 + \cos\theta)$.
8. Find the centroid of the lateral surface of a right circular cone of altitude h.
9. Find the centroid of that portion of the surface of a sphere of radius a cut off by a plane b units from the center (spherical zone of one base).
10. Find the centroid of that portion of a spherical surface cut off by parallel planes at distances b and c from the center (spherical zone).
11. Prove the propositions of Pappus stated in Sec. 125.

Use the propositions of Pappus to find the following (Ex. 12 to 16):

12. The centroid of a semicircular arc
13. The surface of a right circular cone
14. The surface of a torus
15. The volume of a torus
16. The centroid of a semicircular area

17. A buoy consists of a cone of radius 2 ft. and altitude 8 ft. surmounted by a cone of radius 2 ft. and altitude 1 ft. If the buoy is made of sheet metal (whose thickness may be neglected), find the centroid.
18. A buoy consists of a cone of radius 2 ft. and altitude 8 ft. surmounted by a hemisphere of radius 2 ft. If the buoy is made of sheet metal, find the centroid.

126. Fluid Pressure. If a plane surface is submerged vertically in a liquid, then the liquid exerts a certain total force upon it. To find this force, we shall imagine the surface divided into n elements of area $\Delta_1 A, \Delta_2 A, \ldots, \Delta_n A$ and find the force on each of these elements. The resultant of these *elements of force* will be the total force upon the surface. The *pressure* exerted by a liquid is defined as the *force* that it exerts *per unit area*. In computing our elements of force, we shall use the fact that, at a given point in the liquid, the force is exerted equally in all directions.

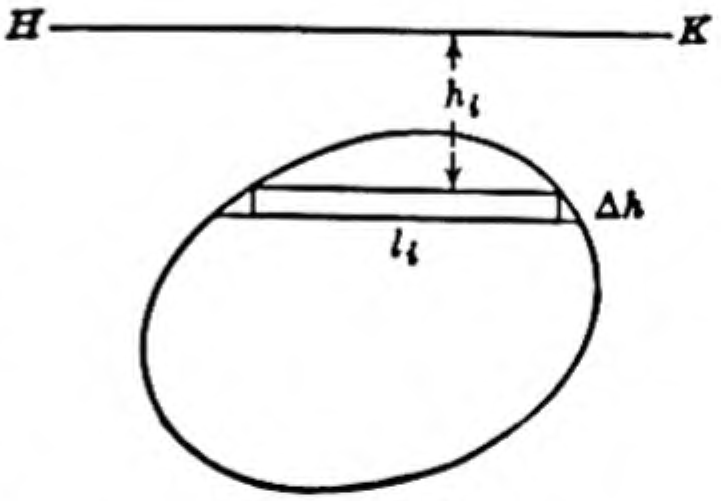


FIG. 171.

Consider a given plane area A, submerged in a liquid so that its plane is vertical. To find the total force exerted by the liquid on this plane area, consider n horizontal rectangular elements of area of lengths $l_1, l_2, \ldots, l_n$ and heights Δh (Fig. 171). Let the depths of the tops of these elements

below the line HK in which the plane of A cuts the surface of the liquid be, respectively, $h_1, h_2, \ldots, h_n$. Now, imagine the ith element of area rotated through 90 deg. about its top edge. Every point of the element would then be at a depth h_i below the surface of the liquid, and the force exerted on such an element would be simply the weight of the column of liquid standing upon it. The volume of such a column would be $h_i l_i \Delta h = h_i \Delta_i A$ if we designate the area of the element by $\Delta_i A$. If we denote by w the weight per unit volume of the liquid, then the force would be $wh_i \Delta_i A$. Next, imagine the ith element of area rotated through 90 deg. about its lower edge, and consider the force exerted upon it in this position. Every point would be at a depth h_{i+1}. The force would, therefore, be $wh_{i+1} \Delta_i A$. Now, since the force at any given depth is exerted equally in all directions, it is clear that the force $\Delta_i F$ on the vertical plane element of area is between the two forces just calculated; that is,

$$wh_i \Delta_i A < \Delta_i F < wh_{i+1} \Delta_i A$$

Hence $\qquad \Delta_i F = w\eta_i \Delta_i A = w\eta_i l_i \Delta h \qquad$ where $h_i < \eta_i < h_{i+1}$

Since the lines of action of these forces are all perpendicular to the submerged plane, an approximation to the total force on the submerged plane area is, therefore

$$\sum_{i=1}^{n} w\eta_i \Delta_i A = \sum_{i=1}^{n} w\eta_i l_i \Delta h$$

Hence, by the fundamental theorem,

$$F = \int wh\, dA = \int whl\, dh$$

where limits are chosen to include the entire area.

If we designate by $\bar{h}$ the depth of the centroid of the area A below the surface of the liquid, we have

$$\bar{h} = \frac{\int h\, dA}{A}$$

that is, $\int h\, dA = \bar{h}A$, with limits chosen to extend the integration over the whole area. Consequently

$$F = w\bar{h}A$$

and we may state the following **theorem**: *The force on a submerged vertical plane area equals the pressure $w\bar{h}$ at the centroid of the submerged area, multiplied by the area.*

Example. A circular water main 4 ft. in diameter is closed by a bulkhead whose center is 40 ft. below the surface of the water in the reservoir. Find the force on the bulkhead. Choose coordinate axes with origin at the center of the bulkhead (Fig.

172). The area of a typical horizontal element of area is then $\Delta A = 2x\,\Delta y$, and the depth below the surface of the water is $h = 40 - y$. The element of force is

$$\Delta F = w(40 - y)2x\,\Delta y$$

and therefore

$$F = 2w \int_{-2}^{2} (40 - y)x\,dy$$

Since $x^2 + y^2 = 4$ is the equation of the circle that forms the boundary of the bulkhead, we have

$$F = 2w \int_{-2}^{2} (40 - y)\sqrt{4 - y^2}\,dy$$
$$= 160\pi w \text{ lb.}$$

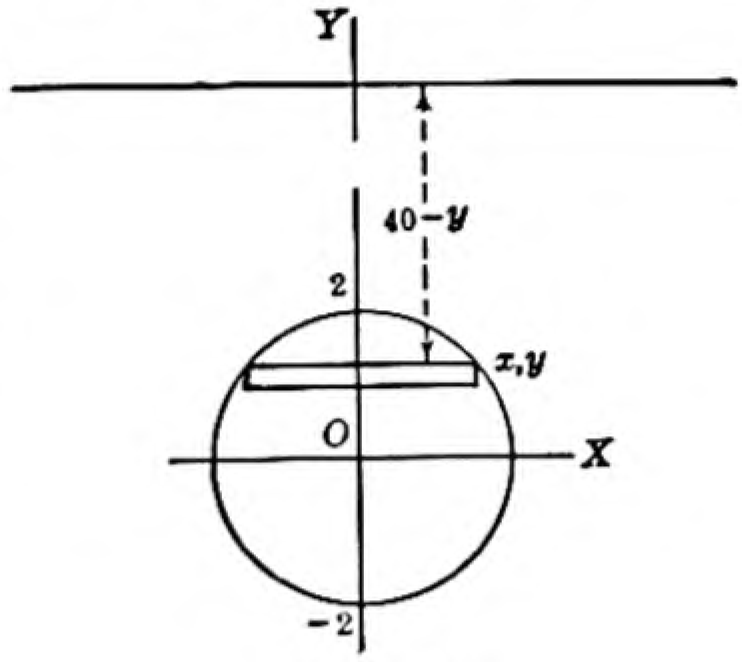


FIG. 172.

If we take $w = 62.4$ lb. per cubic foot

$$F = 31{,}300 \text{ lb. approximately}$$
$$= 15.7 \text{ tons}$$

Using the theorem, we note that the centroid of the area of the bulkhead is its geometrical center. Hence $\bar{h} = 40$ ft. The area of the bulkhead is 4π ft. Therefore

$$F = w\bar{h}A$$
$$= 160\pi w \text{ lb.}$$

127. Center of Pressure. We shall make use of the following principles of mechanics to solve a problem involving the force upon a submerged vertical plane area:

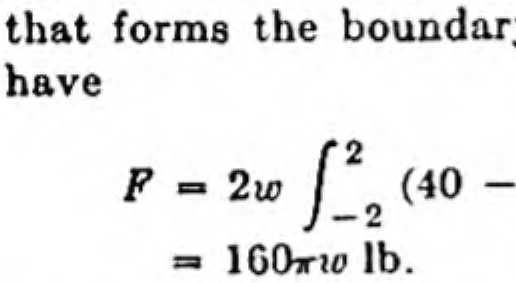


FIG. 173.

1. The resultant of a set of parallel forces is equal to the sum of the forces.

2. Given a line HK and a force F whose line of action LM is at right angles to HK (Fig. 173). Let the shortest distance from LM to HK be l. The *moment of the force F* about the line HK is defined to be the product $l \cdot F$.

3. The moment about a given line of the resultant of a set of forces is equal to the sum of the moments of the separate forces.

In the preceding section, we found the total force on a vertically submerged plane area A (Fig. 171). Let us consider again the elements of force of which a typical one was

$$\Delta_i F = w\eta_i\,\Delta_i A \qquad \text{where } h_i < \eta_i < h_{i+1}$$

This force may be regarded as acting at right angles to the plane of the area A, with its line of action at a distance η_i from HK (Fig. 171). Hence, by (2), the moment of this force is $\eta_i\,\Delta_i F = w\eta_i^2\,\Delta_i A$. The sum of the moments of the n elements of force is

$$\sum_{i=1}^{n} w\eta_i^2\,\Delta_i A = \sum_{i=1}^{n} w\eta_i^2 l_i\,\Delta h$$

If we increase the number of elements of area and take the limit as Δh is made to approach zero, the sum of the moments has a limit (by the fundamental theorem)

$$E = \int wh^2\,dA = \int wh^2 l\,dh$$

with limits chosen so as to extend the integration over the entire area A.

Now, since all the elements of force act at right angles to the plane of A, their resultant is, by (1), just the sum

$$\sum_{i=1}^{n} w\eta_i\,\Delta_i A$$

and the limit of this sum is $F = \int wh\,dA = \int whl\,dh$ with suitable limits of integration.

We now find the depth $\bar{h}$ at which the resultant force F must act to make its moment about HK equal to E. By (3) this gives

$$\bar{h}F = \int wh^2\,dA \tag{4}$$

and from this we can calculate $\bar{h}$. The point P at which F acts is called the *center of pressure* of the area A, and formula (4) gives the *depth* of the center of pressure.

Example. Find the center of pressure for the bulkhead in the Example of Art. 126. We have

$$F = 160\pi w \qquad dA = 2x\,dy \qquad h = 40 - y$$

Hence

$$\begin{aligned} 160\pi w \cdot \bar{h} &= \int_{-2}^{2} w(40 - y)^2 2x\,dy \\ &= 2w \int_{-2}^{2} (40 - y)^2 \sqrt{4 - y^2}\,dy \\ &= 6404\pi w \end{aligned}$$

Therefore

$$\bar{h} = \tfrac{6404}{160} = 40.025 \text{ ft.}$$

That is, the center of pressure P is 40.025 ft. below the surface of the water. It is, therefore, 0.025 ft. = 0.30 in. below the center of the bulkhead. Because of symmetry, P is situated on the y axis. If a brace were set at P, it would hold the bulkhead against the force exerted by the water, and the bulkhead would have no tendency to turn about any axis.

EXERCISES

Solve the following problems by integration, and check by the theorem of Art. 126 where convenient (Ex. 1 to 11):

1. Find the force on one end of a rectangular tank 2 ft. wide and 18 in. deep if the tank is full of water.

2. Find the force on one face of a plank 10 in. by 8 ft., submerged vertically with its upper end in the surface of the water.

3. Solve Exercise 2 if the top of the plank is 3 ft. below the surface.

4. The ends of a trough are equilateral triangles of side 2 ft. Find the force on one end if the trough is full of water.

5. Solve Exercise 4 if the water in the trough is 1 ft. deep.

6. A trapezoidal dam is 1000 ft. long at the top, 800 ft. long at the bottom, and 60 ft. high. If the face is vertical and the water is level with the top, find the force on the dam.

7. A circular cylindrical boiler is 6 ft. in diameter and is placed with its axis horizontal. If it is full of water, find the force on one end.

8. Solve Exercise 7 if the water is 4 ft. deep.

9. A cylindrical tank with axis horizontal has elliptical cross section with major axis 6 ft. and vertical minor axis 4 ft. Find the force on one end if the tank is full of oil weighing 50 lb. per cubic foot.

10. An opening in a dam is closed by a rectangular gate 3 ft. wide and 2 ft. high. The top of the gate is 50 ft. below the surface of the water. Find the force on the gate.

11. The ends of a trough full of water are parabolic segments. If the trough is 3 ft. wide and 2 ft. deep, find the force on one end.

Find the center of pressure in the following cases (Ex. 12 to 17):

12. Exercise 4
13. Exercise 6
14. Exercise 7
15. Exercise 9
16. Exercise 10
17. Exercise 11

18. A parabolic segment with base $2b$ and altitude a is submerged with the base in the surface of a liquid and its axis vertical. Find the center of pressure.

128. Moment of Inertia. Consider a mass m, concentrated at a point P, and let λ be any line (or HK any plane) at a distance R from P. In Art. 120, we defined the moment of first order of m with respect to the line λ (or to the plane HK) to be the product Rm. We now define the *moment of second order*, or *moment of inertia* of the mass m with respect to λ (or HK), to be the product $I = R^2m$. The distance R is called the *radius of gyration* of the mass with respect to the line (or plane).

Next consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at points whose distances from λ (or HK) are, respectively, $R_1, R_2, \ldots, R_n$. We define the moment of inertia of the system to be the sum of the moments of inertia of the separate particles; thus

$$I = R_1^2m_1 + R_2^2m_2 + \cdots + R_n^2m_n = \sum_{i=1}^{n} R_i^2m_i$$

The total mass of the system of particles is $M = \sum_{i=1}^{n} m_i$. We may determine the distance R from λ (or HK) at which a particle of mass M should be placed in order that its moment of inertia with respect to λ (or HK) should be equal to I. We need merely solve the equation $I = R^2M$ in which I and M are known. The distance R is called the radius of gyration of the system of particles.

Example. Find the moment of inertia of the following system of particles in the xy plane with respect to the y axis: masses proportional to 2 at $(-3,6)$, to 5 at $(1,2)$, to 7 at $(-2,9)$, and to 4 at $(6,2)$. We have

$$m_1 = 2k \qquad m_2 = 5k \qquad m_3 = 7k \qquad m_4 = 4k$$
$$R_1 = -3 \qquad R_2 = 1 \qquad R_3 = -2 \qquad R_4 = 6$$

Hence
$$I = 9 \cdot 2k + 1 \cdot 5k + 4 \cdot 7k + 36 \cdot 4k = 195k$$

We can find the radius of gyration of the system, for $M = 18k$. Therefore

$$195k = R^2 \cdot 18k \qquad R^2 = \tfrac{195}{18} = \tfrac{65}{6} \qquad \text{and} \qquad R = \sqrt{\tfrac{65}{6}}$$

129. Moment of Inertia of a Continuous Mass. To find the moment of inertia of a continuous mass, we shall follow the line of argument used in finding the centroid. We divide the mass M into n elements of mass $\Delta_1 M$, $\Delta_2 M$, . . . , $\Delta_n M$, with corresponding radii of gyration R_1, R_2, . . . , R_n. We may express the moment of inertia of each element. The sum of these moments is

$$I_n = R_1^2\, \Delta_1 M + R_2^2\, \Delta_2 M + \cdots + R_n^2\, \Delta_n M = \sum_{i=1}^{n} R_i^2\, \Delta_i M$$

If we define the moment of inertia I of the mass to be the limit of I_n as the greatest $\Delta_i M$ is made to approach zero, then, by a familiar argument,

$$I = \int R^2\, dM$$

limits of integration being chosen to include the entire mass.

Let a mass M occupy a volume V, and let the volume be divided into n elements of volume $\Delta_1 V$, $\Delta_2 V$, . . . , $\Delta_n V$ of which the masses are, respectively, $\Delta_1 M$, $\Delta_2 M$, . . . , $\Delta_n M$. Let the average density of the ith element be δ_i (see Art. 119). Then

$$\Delta_i M = \delta_i\, \Delta_i V \qquad \text{and} \qquad I_n = \sum_{i=1}^{n} R_i^2 \cdot \delta_i\, \Delta_i V$$

The moment of inertia of the mass is $I = \int R^2 \delta\, dV$, limits of integration being chosen to include the entire volume. This is, in general, a triple integral which will be discussed in Chap. 18. It reduces in many cases, however, to a simple integral which we can handle by methods already studied. If the density is a constant and equal to unity so that

$$I = \int R^2\, dV$$

we call this the *moment of inertia of the volume V.*

We may define moments of inertia of an *area*, an *arc length*, and a *surface of revolution* in a similar way, obtaining

$$\int R^2\, dA \qquad \int R^2\, ds \qquad \int R^2\, dS$$

where limits of integration are chosen to include the entire area, arc, or surface.

As in the finding of centroids, where it was essential to choose an element (of mass, volume, area, arc, or surface) the coordinates of whose centroid could be found , so, in finding moments of inertia, we must choose an element whose radius of gyration can be expressed in terms of the coordinate system employed. This will be made clear by examples.

Example 1. Area. Find the moment of inertia of a rectangular area of base b and altitude a, with respect to the base. Let the base lie in the x axis, and the left side in the y axis (Fig. 174). Divide the area into n horizontal elements of area of width b and height Δy. The ith element is shown in the figure. Let R_i be the radius of gyration of this element. Then the moment of inertia of the area is $\sum_{i=1}^{n} R_i^2 \cdot b\,\Delta y$. Since, evidently, R_i is greater than y_i but less than y_{i+1}, the limit of this sum is, by the fundamental theorem,

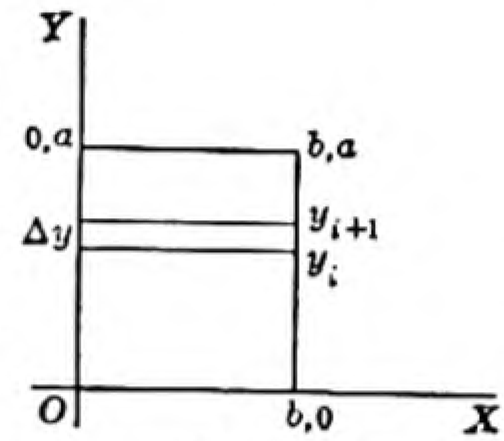


FIG. 174.

$$I = \int_0^a y^2 \cdot b\,dy = \tfrac{1}{3}a^3b$$

If we denote the area of the rectangle by $A = ab$, we may reduce the expression for I, by multiplying it by $A/ab = 1$, to $I = \frac{1}{3}a^2A$.

We recall the concept of a particle of mass, or of a mass concentrated at a point. We may also think of a mass distributed over a surface or along a line and speak of the *surface density* as m units of mass per unit of area. In Example 1, if the surface density is δ units of mass per unit of area, then we may consider a mass of M units distributed evenly over the rectangle, and $M = \delta A$. The moment of inertia of the ith element of mass would be $R_i^2 \cdot \delta \cdot b\,\Delta y$, and

$$I = \int_0^a y^2 \cdot \delta b\,dy = \frac{1}{3}\,\delta a^3 b = \frac{1}{3}\,\delta a^3 b \cdot \frac{M}{\delta ab} = \frac{1}{3}\,a^2 M$$

Note that the radius of gyration of the mass (or of the area) is $R = a/\sqrt{3}$.

We may think of a mass as distributed along a line and speak of the *linear density* as m units of mass per unit of length.

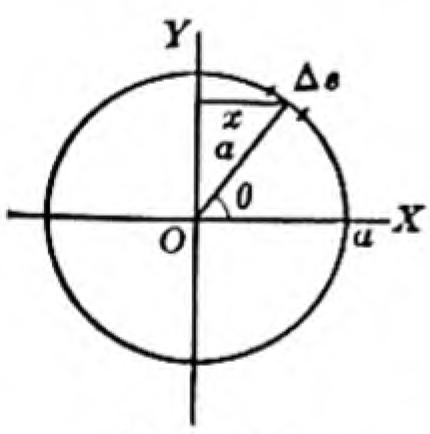


FIG. 175.

Example 2. Arc Length. Find the moment of inertia of the circumference of a circle with respect to a diameter. Let the circle have radius a with center at the origin. We shall illustrate two methods of finding the moment of inertia with respect to the y axis.

Cartesian Coordinates. We have (Fig. 175) $I = \int x^2\,ds$ where limits of integration are chosen to include the entire circle. Also, $x^2 + y^2 = a^2$ and $ds = (a/x)\,dy$. Hence

$$I = 2\int_{-a}^{a} x^2 \cdot \frac{a}{x}\,dy = 2a\int_{-a}^{a} x\,dy$$

$$= 2a\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = a^2\pi$$

If we note that the circumference is $s = 2\pi a$, we have

$$I = \pi a^3 \cdot \frac{s}{2\pi a} = \frac{1}{2} a^2 s$$

If a mass M is distributed evenly along the circumference with linear density δ (for instance, if the circumference represents a thin homogeneous wire), then

$$I = \int x^2 \delta \, ds = \tfrac{1}{2} a^2 \delta s = \tfrac{1}{2} a^2 M$$

Polar Coordinates. Here we have the radius of gyration of the element $x = a \cos \theta$ (Fig. 175), and $ds = a \, d\theta$. Hence

$$I = \int_0^{2\pi} a^2 \cos^2 \theta \cdot a \, d\theta = \frac{a^3}{2} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta = \pi a^3$$

as before.

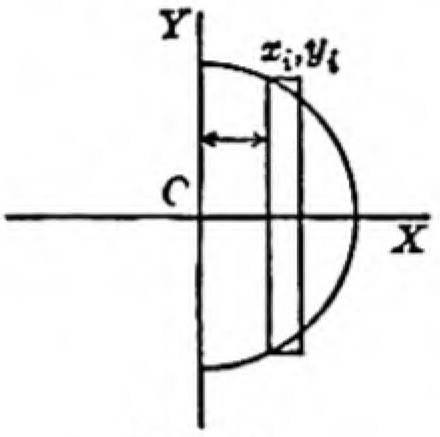


FIG. 176.

Example 3. Volume of Revolution. Find the moment of inertia of a spherical volume with respect to a diameter. If we rotate the area inside the right-hand semicircle $x^2 + y^2 = a^2$ around the y axis, it generates the required volume. We shall find the moment of inertia with respect to the y axis. If we form vertical elements of area (Fig. 176), each one will generate a cylindrical shell, the volume of a typical one being approximately $\Delta_i V = 2\pi x_i \cdot 2y_i \, \Delta x$. Since each point of the shell is at a distance between x_i and x_{i+1} from the axis of moments, we may take $R_i = x_i$ and form the integral $\int R^2 \, dV$. This gives

$$\begin{aligned} I &= \int_0^a x^2 \cdot 2\pi x \cdot 2y \, dx = 4\pi \int_0^a x^3 y \, dx = 4\pi \int_0^a x^3 \sqrt{a^2 - x^2} \, dx \\ &= \tfrac{8}{15} \pi a^5 \end{aligned}$$

Since $V = \frac{4}{3} \pi a^3$, we may write $I = \frac{8}{15} \pi a^5 \cdot \frac{3}{4\pi a^3} \cdot V = \frac{2}{5} a^2 V$.

If the sphere is occupied by a homogeneous mass M of density δ,

$$\int R^2 \, dM = \int R^2 \delta \, dV = \tfrac{2}{5} a^2 \delta V = \tfrac{2}{5} a^2 M$$

Note that the radius of gyration is $\sqrt{\frac{2}{5}} \cdot a$.

Example 4. Find the moment of inertia of a circular sector (area) of radius a and central angle α with respect to a line through the center perpendicular to the plane of the circle. This is often called the *polar moment of inertia* of the sector. Place the sector as shown in Fig. 177. Divide the sector into n elements of area by drawing circles of radii $x_1, x_2, \ldots, x_{n-1}, x_n = a$, at a distance Δx apart. Consider the ith element of area, a segment of a circular ring as shown in the figure. The area of this element is between $\alpha x_i \, \Delta x$ and $\alpha x_{i+1} \, \Delta x$. At the same time, every point of the element is at a distance from O (and therefore from the axis of moments) between x_i and x_{i+1}. Hence, the moment of inertia of the element is between

FIG. 177.

$$x_i^2 \cdot \alpha x_i \, \Delta x = \alpha x_i^3 \, \Delta x \quad \text{and} \quad x_{i+1}^2 \cdot \alpha x_{i+1} \, \Delta x = \alpha x_{i+1}^3 \, \Delta x$$

Therefore, by the familiar argument,

$$I = \int_0^a \alpha x^3 \, dx = \tfrac{1}{4} \alpha a^4$$

Since the area of the sector is $A = \frac{1}{2} \alpha a^2$, we have $I = \frac{1}{2} a^2 A$.

As in the previous examples, if a mass M is distributed over the area with constant surface density δ, then $I = \frac{1}{2}a^2\,\delta A = \frac{1}{2}a^2 M$. The radius of gyration is $R = a/\sqrt{2}$.

Example 5. Find the moment of inertia of the area bounded by a polar curve $r = f(\theta)$ and two radius vectors $\theta = \alpha$ and $\theta = \beta$, with respect to a line through the pole perpendicular to the plane of the curve. Since the element of area in polar coordinates is a circular sector (Fig. 178) with central angle $\Delta\theta$ and radius r_i, the result of Example 4 enables us to write down the moment of inertia of this element of area, namely, $\frac{1}{4}r_i^4\,\Delta\theta$. Consequently, the moment of inertia of the whole area is $\frac{1}{4}\int_\alpha^\beta r^4\,d\theta$.

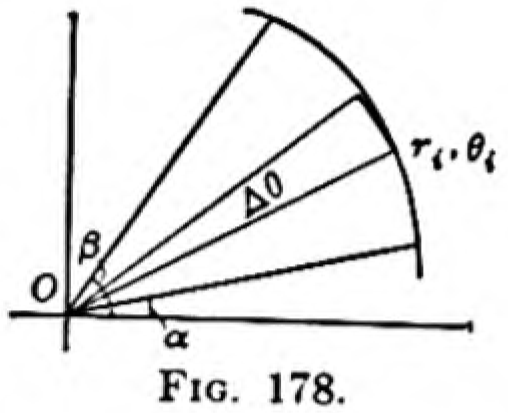


FIG. 178.

The moment of inertia of a body may be interpreted physically as the resistance that the body offers, through its inertia, to rotation about an axis.

In statistics, the "standard deviation" of a distribution represented by a frequency function is simply the radius of gyration, with respect to a vertical line through the centroid (the arithmetic mean) of the area under the frequency curve. Furthermore, it may be interesting to note that, in statistics, moments of higher than second order are frequently used. The moment of kth order about the y axis is defined to be $\int_{-\infty}^{\infty} x^k f(x)\,dx$ where $f(x)$ is the frequency function.

EXERCISES

Find the moment of inertia with respect to the specified line in the following cases.

1. Masses of 3 units at (0,1), 2 units at (−5,2), 4 units at (3,−4), 1 unit at (−1,−2) with respect to each of the coordinate axes

2. Masses of 1 unit at (0,0), 5 units at (−1,1), 3 units at (2,3), 4 units at (3,−2) with respect to the line $y = x$

3. Masses of 2 units at (1,2,1), 3 units at (−3,1,0), 5 units at (−2,−3,4), 4 units at (2,4,−3) with respect to each of the coordinate planes

4. The masses of Exercise 3 with respect to each of the coordinate axes

5. Masses of 5 units at (−1,2,2), 4 units at (5,4,1), 3 units at (6,−2,−1), 2 units at (0,1,−2) with respect to each of the coordinate axes

6. Equal masses concentrated at the corners of a square of side a with respect to one side of the square

7. Equal masses concentrated at the corners of a cube of edge a with respect to an edge of the cube

8. Equal masses concentrated at the corners of a rectangle of length a and width b with respect to a side of length a

9. A straight-line segment (such as a rod or wire) of length l with respect to a perpendicular through one end

10. A wire bent into a rectangle of width 10 in. and length 20 in. with respect to the 10-in. side (use the result of Exercise 9)

11. The area of a circle of radius a with respect to a diameter

12. Solve Exercise 11, using the result of Example 1, Art. 129

13. The area of any triangle of altitude a with respect to the base

14. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ with respect to the coordinate axes

15. The area of Exercise 14 with respect to the line $x = a$

16. The area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ with respect to the y axis

17. The area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$ with respect to the y axis

18. The area of Exercise 17 with respect to the x axis

19. The area in the first quadrant under the curve $y = e^{-x}$ with respect to each of the coordinate axes

20. The length of arc of the curve $y = \ln x$ from $x = 1$ to $x = 2$ with respect to the y axis

21. The circumference of the circle $r = 2a \cos \theta$ with respect to the polar axis

22. The volume of a right circular cone of altitude h and radius of base a with respect to its axis

23. The volume of a right circular cylinder of radius a with respect to its axis

24. Use the result of Exercise 23 to solve Exercise 22.

25. Use the result of Exercise 23 to solve Example 3 of Art. 129.

26. The volume obtained by revolving the area bounded by $y^2 = 4ax$, the line $x = a$, and the x axis about the x axis, with respect to the x axis

27. The volume obtained by revolving the area bounded by $y^2 = 4ax$ and the line $x = a$ about the y axis, with respect to the y axis

28. The volume obtained by revolving the area of Exercise 27 about the line $x = a$, with respect to $x = a$

29. The volume obtained by rotating the triangle whose vertices are (0,2), (0,0), (1,0) about the line $x = 2$, with respect to the line $x = 2$

30. The volume of the torus obtained by rotating the circle $x^2 + y^2 = a^2$ about the line $x = b$ $(b > a)$ with respect to the line $x = b$

31. The volume of the oblate spheroid obtained by revolving the ellipse $x = a \cos \varphi$, $y = b \sin \varphi$ about the y axis, with respect to the y axis

32. The area of a circle of radius a with respect to a line through the center perpendicular to the plane of the circle (polar moment of inertia)

33. The area of the circle $r = 2a \cos \theta$ with respect to a line through the pole perpendicular to the plane of the circle

34. The area bounded by the cardioid $r = a(1 + \cos \theta)$ with respect to a line through the pole perpendicular to the plane of the cardioid

35. The area bounded by one loop of the curve $r = a \cos 3\theta$ with respect to a line through the pole perpendicular to the plane of the curve

36. Solve Exercise 35 for the curve $r = a \cos n\theta$.

37. The surface of a sphere of radius a with respect to a diameter

38. Solve Exercise 37, using polar coordinates.

39. The surface of a right circular cone of altitude h and radius of base a, with respect to its axis

40. The surface of the torus obtained by rotating the circle $x^2 + y^2 = a^2$ about the line $x = b$ $(b > a)$, with respect to the line $x = b$

130. Approximate Integration; Trapezoidal Rule. In many practical problems, it is necessary to calculate the value of a definite integral. As we have seen, this problem is equivalent to the problem of calculating the area bounded by a given curve, the x axis, and two ordinates. Suppose the equation of the curve to be $y = f(x)$ and that we wish to calculate

$\int_a^b f(x)\,dx$. Now it may well happen that $\int f(x)\,dx$ cannot be expressed in terms of elementary functions, so that the usual method of evaluating the definite integral cannot be applied. Or it may happen that we wish to find an area whose bounding curve is determined from experimental data by plotting points and drawing a smooth curve through them. In such a case the curve is specified graphically, but its equation may not be known. Hence, the usual method of evaluating a definite integral to find the area is not available. In all such cases, we must resort to some method of approximate computation of the area or integral in question. The computation of the sum of rectangular areas that underlies the definition of the definite integral as the limit of a sum (Chap. 14) would be one such approximate method. Two other methods that yield simple and easily applied formulas, the trapezoidal rule and Simpson's rule, will be developed in this and the next section. No attempt will be made to discuss the error resulting from the approximation.

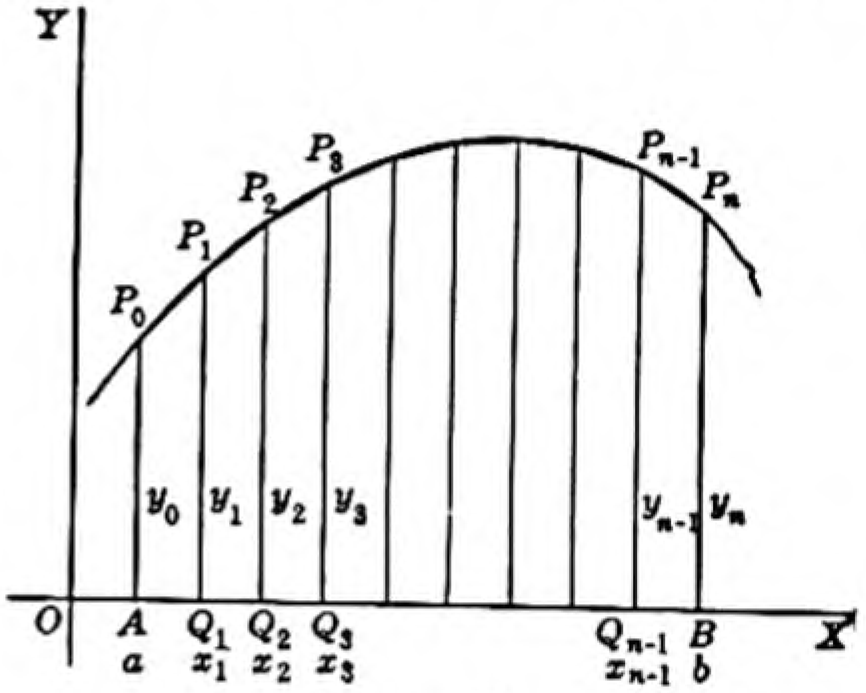


FIG. 179.

Let $y = f(x)$ have a graph as indicated in Fig. 179, and suppose that we are required to find approximately the area bounded by this curve, the x axis, and the ordinates at $x = a$ and $x = b$. Divide the segment AB of the x axis into n subsegments by choosing points with abscissas $x_1, x_2, \ldots, x_{n-1}$. Let $a = x_0$ and $b = x_n$. Let the lengths of the subsegments be

$$\begin{aligned} x_1 - x_0 &= \Delta_1 x \\ x_2 - x_1 &= \Delta_2 x \\ &\cdots\cdots \\ x_n - x_{n-1} &= \Delta_n x \end{aligned}$$

units. Draw ordinates of lengths $y_0, y_1, \ldots, y_n$ units at these points of division, and let them cut the curve in points $P_0, P_1, P_2, \ldots, P_n$ as indicated in the figure. In Art. 104, we next drew horizontal lines through these points to form rectangles and saw that the sum of their areas is an approximation to the required area. Instead of this, draw lines P_0P_1, $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$ forming *inscribed trapezoids*. The sum of the areas of these trapezoids will be an approximation to the area under the curve. We proceed to calculate this sum.

The area of the trapezoid $AQ_1P_1P_0$ is

$$\tfrac{1}{2}(AP_0 + Q_1P_1)\cdot AQ_1 = \tfrac{1}{2}(y_0 + y_1)\,\Delta_1 x$$

The area of the next trapezoid is $\frac{1}{2}(y_1 + y_2)\,\Delta_2 x$. The area of the next is $\frac{1}{2}(y_2 + y_3)\,\Delta_3 x$, and so on. Clearly, the sum of the areas of the trapezoids is

$$S = \tfrac{1}{2}(y_0 + y_1)\,\Delta_1 x + \tfrac{1}{2}(y_1 + y_2)\,\Delta_2 x + \cdots + \tfrac{1}{2}(y_{n-1} + y_n)\,\Delta_n x \tag{5}$$

If we divide the segment AB into n *equal* subsegments so that

$$\Delta_1 x = \Delta_2 x = \cdots = \Delta_n x = \Delta x,$$

formula (5) obviously reduces to

$$\star \qquad S = (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\,\Delta x \tag{6}$$

This is known as the *trapezoidal rule*. If the curve is sufficiently regular to permit a choice of equal intervals Δx, this rule is more convenient than (5). Evidently, if the equation of the curve is known, the ordinates $y_0, y_1, y_2, \ldots, y_n$ can be calculated. If the curve is given graphically, that is, sketched through points plotted from empirical data, the ordinates must be measured or their lengths taken directly from the tabulated data. In the latter case, formula (5) may prove to be more convenient than (6).

Since the value of any definite integral can be represented by the area under a curve, it can be approximated by use of the trapezoidal rule. It is clear from geometrical considerations that the larger the number of divisions chosen the better will be the approximation. We may write

$$\int_a^b f(x)\,dx = [\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)]\,\Delta x \text{ approximately}$$

Example 1. To calculate an approximate value for $\int_1^9 \frac{dx}{x}$ by the trapezoidal rule, we shall make use of eight divisions. This gives $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, . . . , $x_8 = 9$. By calculation we have, since

$$y = \frac{1}{x} = f(x)$$

$$\begin{aligned}
\tfrac{1}{2}y_0 &= \tfrac{1}{2}\cdot\tfrac{1}{1} = 0.5000\\
y_1 &= \tfrac{1}{2} = 0.5000\\
y_2 &= \tfrac{1}{3} = 0.3333\\
y_3 &= \tfrac{1}{4} = 0.2500\\
y_4 &= \tfrac{1}{5} = 0.2000\\
y_5 &= \tfrac{1}{6} = 0.1667\\
y_6 &= \tfrac{1}{7} = 0.1429\\
y_7 &= \tfrac{1}{8} = 0.1250\\
\tfrac{1}{2}y_8 &= \tfrac{1}{2}\cdot\tfrac{1}{9} = \underline{0.0556}\\
& \ \ 2.2735
\end{aligned}$$

Since $\Delta x = 1$, we have

$$S = (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_7 + \tfrac{1}{2}y_8)\,\Delta x = 2.2735$$

We can compare this with the actual value of the integral, namely,

$$\int_1^9 \frac{dx}{x} = \ln x \Big]_1^9 = \ln 9 = 2.1972$$

approximately, so that our calculation gives a result in error by 3 per cent.

Example 2. A farmer owns a piece of land bounded on one side by a river bank and on the other three sides by straight lines, as indicated in Fig. 180. Measurements are made as shown. It is required to find an approximate value for the area of the field. Here the curve $y = f(x)$ is given graphically by plotting points from the measurements. Evidently, formula (5) is more conveniently used than (6). We have $y_0 = 96$,

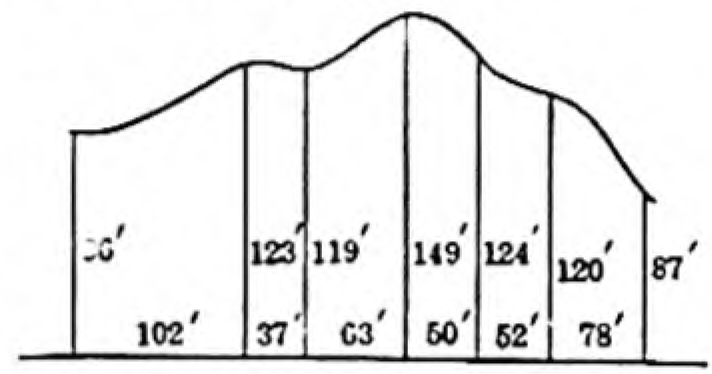


FIG. 180.

$y_1 = 123$, $y_2 = 119$, $y_3 = 149$, $y_4 = 124$, $y_5 = 120$, $y_6 = 87$; $\Delta_1 x = 102$, $\Delta_2 x = 37$, $\Delta_3 x = 63$, $\Delta_4 x = 50$, $\Delta_5 x = 52$, $\Delta_6 x = 78$. Applying formula (5), we get

$$\begin{aligned}
\tfrac{1}{2}(y_0 + y_1)\,\Delta_1 x &= \tfrac{1}{2}(96 + 123)102 = 11{,}169\\
\tfrac{1}{2}(y_1 + y_2)\,\Delta_2 x &= \tfrac{1}{2}(123 + 119)37 = 4{,}477\\
\tfrac{1}{2}(y_2 + y_3)\,\Delta_3 x &= \tfrac{1}{2}(119 + 149)63 = 8{,}442\\
\tfrac{1}{2}(y_3 + y_4)\,\Delta_4 x &= \tfrac{1}{2}(149 + 124)50 = 6{,}825\\
\tfrac{1}{2}(y_4 + y_5)\,\Delta_5 x &= \tfrac{1}{2}(124 + 120)52 = 6{,}344\\
\tfrac{1}{2}(y_5 + y_6)\,\Delta_6 x &= \tfrac{1}{2}(120 + 87)78 = \underline{8{,}073}\\
S &= 45{,}330
\end{aligned}$$

The student will realize that the calculated value of the area of the field is in error, not only because curved lines have been replaced by straight-line segments to form trapezoids whose areas appear in (5), but because the measurements themselves may be in error. Since some measurements are given with two significant figures, others with three, we are not justified in retaining more than two significant figures in our result. We say, therefore, that the area of the field is approximately 45,000 sq. ft.

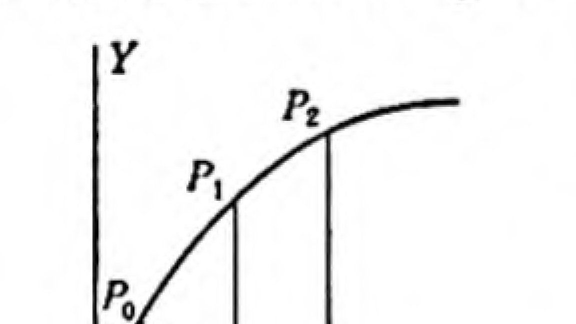


FIG. 181.

131. Simpson's Rule. As a preliminary step in developing Simpson's rule, we shall establish a property of the parabola. Consider the parabola $y = ax^2 + bx + c$ (Fig. 181). Let x_0, x_1, x_2 be the abscissas of points A, Q_1, Q_2 on the x axis, and suppose Q_1 to be the mid-point of AQ_2. Let y_0, y_1, y_2 be corresponding ordinates of points P_0, P_1, P_2 on the parabola, and let $\Delta x = x_2 - x_1 = x_1 - x_0$. We shall show that the area under the parabola from $x = x_0$ to $x = x_2$ is given by the formula

$$AQ_2P_2P_0 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) \tag{7}$$

We have

$$\begin{aligned}
\text{Area} &= \int_{x_0}^{x_2} (ax^2 + bx + c)\,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big]_{x_0}^{x_2} \\
&= \tfrac{1}{6}[2a(x_2^3 - x_0^3) + 3b(x_2^2 - x_0^2) + 6c(x_2 - x_0)] \\
&= \frac{(x_2 - x_0)}{6}[2a(x_2^2 + x_2x_0 + x_0^2) + 3b(x_2 + x_0) + 6c] \\
&= \frac{2\Delta x}{6}[(ax_0^2 + bx_0 + c) + (ax_2^2 + bx_2 + c) + \\
&\qquad a(x_0^2 + 2x_0x_2 + x_2^2) + 2b(x_0 + x_2) + 4c] \\
&= \frac{\Delta x}{3}\left[y_0 + y_2 + 4a\left(\frac{x_0 + x_2}{2}\right)^2 + 4b\left(\frac{x_0 + x_2}{2}\right) + 4c\right] \\
&= \frac{\Delta x}{3}(y_0 + y_2 + 4y_1)
\end{aligned}$$

since
$$x_1 = \frac{x_0 + x_2}{2}$$

This same formula holds in case y is a polynomial of third degree in x (see Exercise 25 below).

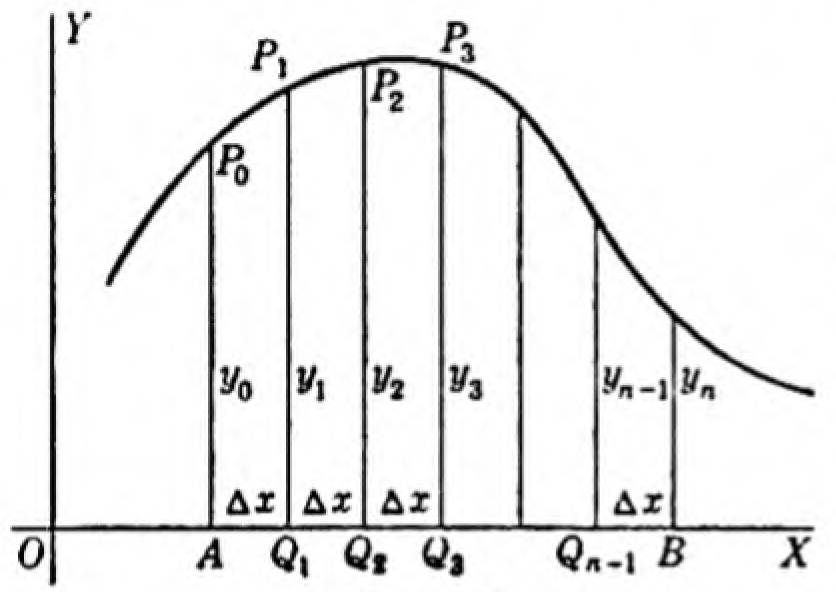


FIG. 182.

Now consider the area shown in Fig. 182. Let us suppose n to be *even*, so that there is an even number of subsegments AQ_1, Q_1Q_2, . . . $Q_{n-1}B$. Furthermore, let the subsegments be *equal in length*, say Δx units long. Instead of joining points $P_0, P_1, P_2, \ldots, P_n$ by straightline segments, let us pass a parabola with vertical axis through P_0, P_1, P_2; another through P_2, P_3, P_4; another through P_4, P_5, P_6; and so on. We shall add together the areas bounded by these parabolic arcs, the even-numbered ordinates, and the x axis.

The area of the first strip $AQ_2P_2P_0$ is, by (7),

$$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the areas of the second, third, . . . , nth strips are, respectively

$$\frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$
$$\frac{\Delta x}{3}(y_4 + 4y_5 + y_6)$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding these together, we get for our approximation to the area under the curve

$$\star\ S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$
$$= \frac{\Delta x}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})]$$

This is called *Simpson's rule.** Note that n is required to be *even* and that the subdivisions of the segment AB of the x axis are all Δx units in length.

We may, therefore, write as an approximate formula for a definite integral

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\{f(x_0) + f(x_n) + 2[f(x_2) + f(x_4) + \cdots + f(x_{n-2})] + 4[f(x_1) + f(x_3) + \cdots + f(x_{n-1})]\} \text{ approximately}$$

As in the case of the trapezoidal rule, the larger n and the smaller Δx, the better will be the approximation obtained. The result is exact if $f(x)$ is a polynomial of third or lower degree (see Exercise 25 below).

Example. To calculate an approximate value for $\int_1^9 \frac{dx}{x}$ by use of Simpson's rule, we may take eight divisions. Values of $x_0, x_1, \ldots, x_n$ and of $y_0, y_1, \ldots, y_n$ are listed in Example 1 of the preceding section. We have

$$\begin{aligned} y_0 &= 1.0000 \\ y_8 &= 0.1111 \\ 2y_2 &= 0.6666 \\ 2y_4 &= 0.4000 \\ 2y_6 &= 0.2858 \\ 4y_1 &= 2.0000 \\ 4y_3 &= 1.0000 \\ 4y_5 &= 0.6668 \\ 4y_7 &= \underline{0.5000} \\ & \ 6.6303 \end{aligned}$$

Also, $\frac{\Delta x}{3} = \frac{1}{3}$. Hence, $S = \frac{1}{3}(6.6303) = 2.2101$. We may compare the results of our methods for calculating this integral, as follows:

Trapezoidal rule......	2.2735 with error of 3 per cent
Simpson's rule........	2.2101 with error of 0.6 per cent
Actual value..........	2.1972 (from a four-place table of logarithms)

EXERCISES

Compute the following integrals by use of the trapezoidal rule as indicated. Check the results by performing the integration where convenient (Ex. 1 to 12).

* Named after Thomas Simpson (1710–1761).

1. $\int_0^5 x^2\,dx$; use $n = 5$.

2. $\int_1^4 \frac{dx}{x^2}$; use $n = 6$.

3. $\int_0^4 x\sqrt{16 - x^2}\,dx$; use $n = 8$.

4. $\int_0^4 \sqrt{16 - x^2}\,dx$; use $n = 8$.

5. $\int_0^9 \sqrt{x^2 + 25}\,dx$; use $n = 9$.

6. $\int_0^8 (64 - x^2)^{1/3}\,dx$; use $n = 8$.

7. $\int_0^4 \sqrt{64 - x^3}\,dx$; use $n = 8$.

8. $\int_{-1}^2 \sqrt{16 + x^4}\,dx$; use $n = 6$.

9. $\int_0^4 \frac{dx}{\sqrt{25 - x^2}}$; use $n = 8$.

10. $\int_0^3 x\sqrt{8 + x^3}\,dx$; use $n = 6$.

11. $\int_0^{\pi/2} \sin\theta\,d\theta$; use $n = 9$.

12. $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$; use $n = 9$.

Use Simpson's rule to compute approximate values for the following integrals as indicated. Check results by performing the integration where convenient (Ex. 13 to 22).

13. $\int_0^4 x^3\,dx$; use $n = 4$. Explain why the result is exact.

14. $\int_{-2}^4 x^4\,dx$; use $n = 6$.

15. The integral of Exercise 4

16. The integral of Exercise 6

17. The integral of Exercise 7

18. The integral of Exercise 8

19. The integral of Exercise 12; use $n = 6$.

20. $\int_0^{\pi/3} \sqrt{\tan\theta}\,d\theta$; use $n = 6$.

21. $\int_0^{\pi/2} \sqrt{2 + \sin^2\theta}\,d\theta$; use $n = 6$.

22. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{2 - \cos^2\theta}}$; use $n = 6$.

23. Integrals like $\int_0^2 e^{-x^2}\,dx$ must frequently be evaluated in statistical work. Tables of this so-called "probability integral" are available, but Simpson's rule may also be used. Calculate an approximate value for this integral, using $n = 10$.

24. Find the length of the ellipse whose parametric equations are $x = 2\cos\varphi$, $y = \sin\varphi$. Use Simpson's rule to evaluate the integral, choosing some convenient value of n.

25. Verify that formula (7) is correct in each of the following cases: (a) $y = k$, (b) $y = mx$, (c) $y = ax^3$. Hence, show that Simpson's rule is exact for $f(x)$ a polynomial of third or lower degree.

26. A *prismatoid* is a solid having as bases two polygons in parallel planes. The other faces are triangles, each of which has a side of one polygon for base and a vertex of the other polygon as its vertex. Special cases of prismatoids occur when two of the adjoining triangles are in the same plane, or when one of the polygonal bases degenerates to a line or point. If the areas of the polygonal bases are A_0 and A_2, respectively, if the distance between these bases is h, and if A_1 is the area of the cross section of the prismatoid made by a plane parallel to the bases and midway between them, then the volume of the solid is given by the *prismoidal formula* $V = (h/6)(A_0 + 4A_1 + A_2)$.

This formula is proved in books on solid geometry. Now let a solid have parallel end areas, and let these be parallel to one of the coordinate planes, say the yz plane. Suppose, further, that the cross-sectional area made by any plane parallel to the end planes is a polynomial function of degree less than 4 of the distance x from the yz plane to the cutting plane: $A = a_0x^3 + a_1x^2 + a_2x + a_3$. Let x_0 and x_2 be the distances from the yz plane to the ends of the solid. Then $V = \int_{x_0}^{x_2} A\,dx$. *Show that* the value of V obtained by using the prismoidal formula is the same as that obtained from the integral. Hence, the prismoidal formula gives an exact result not only for a prismatoid, but for a volume of the type described.

27. Explain why the prismoidal formula can be used to evaluate a definite integral approximately:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{x_2 - x_0}{6}\,[f(x_0) + 4f(x_1) + f(x_2)]$$

(compare Exercise 25). The result is exact if $f(x)$ is a polynomial of degree less than or equal to 3.

28. In Simpson's rule, set $n = 2$, and show that the result is the prismoidal formula.

MISCELLANEOUS EXERCISES

1. The force required to stretch a spring 3 in. is 2 lb. How much work must be done to stretch the spring a distance of 1 ft.?

2. If it requires 16 ft.-lb. of work to stretch a spring 2 ft., find the "spring constant" (see Example 1, Art. 118).

3. The top of a conical reservoir is a circle of radius a ft. The depth is h ft., and the reservoir is full of liquid of density w lb. per cubic foot. Find the work done in pumping the contents to a delivery point at the top of the reservoir.

4. A hemispherical reservoir has radius a ft. and is full of liquid of density w lb. per cubic foot. Find the work done in pumping the contents to a delivery point at the top of the reservoir.

5. Masses of 3, 2, 4, 7 units are placed at points (4,1), (−2,2), (−5,−3), and (3,−1), respectively. Find the centroid.

6. Masses of m, $3m$, $6m$, $8m$, $8m$ are placed at points (1,2,4), (−2,−3,−1), (6,−5,2), (5,4,−2), (−1,2,2), respectively. Find the centroid.

7. A table has legs 3 ft. long weighing 12 lb. each and a top weighing 52 lb. Find the centroid.

8. Find the centroid of an open rectangular box 1 ft. wide, 2 ft. long, and 6 in. deep.

9. A solid sphere of radius 3 in. surmounts a solid circular cylinder of the same material and of radius 4 in. and altitude 6 in. The center of the sphere is in line with the axis of the cylinder. Find the centroid.

10. A rectangular plate has vertices (0,0), (14,0), (14,10), (0,10). A circular hole of radius 2 in. and center at (9,3) is cut from the plate. Find the centroid of the remaining object.

11. Find the centroid of the area bounded by the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

12. Find the centroid of the area between the parabolas $9y = 2x^2$ and $y = 11 - x^2$.

13. Find the centroid of the area under one arch of the curve $y = \sin^2 x$.

14. Find the centroid of the area included between the two curves $y^2 = 8x$ and $y^2 + 2x - 20 = 0$.

15. Draw any quadrilateral; then find the centroid of its area by a graphical method.

16. Find the centroid of the area bounded by the x axis, the lines $x = a$, $x = 2a$, and the hyperbola whose parametric equations are $x = a \sec \varphi$, $y = a \cos \varphi$.

17. Find the centroid of the area in the first quadrant bounded by the hypocycloid $x = a \cos^3 \varphi$, $y = a \sin^3 \varphi$.

18. Find the centroid of the area of the upper half of the cardioid $r = a(1 + \cos \theta)$.

19. Find the centroid of the area of half a circular ring of inside radius a and outside radius b.

20. The area in the first quadrant bounded by the ellipse $x = a \cos \varphi$, $y = b \sin \varphi$ rotates about the y axis (half an oblate spheroid). Find the centroid of the volume.

21. Find the centroid of the volume inside the sphere $x^2 + y^2 + z^2 = 9a^2$, outside the hyperboloid $x^2 + y^2 - z^2 = a^2$, and above the xy plane.

22. Find the centroid of the volume of a spherical segment of one base of radius a and altitude h.

23. A right pyramid of altitude h has a rectangular base of sides a and b. Find the centroid of the volume.

24. Find the centroid of the smaller volume bounded by the surfaces $x^2 + y^2 = 25$, $4y^2 = 9x$, $z = y$, $z = 0$ (see Exercise 11, page 311).

25. A wooden top consists of a cone of radius 1 in. and altitude 3 in. surmounted by a hemisphere of radius 1 in. Find the centroid.

26. Find the centroid of the arc of the parabola $y^2 = 4x$ from the origin to the point (4,4).

27. Find the centroid of the surface obtained by rotating one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the polar axis.

28. A trough 4 ft. deep and 6 ft. wide has semielliptical ends. Find the force on one end if the trough is full of water.

29. The ends of a trough are parabolic segments. If the trough is 2 ft. wide and 2 ft. deep, find the force on one end.

30. An opening in a dam is closed by a circular bulkhead 2 ft. in diameter. If the center is 60 ft. below the surface of the water, find the force on the bulkhead.

31. Find the moment of inertia of the area of an ellipse with respect to its major and its minor axes.

32. Find the moment of inertia of the area under the curve $y = \cos x$ from $x = -\pi/2$ to $x = \pi/2$ with respect to the x axis.

33. Find the moment of inertia with respect to the y axis of the area in the first quadrant bounded by the hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

34. Find the moment of inertia with respect to the z axis of the volume bounded by the surfaces $x^2 + y^2 = 4az$ and $x^2 + y^2 = z^2$.

35. The area in the first quadrant bounded by the hypocycloid of Exercise 33 revolves about the y axis. Find the moment of inertia of the volume with respect to the y axis.

36. Find the moment of inertia of the area bounded by the curve $r^2 = a^2 \cos \theta$ with respect to a line through the pole perpendicular to the plane of the curve.

37. Find the moment of inertia of the area bounded by the curve $r^2 = a^2 \sin 2\theta$ with respect to a line through the pole perpendicular to the plane of the curve.

38. Evaluate the integral $\int_0^4 \sqrt{x^2 + 1}\, dx$. Compute approximate values by the trapezoidal rule and by Simpson's rule, using $n = 4$. Compare the results.

39. Same as Exercise 38 for $\int_0^{\pi/2} \sin^2 x\, dx$. Use $n = 6$.

40. Same as Exercise 38 for $\int_0^4 xe^{-x}\, dx$. Use $n = 8$.

CHAPTER 17

PARTIAL DIFFERENTIATION

132. Functions of More than One Variable. In all of our work so far, we have been concerned with functions of a single independent variable. We can, however, think of any number of examples of quantities that depend upon two or more independent variables. For instance, the area of a rectangle is a function of its base and altitude; the mass of a rectangular solid of homogeneous material is a function of its length, breadth, height, and density; the present value of an annuity is a function of the size and frequency of the payments, the interest rate, and the length of time during which payments are to be made.

If z is a function of two variables x and y, we write $z = f(x,y)$. If u is a function of three variables x, y, z, we write $u = f(x,y,z)$, and similarly for functions of any number of variables. A function of two variables $z = f(x,y)$ can be conveniently represented geometrically by the z coordinate of a point on a surface in space of three dimensions. We shall confine our discussion largely to functions of two variables and give a brief treatment of some of the most important principles involved; a more extensive discussion of the subject is best deferred to a more advanced course.

133. Limit of a Function of More than One Variable. The idea of the limit of a function of a single variable can be extended to the case of the limit of a function of two or more variables. We say that the *limit of $f(x,y)$ is l as x approaches a and y approaches b*, provided that the difference between l and $f(x,y)$ can be made numerically as small as we please by taking x close enough to a and y close enough to b. Note that we are in no way concerned with what happens when $x = a$ and $y = b$. Geometrically, this means that the z coordinate QP of the point $P(x,y,z)$ of the surface $z = f(x,y)$ approaches a length $l = RS$ as a limit as the point $Q(x,y,0)$ approaches the point $R(a,b,0)$ *along any path whatever in the xy plane* (Fig. 183). The point S may or may not be a point of the surface. If S is a point of the surface so that $l = f(a,b)$, then the function $f(x,y)$ has the property that $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b)$ and is said to be *continuous* at the point (a,b). If $f(x,y)$ is continuous at each point of a certain region in the xy plane, it is said to be continuous throughout this

region. In the following discussion the functions are assumed to be continuous at all points involved unless otherwise expressly stated. The student should compare this definition with that given in Art. 7 and Art. 11 for the limit and continuity of a function of a single variable.

It is very important to observe that, in defining $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$, it is understood that the limiting value of the function is to be quite independent of the manner in which x approaches a and y approaches b.

We may state our definition of the limit of $f(x,y)$ in precise language as follows: The limit of $f(x,y)$ as x approaches a and y approaches b is l provided that, for any given positive number ϵ, however small, there exists a number δ such that $|f(x,y) - l| < \epsilon$ for all values of $x \neq a$ and $y \neq b$ such that $|x - a| < \delta$ and $|y - b| < \delta$.

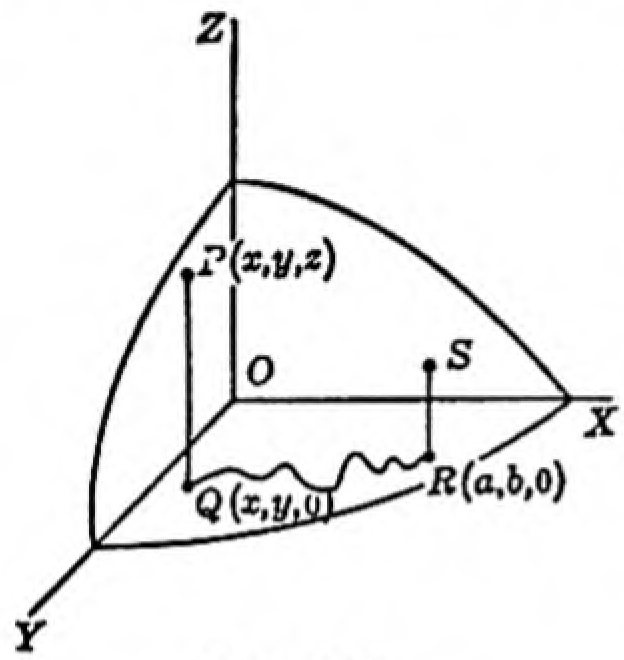


FIG. 183.

The limit of a function of more than two variables is similarly defined, although the geometrical interpretation is no longer convenient. For example,

$$\lim_{\substack{x \to a \\ y \to b \\ z \to c}} F(x,y,z) = L$$

provided that the difference between L and $F(x,y,z)$ can be made numerically as small as we please by taking x close enough to a, y close enough to b, and z close enough to c. If $L = F(a,b,c)$, then $F(x,y,z)$ is said to be continuous at (a,b,c).

134. Partial Derivatives. Let z be a function of two variables, $z = f(x,y)$. If y is held fixed, z becomes a function of x alone, and its derivative (if it exists) can be found. The result is called the *partial derivative of z with respect to x* and is denoted by the symbol $\frac{\partial z}{\partial x}$. Other symbols frequently used for this partial derivative are $f_x(x,y)$, f_x, $\frac{\partial f}{\partial x}$, z_x. Thus

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

If x, instead of y, is held fixed, then $f(x,y)$ becomes a function of y alone, and its derivative (if it exists) can be found. The result is called the partial derivative of z with respect to y and is denoted by the symbol $\frac{\partial z}{\partial y}$, or by $f_y(x,y)$, f_y, $\frac{\partial f}{\partial y}$, z_y. Thus

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

If z is given as an implicit function of x and y by an equation

$$F(x,y,z) = 0$$

the partial derivative of z with respect to x can be found by holding y constant, keeping in mind that z is now a function of x, and applying the former rule (Art. 27) for finding the derivative of an implicit function. The differentiation of implicit functions of more than one variable will be more fully discussed in Art. 142.

Example 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = 3x^3y + x^2y^2 + 7y^4$. Regarding y as a constant, we have

$$\frac{\partial z}{\partial x} = 9x^2y + 2xy^2$$

Regarding x as a constant, we have

$$\frac{\partial z}{\partial y} = 3x^3 + 2x^2y + 28y^3$$

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $5x^3yz^2 + 11y^4z + z^5 - 10 = 0$. Regarding y as a constant and differentiating with respect to x, remembering that z is a function of x, we obtain

$$10x^3yz\frac{\partial z}{\partial x} + 15x^2yz^2 + 11y^4\frac{\partial z}{\partial x} + 5z^4\frac{\partial z}{\partial x} = 0$$

Hence
$$\frac{\partial z}{\partial x} = -\frac{15x^2yz^2}{10x^3yz + 11y^4 + 5z^4}$$

Regarding x as a constant and differentiating with respect to y, remembering that z is a function of y, we obtain

$$10x^3yz\frac{\partial z}{\partial y} + 5x^3z^2 + 11y^4\frac{\partial z}{\partial y} + 44y^3z + 5z^4\frac{\partial z}{\partial y} = 0$$

Hence
$$\frac{\partial z}{\partial y} = -\frac{5x^3z^2 + 44y^3z}{10x^3yz + 11y^4 + 5z^4}$$

In finding partial derivatives, it is very important to keep clearly in mind which are the independent variables. Consider the following example.

Example 3. Let x and y be given as functions of independent variables r, θ by the equations

$$x = e^r\cos\theta \qquad y = e^r\sin\theta \tag{1}$$

We have at once

$$\frac{\partial x}{\partial r} = e^r\cos\theta = x \qquad \frac{\partial y}{\partial r} = e^r\sin\theta = y$$

$$\frac{\partial x}{\partial \theta} = -e^r\sin\theta = -y \qquad \frac{\partial y}{\partial \theta} = e^r\cos\theta = x$$

If, however, we choose to regard (1) as simultaneous equations giving r and θ in terms of independent variables x, y, we may solve for r and θ. To this end, we have

$$x^2 + y^2 = e^{2r} \qquad \frac{y}{x} = \tan\theta$$

and

$$r = \frac{1}{2}\ln(x^2 + y^2) \qquad \theta = \arctan\frac{y}{x} \tag{2}$$

From (2), we have at once

$$\frac{\partial r}{\partial x} = \frac{x}{x^2 + y^2} \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{x^2 + y^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

Note particularly that here $\dfrac{\partial x}{\partial r}$ is *not* equal to $\dfrac{1}{\dfrac{\partial r}{\partial x}}$, and similarly for the other partial derivatives.

An alternative method for finding the partial derivatives of r and θ from equations (1) is to regard x and y as independent variables and differentiate as follows: First, hold y constant, and differentiate with respect to x, thus

$$\begin{aligned} 1 &= -e^r \sin\theta \frac{\partial \theta}{\partial x} + e^r \cos\theta \frac{\partial r}{\partial x} \\ 0 &= e^r \cos\theta \frac{\partial \theta}{\partial x} + e^r \sin\theta \frac{\partial r}{\partial x} \end{aligned} \tag{3}$$

These are simultaneous equations from which to find $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial \theta}{\partial x}$. Multiplying the first equation through by $\cos\theta$, the second by $\sin\theta$, and adding the results gives

$$\cos\theta = e^r(\cos^2\theta + \sin^2\theta)\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\cos\theta}{e^r} = \frac{e^r\cos\theta}{e^{2r}} = \frac{x}{x^2 + y^2}$$

Multiplying the first of equations (3) by $\sin\theta$ and the second by $\cos\theta$ and subtracting gives $\dfrac{\partial \theta}{\partial x}$.

Now hold x constant, and differentiate each of equations (1) with respect to y. The resulting equations can be solved for $\dfrac{\partial r}{\partial y}$ and $\dfrac{\partial \theta}{\partial y}$.

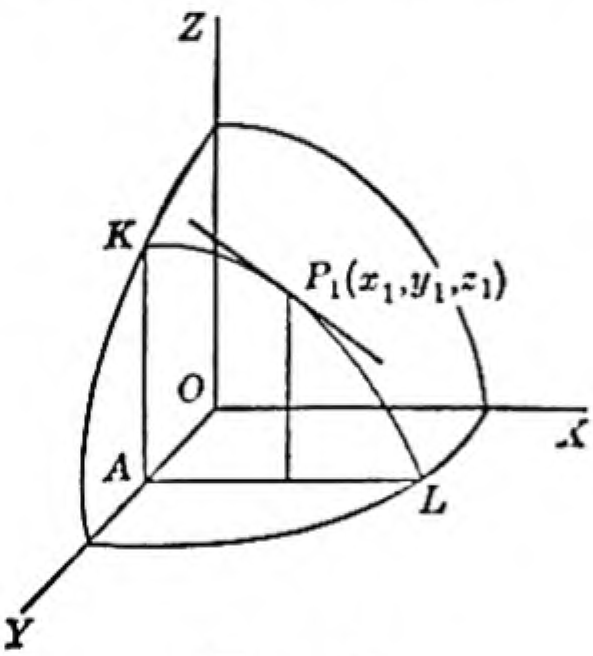


FIG. 184.

135. Geometrical Interpretation. Suppose that $z = f(x,y)$ is represented by a surface as shown in Fig. 184. Consider the point $P_1(x_1,y_1,z_1)$ of the surface. Cut the surface by the plane $y = y_1$; the curve of intersection is KL. In this plane the z coordinate of a point on KL is a function of x; therefore, the slope of KL at any point is the value of

the derivative of z with respect to x at that point. Consequently, the slope of the tangent line to KL at P_1 is $\left.\dfrac{\partial z}{\partial x}\right]_{\substack{x=x_1\\y=y_1}}$ In general, therefore, $\dfrac{\partial z}{\partial x}$ is the slope of the curve cut from the surface $z = f(x,y)$ by a plane parallel to the xz plane.

Similarly, it is clear that $\dfrac{\partial z}{\partial y}$ is the slope of the curve cut from the surface by a plane parallel to the yz plane.

EXERCISES

Find the partial derivatives of each of the following functions (Ex. 1 to 24):

1. $z = x^3 + 4x^2y - 8$

2. $z = x^2y^2 + xy^4$

3. $z = 2x + \dfrac{y}{x}$

4. $z = \dfrac{x}{y} + \dfrac{y}{x}$

5. $z = \ln (x^3 + y^2)$

6. $z = \log (xy^2 - 1)$

7. $u = e^{x^2+2u}$

8. $u = e^{\frac{y}{x}}$

9. $u = 10^{x^2+xy+z^2}$

10. $w = a^{bx+\frac{c}{y}}$

11. $w = x \sin y + y \sec x$

12. $w = \tan \left(x + \dfrac{z^2}{2}\right)$

13. $x = (s^2 - t^2)^{1/2}$

14. $y = \dfrac{s + t}{s - t}$

15. $v = xyz + x^2y - xz^3 + y^4$

16. $v = w^2 + \dfrac{uz}{y}$

17. $w = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

18. $z = xe^{\sin y}$

19. $z = \cosh (x^2 + 3y)$

20. $z = \sinh (xe^y)$

21. $s = \arctan (y/x)$

22. $s = \arcsin xy$

23. $Q = \dfrac{\sin \theta}{r}$

24. $Q = e^{r^2} \cos (\theta/2)$

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ (Ex. 25 to 32).

25. $x^2 + y^2 - z^2 = 4$

26. $x^2 + y^2 + z^2 = a^2$

27. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

28. $xy + z^2 = 16$

29. $xyz = a^3$

30. $x^4y^2 + 5y^2z^3 + z^6 = a^6$

31. $4x^2y - 3x^3y^5 - 17z^4 + 1 = 0$

32. $xe^z + y^2 = 1$

Find $\dfrac{\partial x}{\partial r}, \dfrac{\partial y}{\partial r}, \dfrac{\partial x}{\partial \theta}, \dfrac{\partial y}{\partial \theta}, \dfrac{\partial r}{\partial x}, \dfrac{\partial \theta}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial y}$, using the methods of Example 3, Art. 134 (Ex. 33 to 36).

33. $x = r \cos \theta \qquad y = r \sin \theta$

34. $x = r \cosh \theta \qquad y = r \sinh \theta$

35. $x = re^{\theta} \qquad y = re^{-\theta}$

36. $x = r \sec \theta \qquad y = r \tan \theta$

37. Find the slopes at $(2,-1,8)$ of the curves cut from the surface $z = x^2 + 4y^2$ by planes $x = 2$ and $y = -1$.

38. Find the slopes at $(-1,3,-17)$ of the curves cut from the surface $z = x^2 - 2y^2$ by planes $x = -1$ and $y = 3$.

39. Find the equations of the tangent to the circle $x^2 + y^2 + z^2 = 14$, $x = 2$ at the point $(2,3,1)$.

40. Find the equations of the tangent to the ellipse $4x^2 + 9y^2 + z^2 = 29$, $y = 1$ at the point $(-1,1,4)$.

41. Find the equations of the tangent to the hyperbola $xyz = 6$, $x = 2$ at the point $(2,-3,-1)$.

42. Find the equations of the tangent to the parabola $x^2y + zy^2 = 10$, $y = -2$ at the point $(1,-2,3)$.

43. If $z = \dfrac{x^2 - y^2}{xy}$, verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

44. If $z = \dfrac{x^4 - y^4}{xy}$, verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$.

45. If $z = \dfrac{x^n - y^n}{xy}$, verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = (n-2)z$.

46. If $z = \dfrac{y^2}{x}$, verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

47. If $u = \dfrac{xz + y^2}{yz}$, verify that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

48. If $u = x^2 - y^2 - 2xy + y + z$, verify that

$$(x + y)\frac{\partial u}{\partial x} + (x - y)\frac{\partial u}{\partial y} + (y - x)\frac{\partial u}{\partial z} = 0$$

136. Partial Derivatives of Higher Order. If $z = f(x,y)$, it is clear that $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are, in general, functions of x and y. They may possess partial derivatives with respect to x or y. Differentiating with respect to x gives

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

the second partial derivative of z with respect to x. Here the symbol $\dfrac{\partial}{\partial x}$ means to differentiate partially with respect to x whatever follows. Various symbols are used to denote this derivative, for example,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = f_{xx} = z_{xx}$$

Similarly $$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}(x,y) = f_{yy} = z_{yy}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially with respect to y gives

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy}(x,y) = f_{xy} = z_{xy}$$

the second partial derivative of z with respect to x and y. If we differentiate $\frac{\partial z}{\partial y}$ with respect to x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx}(x,y) = f_{yx} = z_{yx}$$

the second partial derivative of z with respect to y and x. It is shown in more advanced courses that, if the derivatives involved are continuous, then $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$, that is, the *order of differentiation is immaterial.* Similarly, if the derivatives concerned are continuous,

$$\frac{\partial^3 z}{\partial y\, \partial x^2} = \frac{\partial^3 z}{\partial x\, \partial y\, \partial x} = \frac{\partial^3 z}{\partial x^2\, \partial y}$$

In general, $\frac{\partial^{h+k} z}{\partial x^h\, \partial y^k}$ means the result obtained if z is differentiated h times with respect to x and k times with respect to y, the order of differentiation being immaterial if the derivatives concerned are continuous.

Similar remarks apply to functions of more than two variables.

Example 1. Verify that $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$ if $z = \sin(2x + 3y)$. We have

$$\frac{\partial z}{\partial x} = 2\cos(2x + 3y) \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y\, \partial x} = -6\sin(2x + 3y)$$

Also
$$\frac{\partial z}{\partial y} = 3\cos(2x + 3y) \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x\, \partial y} = -6\sin(2x + 3y)$$

Example 2. Find $\frac{\partial^4 u}{\partial x\, \partial y\, \partial z^2}$ if $u = x^2yz^3 + z^6$. We have

$$\frac{\partial u}{\partial z} = 3x^2yz^2 + 6z^5 \qquad \frac{\partial^2 u}{\partial z^2} = 6x^2yz + 30z^4$$

$$\frac{\partial^3 u}{\partial y\, \partial z^2} = 6x^2z \qquad \frac{\partial^4 u}{\partial x\, \partial y\, \partial z^2} = 12xz$$

EXERCISES

Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\, \partial y}, \frac{\partial^2 z}{\partial y^2}$ in the following cases (Ex. 1 to 8):

1. $z = x^3y^2 + y^5$
2. $z = 4x^3 + xy^2 + 10$
3. $z = x\sin y$
4. $z = \sin 3x \cos 2y$
5. $z = e^{x+y^2}$
6. $z = \ln\sqrt{x^2 + y^4}$
7. $x^2 + 4y^2 + 16z^2 - 64 = 0$
8. $xy + yz + xz = 1$
9. Verify that $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$ if $z = x^3y + y^4$.
10. Verify that $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$ if $z = \frac{x + y}{x - y}$.

11. Verify that $\dfrac{\partial^3 z}{\partial x^2\, \partial y} = \dfrac{\partial^3 z}{\partial x\, \partial y\, \partial x} = \dfrac{\partial^3 z}{\partial y\, \partial x^2}$ if $z = e^{xy}$.

12. Same as Exercise 11 if $z = x^4 + 2xy^3 + 3x^3y + 4y^4$

13. Same as Exercise 11 if $z = y^3 \sin (2x + y)$

14. Same as Exercise 11 if $z = \ln (x^2 + 3y^2)^4$

15. Verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = \ln \sqrt{x^2 + y^2}$.

16. Verify that $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = \cos (x + y) + \cos (x - y)$.

17. Verify that $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\, \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = 13x^2 + 10xy + 2y^2$.

18. Verify that $\dfrac{\partial^3 z}{\partial y\, \partial x^2} - 2\dfrac{\partial^3 z}{\partial y^2\, \partial x} + \dfrac{\partial^3 z}{\partial y^3} = \dfrac{1}{x^2}$ if $z = x^4 + \sin (x + y) - y \ln x$.

19. Verify that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x\, \partial y} + \dfrac{\partial u}{\partial y} - u = \cos (x + 2y) + e^y$ if

$$u = y^2e^x + (x + y)e^{-x} + \tfrac{1}{3} \sin (x + 2y) - xe^y$$

20. Verify that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\, \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = x^2 + y^2$ if $u = \dfrac{y^2}{x^2} + y + \dfrac{1}{2}(x^2 + y^2)$.

137. The Differential of a Function. Let $z = f(x,y)$ be represented by a surface, and let $P(x,y,z)$ be any point on this surface. Let x be given an increment Δx and y an increment Δy. This will produce an increment Δz in z, and we shall designate by Q the point

$$(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$$

of the surface. Pass planes through P and Q parallel to the coordinate planes as shown in Fig. 185. From the figure, we have

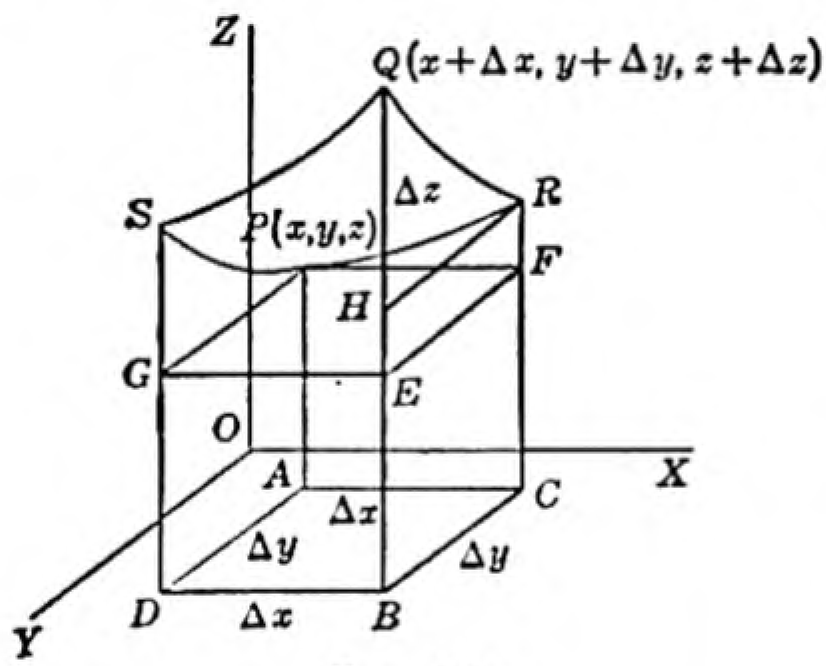


FIG. 185.

$$\Delta z = EQ = BQ - AP = f(x + \Delta x, y + \Delta y) - f(x,y)$$

Note that RH is drawn parallel to FE and that

$$EQ = EH + HQ$$

Now
$$EH = FR$$

Furthermore
$$\lim_{\Delta x \to 0} \frac{FR}{PF} = \frac{\partial z}{\partial x}$$

(the slope at P of curve PR), and therefore

$$\frac{FR}{PF} = \frac{FR}{\Delta x} = \frac{\partial z}{\partial x} + \epsilon_1$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Hence

$$FR = \left(\frac{\partial z}{\partial x} + \epsilon_1\right) \Delta x$$

Next we may consider the ratio HQ/RH. The limit of this ratio as $RH = \Delta y \to 0$ is the partial derivative of z with respect to y *evaluated at point* R, that is, the slope of the curve RQ at R. But if we maintain the configuration, as $PF = \Delta x$ is made to approach zero, the curve RQ approaches coincidence with the curve PS, and the slope of RQ at R approaches as a limit the slope of the curve PS at P, if $f(x,y)$ and its derivatives are assumed continuous. That is to say,

$$\lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{HQ}{RH} = \frac{\partial z}{\partial y}$$

Consequently $$\frac{HQ}{RH} = \frac{HQ}{\Delta y} = \frac{\partial z}{\partial y} + \epsilon_2$$

and $$HQ = \left(\frac{\partial z}{\partial y} + \epsilon_2\right) \Delta y$$

where ϵ_2 approaches zero when *both* Δy and Δx approach zero. Therefore

$$\Delta z = \left(\frac{\partial z}{\partial x} + \epsilon_1\right) \Delta x + \left(\frac{\partial z}{\partial y} + \epsilon_2\right) \Delta y$$

and $$\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{4}$$

where $\epsilon_1 \Delta x$ and $\epsilon_2 \Delta y$ are quantities that are small in comparison with Δx and Δy.* The derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are, of course, evaluated at point $P(x,y,z)$.

In analogy with the case of a function of a single variable (Art. 56) we may call $\frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$ the *principal part* and $\epsilon_1 \Delta x + \epsilon_2 \Delta y$ the "negligible part" of Δz, and imagine quantities dz, dy, dx instead of Δz, Δy, Δx for which

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{5}$$

As in Art. 56, we call dz, dy, and dx *differentials* whereas Δz, Δy, and Δx are *differences* (that is, "increments"). Purely logically, dz, dy, dx are variable quantities which are required to satisfy the relation (5), nothing else.

The advantage of the differential notation, especially for computational purposes, is the increased freedom in algebraic manipulation that results

* It is customary to compare these quantities with $|\Delta x| + |\Delta y|$ or with $\sqrt{\Delta x^2 + \Delta y^2}$.

from its use. This will appear in the examples and discussion of this and the following sections.

Here we have supposed x and y to be independent variables. In Art. 141 we shall show that formula (5) holds even when x and y are not independent variables but are functions of other independent variables.

Formula (4) for Δz was obtained by reference to a geometrical situation. We may express each step in analytical terms in the following way:

$$\begin{aligned} EQ &= f(x + \Delta x, y + \Delta y) - f(x,y) \\ FR &= CR - CF = CR - AP \\ &= f(x + \Delta x, y) - f(x,y) \\ HQ &= BQ - BH = BQ - CR \\ &= f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) \end{aligned}$$

Observe that

$$EQ = FR + HQ = \frac{FR}{\Delta x}\,\Delta x + \frac{HQ}{\Delta y}\,\Delta y$$

Therefore

$$\Delta z = \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}\,\Delta x + \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y}\,\Delta y \tag{6}$$

Now, in the first term of the right-hand member, we have

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x} = \frac{\partial z}{\partial x}$$

$$\frac{f(x + \Delta x, y) - f(x,y)}{\Delta x} = \frac{\partial z}{\partial x} + \epsilon_1 \qquad \text{where } \epsilon_1 \to 0 \text{ when } \Delta x \to 0$$

In the second member,

$$\lim_{\Delta y \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} = f_y(x + \Delta x, y)$$

where we use the notation $f_y(x + \Delta x, y)$ to denote the partial derivative with respect to y evaluated at the point whose x coordinate is $x + \Delta x$ and y coordinate is y. Since we suppose the partial derivatives continuous,

$$f_y(x + \Delta x, y) = f_y(x,y) + \epsilon = \frac{\partial z}{\partial y} + \epsilon \qquad \text{where } \epsilon \to 0 \text{ when } \Delta x \to 0$$

Hence

$$\begin{aligned} \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} &= f_y(x + \Delta x, y) + \epsilon' \\ &= \frac{\partial z}{\partial y} + \epsilon + \epsilon' = \frac{\partial z}{\partial y} + \epsilon_2 \qquad \text{where } \epsilon_2 = \epsilon + \epsilon' \end{aligned}$$

Since $\epsilon' \to 0$ when $\Delta y \to 0$, $\epsilon_2 \to 0$ when both Δx and Δy approach zero. Hence

$$\Delta z = \left(\frac{\partial z}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial z}{\partial y} + \epsilon_2\right)\Delta y$$

and formula (4) is established.

Note particularly that the analytical formulation can be made entirely independently of any geometrical interpretation, since (6) is simply an identity for all Δx and Δy different from zero. If $u = F(x,y,z, \ldots, t)$ so that the geometrical interpretation cannot be used, Δu may be found from similar analytical considerations. The differential of u can then be defined, and we have the formula

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz + \cdots + \frac{\partial u}{\partial t}\,dt$$

Here again this formula can be shown to hold whether $x, y, z, \ldots, t$ are independent variables or not.

Example 1. If $z = x^2y^3 + 2xy - 5x^4 - 11y^2$, find dz. We have

$$\frac{\partial z}{\partial x} = 2xy^3 + 2y - 20x^3$$

$$\frac{\partial z}{\partial y} = 3x^2y^2 + 2x - 22y$$

Therefore $\quad dz = (2xy^3 + 2y - 20x^3)\,dx + (3x^2y^2 + 2x - 22y)\,dy$

Example 2. A triangle has sides 22 and 15 ft. with included angle 30 deg. If the linear measurements may be in error by as much as 1 in. and the angle by as much as 10 min., find the maximum error in the computed area. If two sides of a triangle are x and y units long and the included angle is θ radians, the area is $A = \frac{1}{2}xy \sin\theta$ (Fig. 186). If x, y, and θ receive certain increments Δx, Δy, $\Delta\theta$ which we shall replace by dx, dy, $d\theta$, then

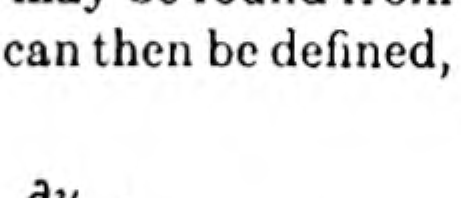


FIG. 186.

$$dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy + \frac{\partial A}{\partial \theta}\,d\theta$$
$$= \tfrac{1}{2}y \sin\theta\,dx + \tfrac{1}{2}x\sin\theta\,dy + \tfrac{1}{2}xy\cos\theta\,d\theta$$

If we evaluate this for $x = 22$ ft., $y = 15$ ft., $\theta = 30$ deg. and write

$$dx = 1 \text{ in.} = \tfrac{1}{12} \text{ ft.} = dy$$

and $d\theta = 10' = \frac{10}{60}\cdot\frac{\pi}{180}$ radian, we shall have a reasonable approximation to the error in A. Thus,

$$dA = \frac{1}{2}\cdot 15\cdot\frac{1}{2}\cdot\frac{1}{12} + \frac{1}{2}\cdot 22\cdot\frac{1}{2}\cdot\frac{1}{12} + \frac{1}{2}\cdot 22\cdot 15\cdot\frac{\sqrt{3}}{2}\cdot\frac{10}{60}\cdot\frac{\pi}{180}$$
$$= 1.2 \text{ sq. ft.}$$

Example 3. A covered rectangular box with inside dimensions 3 by 4 by 5 ft. is lined with felt $\frac{1}{8}$ in. thick. Find approximately the percentage reduction in volume. If

the box is x ft. long, y ft. wide, and z ft. deep, its volume is $V = xyz$. Since we want the *percentage* decrease in volume, we shall calculate $100\frac{dV}{V}$. To do this most conveniently, we take the logarithm of V and then find the differential of $\ln V$. We obtain

$$\ln V = \ln x + \ln y + \ln z$$

$$100d(\ln V) = 100\frac{dV}{V} = 100\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right)$$

Evaluating this for $x = 5$, $y = 4$, $z = 3$ and putting

$$dx = dy = dz = -\tfrac{1}{4} \text{ in.} = -\tfrac{1}{48} \text{ ft.}$$

we get $$100\frac{dV}{V} = -100\left(\frac{1}{5}\cdot\frac{1}{48} + \frac{1}{4}\cdot\frac{1}{48} + \frac{1}{3}\cdot\frac{1}{48}\right) = -1.6 \text{ per cent}$$

that is, a decrease of 1.6 per cent in volume. Let the student compute the precise change in V and compare with this result.

EXERCISES

Find the differential of each of the following functions (Ex. 1 to 18):

1. $z = x^3y + x^2y^2 + 1$

2. $z = \dfrac{x}{y} + \sqrt{xy}$

3. $z = 2x^4 - 3x^3y + x^2y^2 + 4xy^3 + y^4$

4. $z = (x^2 + y^3)^4$

5. $u = e^{x^2-y^2-2z}$

6. $u = xe^{y+4z}$

7. $w = e^{\frac{r}{s}}$

8. $v = s^2e^{\sin t}$

9. $s = \ln (r^2 + t^2)^{3/2}$

10. $s = \ln xyz^2$

11. $z = \dfrac{x}{\sqrt{x^2 + y^2}}$

12. $z = \dfrac{\sqrt{x^2 + y^2}}{y}$

13. $w = \tan\left(2t + \dfrac{r}{3}\right)$

14. $w = y \csc xz$

15. $z = \arctan (y/x)$

16. $z = \arccos (2x - y)$

17. $u = \sinh (x/y)$

18. $v = \cosh (x^2 + y^2)$

19. The sides of a rectangle are found to be 7 ft. and 12 ft. with possible errors of 1 in. in each measurement. Find approximately the possible error in the computed area.

20. In Exercise 19, find approximately the percentage error in the area.

21. Find approximately the percentage error in the computed area of a rectangle if the measurements of the base and altitude are in error by 1 per cent and 3 per cent, respectively.

22. The gravitational attraction between a particle of mass m and a particle of unit mass r units distant is $F = km/r^2$. Find approximately the percentage change in F resulting from an increase of 2 per cent in m and a decrease of 3 per cent in r.

23. The base of a rectangular box is a square of side 12.01 in. The depth is 6.98 in. Find the volume, using differentials.

24. The sides of a right triangle ABC are measured and found to be $a = 5.7$ ft. and $b = 8.3$ ft., respectively, with an error of ± 0.1 ft. in each. Find approximately the maximum error in the computed hypotenuse c. Find also the percentage error in c.

25. In the triangle of Exercise 24, $\tan A = a/b$ is computed. Find approximately the maximum error and percentage error in $\tan A$.

26. In the triangle of Exercise 24, angle A is found from a table of tangents. Find approximately the maximum error in A.

27. The distance from C to an inaccessible point B is found by constructing a right triangle ACB (right angle at C). AC is laid off to be 1000 ft., with a possible error of 0.5 ft. Angle CAB is measured to be $59°37'$ with a possible error of 1 minute. Find approximately the greatest possible error in the computed value of CB.

28. In Exercise 27, find the percentage error in CB.

29. The distance from a point A to an inaccessible point B is found by measuring a base line $AC = 500$ ft. and angles $CAB = \theta = 45°$ and $ACB = \varphi = 60°$. Find approximately the greatest possible error in AB caused by errors of 2 minutes in θ and φ.

30. Solve Exercise 29 if the length of the base line AC may be in error by as much as 6 in. with possible errors of 2 minutes in θ and φ.

31. The sine of an acute angle of a right triangle is found by measuring the hypotenuse and opposite side. If these are found to be, respectively, 5 ft. and 4 ft. with possible errors of 3 in. in each, find approximately the possible error in the computed sine.

32. In Exercise 31, find the possible error in the angle as found from its sine (nearest 10 minutes).

33. A triangle is found to have sides 53 ft. and 41 ft., with included angle 37 deg. If the sides may be in error by 6 in. and the angle by 30 minutes, find approximately the maximum error in the computed area.

34. In Exercise 33, find approximately the maximum error in the computed third side.

138. Directional Derivative. Let $z = f(x,y)$, and let $P(x,y,z)$ be some fixed point of the surface, Fig. 187. Let $Q(x + \Delta x, y + \Delta y, z + \Delta z)$ be any nearby point on the surface. Consider the plane through P and Q parallel to the z axis. It cuts the surface in the curve PQ, the xy plane in $AB = \Delta l$, and the plane GF in $PE = \Delta l$, as shown in the figure. If this plane makes an angle α with the xz plane, then $\angle CAB = \alpha$ as shown. The limit of the quotient EQ/PE as $PE = \Delta l$ approaches zero is the slope at P of the curve PQ cut from the surface by the plane that makes the angle α with the xz plane.

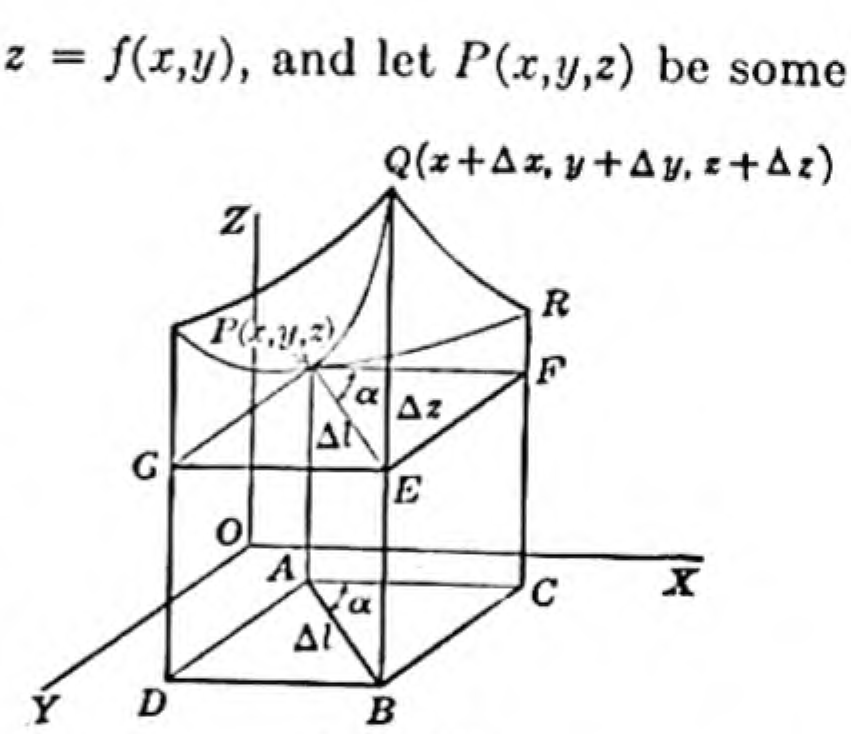


FIG. 187.

This situation may be described in slightly different terms. As a point moves from A to B in the xy plane along the line AB, the corresponding point on the surface moves from P to Q, and the value of the

function changes from z to $z + \Delta z$. The average rate of change of the function over this interval is $\frac{\Delta z}{\Delta l} = \frac{EQ}{AB} = \frac{EQ}{PE}$, and the limit of the ratio as $\Delta l = AB = PE$ is made to approach zero is the rate of change of the function at P in the direction AB. This direction is completely specified by the angle α measured in the positive sense from the positive direction of the x axis. Note that $0 \leqq \alpha < 360°$. We call

$$\lim_{\Delta l \to 0} \frac{\Delta z}{\Delta l} = \frac{dz}{dl}$$

the *directional derivative of* $z = f(x,y)$ in the direction of the line segment AB.

We proceed to the calculation of the directional derivative. From Art. 137, we have

$$\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 approach zero if Δx and Δy both approach zero. Therefore

$$\frac{\Delta z}{\Delta l} = \frac{\partial z}{\partial x} \frac{\Delta x}{\Delta l} + \frac{\partial z}{\partial y} \frac{\Delta y}{\Delta l} + \epsilon_1 \frac{\Delta x}{\Delta l} + \epsilon_2 \frac{\Delta y}{\Delta l}$$

Note that $\frac{\Delta x}{\Delta l} = \cos \alpha$, $\frac{\Delta y}{\Delta l} = \sin \alpha$. Therefore

$$\frac{\Delta z}{\Delta l} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha + \epsilon_1 \cos \alpha + \epsilon_2 \sin \alpha$$

If Δl is made to approach zero, then Δx and Δy both approach zero, and

$$\star \qquad \frac{dz}{dl} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha \tag{7}$$

In this formula, of course, the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are to be evaluated *at the point P*.

Note that, if $\alpha = 0$, then $\frac{dz}{dl} = \frac{\partial z}{\partial x}$, and the partial derivative of z with respect to x is seen to be a special case of the directional derivative, namely, in the direction of the positive x axis. If we ask for the directional derivative in the direction of the negative x axis, then $\alpha = 180°$, and $\frac{dz}{dl} = -\frac{\partial z}{\partial x}$. Similarly, if $\alpha = 90°$, $\frac{dz}{dl} = \frac{\partial z}{\partial y}$, and the partial derivative with respect to y is seen to be the directional derivative in the direction of the positive y axis.

Example 1. The temperature at any point of a rectangular plate lying in the xy plane is given by the formula

$$z = \frac{1}{x^2 + y^2 + 1}$$

At the point (3,2), find the rate of change of temperature in the direction making an angle of 60 deg. with the x axis (Fig. 188). We shall find the directional derivative of this function for $\alpha = 60°$. We have

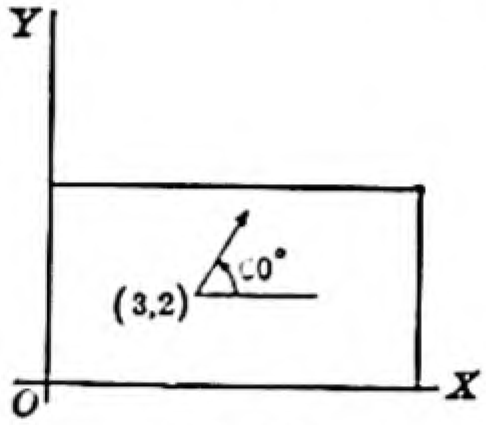


FIG. 188.

$$\left.\frac{\partial z}{\partial x}\right]_{3,2} = -\left.\frac{2x}{(x^2+y^2+1)^2}\right]_{3,2} = -\frac{3}{98}$$

$$\left.\frac{\partial z}{\partial y}\right]_{3,2} = -\left.\frac{2y}{(x^2+y^2+1)^2}\right]_{3,2} = -\frac{2}{98}$$

Therefore $$\frac{dz}{dl} = -\frac{3}{98}\cos 60° - \frac{2}{98}\sin 60°$$

$$= -\frac{3}{98}\cdot\frac{1}{2} - \frac{2}{98}\cdot\frac{\sqrt{3}}{2} = -\frac{3+2\sqrt{3}}{196} = -0.033 \text{ approximately}$$

If x and y are measured in inches and temperature is measured in degrees, this means that, at (3,2) in the direction shown (Fig. 188), the temperature is decreasing at the rate of 0.033 deg. per inch.

Example 2. In Example 1, find the directions of maximum and of minimum numerical values of the rate of change of temperature at (3,2). We have, from Example 1,

$$\frac{dz}{dl} = -\frac{3}{98}\cos\alpha - \frac{2}{98}\sin\alpha = F(\alpha)$$

for the directional derivative in the direction making an angle α with the x axis, and we are to find α so that $F(\alpha)$ will have a maximum or minimum numerical value. Differentiating $F(\alpha)$ with respect to α, we obtain

$$F'(\alpha) = -\tfrac{1}{98}(-3\sin\alpha + 2\cos\alpha)$$

from which we find

$$-3\sin\alpha + 2\cos\alpha = 0$$
$$-3\tan\alpha + 2 = 0$$
$$\tan\alpha = \tfrac{2}{3} \qquad \text{and} \qquad \alpha = \arctan\tfrac{2}{3}$$

The value of $\frac{dz}{dl}$ for this value of α is, if α is taken in the first quadrant,

$$\left.\frac{dz}{dl}\right]_{\alpha=\arctan\frac{2}{3}} = -\frac{1}{98}\left(3\cdot\frac{3}{\sqrt{13}} + 2\cdot\frac{2}{\sqrt{13}}\right) = -\frac{\sqrt{13}}{98}$$

If α is taken to be an angle in the third quadrant, the direction is reversed, and the directional derivative has a value

$$-\frac{1}{93}\left[3\left(-\frac{3}{\sqrt{13}}\right) + 2\left(-\frac{2}{\sqrt{13}}\right)\right] = \frac{\sqrt{13}}{98}$$

In the first case, $F''(\alpha) = \frac{1}{98}(3\cos\alpha + 2\sin\alpha)$ is positive, and we have a minimum value. In the second case, $F''(\alpha)$ is negative, and we have a maximum value. But note that we are interested in the maximum or minimum *numerical* value of $\frac{dz}{dl}$, rather than in the algebraically largest or smallest value. The two preceding results are numerically equal, and either one, say the first, gives the numerical maximum.

Since we also desire the minimum numerical value, we shall see what value of α makes $\frac{dz}{dl}$ equal to zero. We have

$$\frac{dz}{dl} = -\frac{3}{98}\cos\alpha - \frac{2}{98}\sin\alpha = 0$$

if $3 + 2\tan\alpha = 0$, that is, if $\tan\alpha = -\frac{3}{2}$. If, therefore, $\alpha = \arctan\left(-\frac{3}{2}\right)$, the directional derivative is zero. Note that, if the value of α giving the maximum numerical value of $\frac{dz}{dl}$ is denoted by α_1 and that giving $\frac{dz}{dl}$ a value zero is denoted by α_2, then

$$\tan\alpha_1 \cdot \tan\alpha_2 = \tfrac{2}{3}\left(-\tfrac{3}{2}\right) = -1$$

Hence the two directions are at right angles to one another.

EXERCISES

1. The temperature at any point of a rectangular plate lying in the xy plane is given by the formula $T = x^2 + y^2$. Find the rate of change of temperature at (1,5) in the direction making an angle of 30 deg. with the x axis.

2. In Exercise 1, find the rate of change of temperature at point (3,4) in the direction toward the origin.

3. In Exercise 1, find the maximum and minimum numerical values of the rate of change of temperature at (3,4). Find the directions for these rates.

4. The density at each point of a thin rectangular plate is given by the formula $\delta = \dfrac{1}{\sqrt{x^2 + y^2 + 1}}$. Find the rate of change of density at the point (2,1) in the direction of a line making an angle of 60 deg. with the x axis.

5. In Exercise 4, find the direction and magnitude of the maximum rate of change of δ at (2,1).

6. The electric potential at any point of the xy plane is given by the formula $V = \ln\sqrt{x^2 + y^2}$. Find the rate of change of the potential at the point (3,4) in a direction making an angle of 30 deg. with the x axis.

7. In Exercise 6, find the rate of change of the potential at the point $P_1(x_1,y_1)$ in the direction P_1O toward the origin; in a direction at right angles to P_1O.

8. In Exercise 7, find the magnitude and direction of the maximum rate of change of potential at $P_1(x_1,y_1)$.

9. The paraboloid $z = x^2 + 4y^2$ is cut by a plane perpendicular to the xy plane and passing through the point (2,1,8) of the surface. What is the equation of this plane if the curve it cuts from the paraboloid has slope zero at this point?

10. The paraboloid $36z = 4x^2 + 9y^2$ is cut by a plane perpendicular to the xy plane and passing through the point (3,2,2) of the surface. What is the equation of the plane if the curve it cuts from the paraboloid has slope zero at this point?

11. In what direction from the point $P_1(x_1,y_1)$ is the directional derivative of $z = \sqrt{x^2 + y^2}$ a maximum? Find this maximum value.

12. In what direction from the point $P_1(x_1,y_1)$ is the directional derivative of $z = axy$ a maximum? Find this maximum value.

13. Find the directional derivative of the function $z = e^x \cos y$ at $(1,\pi/4)$ in the direction making an angle of 30 deg. with the x axis.

14. Show that the maximum value of the directional derivative at (x_1,y_1) of the function $z = f(x,y)$ is $\sqrt{\left(\frac{\partial z}{\partial x}\right)_1^2 + \left(\frac{\partial z}{\partial y}\right)_1^2}$ where the subscripts indicate that the derivatives are to be evaluated at (x_1,y_1). This is called the *gradient* of the function.

15. If α_1 is the value of α for which the directional derivative is zero and if α_2 is the value of α for which the directional derivative is numerically a maximum, show that $\tan \alpha_1 \tan \alpha_2 = -1$. Interpret geometrically.

16. If $w = f(x,y,z)$, show that the directional derivative has the value

$$\frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \cos \beta + \frac{\partial w}{\partial z} \cos \gamma$$

where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are direction cosines of the desired direction.

139. The Total Derivative of a Function. Let us consider a function $z = f(x,y)$. Now, if we suppose that x and y are in turn functions of another variable, say t, then z is actually a function of t, and its derivative with respect to t (if it exists) can be found. We could express z in terms of t by substituting for x and y their values in terms of t in $z = f(x,y)$, and then calculate $\frac{dz}{dt}$ by the usual rules. But this derivative can be found in a somewhat different way by making use of partial derivatives as follows:

Let t receive an increment Δt; x and y then receive increments Δx and Δy, and we can calculate Δz as in Art. 137:

$$\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Dividing by Δt, we obtain

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

Assuming the existence of the derivatives involved and taking limits as Δt is made to approach zero, we get

$$\star \qquad \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \tag{8}$$

The derivative $\frac{dz}{dt}$ given by (8) is called the *total derivative* of z with respect to t.

Observe that we can obtain formula (8) formally from (5) (Art. 137) by dividing through by dt and then interpreting $\frac{dz}{dt}, \frac{dx}{dt}, \frac{dy}{dt}$ as ordinary derivatives.

Example 1. If $z = x^2 + xe^y$ and $x = \sin t$, $y = \ln t$, we have

$$\frac{\partial z}{\partial x} = 2x + e^y, \qquad \frac{\partial z}{\partial y} = xe^y, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = \frac{1}{t}$$

and

$$\frac{dz}{dt} = (2x + e^y) \cos t + xe^y \cdot \frac{1}{t}$$

If desired, this can be expressed in terms of t, thus

$$\frac{dz}{dt} = (2 \sin t + t) \cos t + \sin t$$

Let the student verify this result by first expressing z in terms of t and then differentiating.

If $z = f(x,y)$ where y is a function of x, say $y = g(x)$, then z is in reality a function of x alone, and we can calculate $\frac{dz}{dx}$, the total derivative of z with respect to x. This is simply the situation covered in formula (8) if we take $x = t$. Hence

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \tag{9}$$

It is very important to understand the meaning of the notation involved: $\frac{\partial z}{\partial x}$ is calculated by differentiating $f(x,y)$ with y held fast; $\frac{\partial z}{\partial y}$ is calculated by differentiating $f(x,y)$ with x held fast; $\frac{dy}{dx}$ is simply the ordinary derivative of $g(x)$; $\frac{dz}{dx}$ is what results from differentiating a function of x alone.

Example 2. If $z = \frac{1}{3}\sqrt{36 - x^2 - 4y^2}$ and $y = x^2$, find $\frac{dz}{dx}$. We have

$$f(x,y) = \frac{1}{3}\sqrt{36 - x^2 - 4y^2}$$

so that

$$\frac{\partial z}{\partial x} = \frac{-x}{3\sqrt{36 - x^2 - 4y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{-4y}{3\sqrt{36 - x^2 - 4y^2}}, \qquad \text{and} \qquad \frac{dy}{dx} = 2x$$

Therefore

$$\frac{dz}{dx} = -\frac{x}{3\sqrt{36 - x^2 - 4y^2}} - \frac{8xy}{3\sqrt{36 - x^2 - 4y^2}} = \frac{-x(1 + 8y)}{3\sqrt{36 - x^2 - 4y^2}}$$

Observe that, if this result is expressed in terms of x by using $y = x^2$, we obtain

$$\frac{dz}{dx} = \frac{-x(1 + 8x^2)}{3\sqrt{36 - x^2 - 4x^4}}$$

Let the student verify this result by first expressing z in terms of x and then finding $\frac{dz}{dx}$.

140. Geometrical Interpretation of the Total Derivative. Suppose $z = f(x,y)$ where $y = g(x)$. The first equation is represented by a surface, and the second by a cylindrical surface whose elements are parallel to the z axis. These two surfaces intersect in a space curve. In Fig. 189 the surfaces are those of Example 2 of the preceding section. The

curve $\widehat{CD}$ is the curve of intersection of the two surfaces. The z coordinate QP of a point P of this curve is a function of x alone; for if x is given, y is determined by $y = g(x)$, and hence z is determined by $z = f(x,y)$. The derivative of z with respect to x, given by formula (9) (Art. 139), is the rate of change with regard to x of this z coordinate of P.

If $z = f(x,y)$ and x and y are functions of t, then the curve $\widehat{OD}$ in the xy plane (Fig. 189) is given in parametric form. The derivative $\frac{dz}{dt}$ is the rate of change, in regard to the parameter t, of the z coordinate of P.

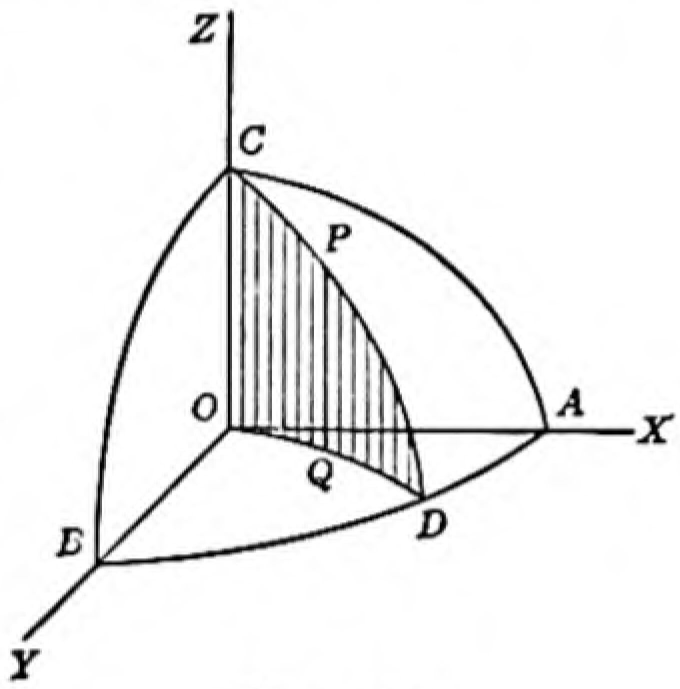


FIG. 189.

141. Several Variables. Suppose

$$z = f(x,y)$$

and that x and y are functions of several other variables. To make the situation definite, suppose $x = g_1(s,t)$ and $y = g_2(s,t)$. If t, say, is held fixed, then z becomes a function of s alone, and we may calculate the partial derivative of z with respect to s. The situation is covered by formula (8) (Art. 139) except that we now have partial derivatives with respect to s instead of total derivatives; thus

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{10}$$

In the same way,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{11}$$

Similar formulas hold if x and y are functions of more than two independent variables. Also, if $u = f(x,y,z, \ldots)$ is a function of several variables each of which is, in turn, a function of several variables

$$x = g_1(r,s,t, \ldots),$$

$y = g_2(r,s,t, \ldots)$, $z = g_3(r,s,t, \ldots)$, etc., then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} + \cdots$$

with similar formulas for $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \ldots$. It is assumed that all derivatives involved are continuous.

We can now show that formula (5) of Art. 137, namely,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

holds whether x and y are independent variables or not. Suppose, as above, that $z = f(x,y)$ where $x = g_1(s,t)$ and $y = g_2(s,t)$ with s, t inde-

pendent variables. Then, in reality, $z = F(s,t)$. Consequently, by (5), since s, t are *independent* variables,

$$dz = \frac{\partial z}{\partial s} ds + \frac{\partial z}{\partial t} dt$$

By (10) and (11), this becomes

$$\begin{aligned} dz &= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right) ds + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) dt \\ &= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right) \\ &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \end{aligned}$$

The extension to cases of functions of more variables is evident.

EXERCISES

In the following cases, find the total derivative of z with respect to t, φ, or θ as indicated (Ex. 1 to 6):

1. $z = x^2 + 4y^2$ where $x = \sin t$, $y = \cos t$
2. $z = xy^2 + x + 1$ where $x = e^t$, $y = e^{-t}$
3. $z = e^x \sin y$ where $x = \ln t$, $y = t^2$
4. $z = \tan (x^2 + y^2)$ where $x = 3\varphi$, $y = e^\varphi$
5. $z = \cosh (y/x)$ where $x = \varphi^2$, $y = e^\varphi$
6. $z = \arctan (y/x)$ where $x = \ln \theta$, $y = e^\theta$

Use the method of Art. 139 to find $\dfrac{dz}{dx}$ in the following cases. Check by expressing z in terms of x and differentiating (Ex. 7 to 12).

7. $z = (x^2 + 4y^2 - 16)^{3/2}$ where $y = \sqrt{x}$
8. $z = \cos (y/x)$ where $y = x^2 + 1$
9. $z = \ln (x^2 + y^2)$ where $y = \sin x$
10. $z = e^{\frac{y}{x}}$ where $y = \tan x$
11. $z = \operatorname{arcsec} xy$ where $y = e^x$
12. $z = \tanh (x^2 - y^2)$ where $y = e^{-x}$

In Exercises 13 to 17, use the methods of Art. 141 to find the derivatives indicated.

13. If $z = 4x^2 - 9y^2$ where $x = \dfrac{t}{s}$ and $y = s^2 + t^2$, find $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$.

14. If $z = x^2y^3$ where $x = \dfrac{\sin t}{s}$, $y = \dfrac{s}{t}$ find $\dfrac{\partial z}{\partial t}$, $\dfrac{\partial z}{\partial s}$.

15. If $z = \ln \sqrt{x^2 + y^2}$ where $x = se^t$, $y = se^{-t}$, find $\dfrac{\partial z}{\partial t}$, $\dfrac{\partial z}{\partial s}$.

16. If $z = e^{\frac{y}{x}}$ where $x = s \cos t$, $y = s \sin t$, find $\dfrac{\partial z}{\partial t}$, $\dfrac{\partial z}{\partial s}$.

17. If $z = u^2 + v^2 + w^2$ where $u = ye^x$, $v = xe^{-y}$, $w = \frac{y}{x}$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

18. If $z = f(x - y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

19. If $z = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\cos\theta - \frac{\partial z}{\partial \theta}\cdot\frac{\sin\theta}{r}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\sin\theta + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

(transformation from rectangular to polar coordinates). [*Hint:* Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ where $z = \varphi(r,\theta)$ and $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$.]

20. If $z = f(x,y)$ and $x = g_1(t)$, $y = g_2(t)$, show that

$$\frac{d^2z}{dt^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

21. If $z = f(x,y)$ and $x = g_1(t,s)$, $y = g_2(t,s)$, show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

22. If $z = f(x,y)$ and $x = g_1(t,s)$, $y = g_2(t,s)$, show that

$$\frac{\partial^2 z}{\partial s\,\partial t} = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right) + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}$$

23. If $z = e^x \sin y$ where $x = t^2$, $y = 3t$, find $\frac{d^2z}{dt^2}$. Verify by using the formula of Exercise 20.

24. If $z = x^2 + 4xy$ where $x = se^t$ and $y = se^{-t}$, find $\frac{\partial^2 z}{\partial t^2}, \frac{\partial^2 z}{\partial s\,\partial t}, \frac{\partial^2 z}{\partial s^2}$ and verify by use of the formulas in Exercises 21 and 22.

25. Similar to Exercise 24 for $z = x^2 - 4y^2$ where $x = r\cos\theta$ and $y = r\sin\theta$

142. Differentiation of Implicit Functions. Suppose that y is given as an implicit function of x by the equation

$$f(x,y) = 0 \tag{12}$$

This means that $y = \varphi(x)$ where $\varphi(x)$ is defined by equation (12) (see Art. 5). We may find $\frac{dy}{dx}$ from the following considerations. Let us, temporarily, set $z = f(x,y)$. Then, for all x, y, z satisfying this equation,

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy^*$$

* Note that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are identical with $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

whether x and y are independent variables or not. If we consider x an independent variable and allow y to take only those values for which (12) is satisfied, that is, for which $z = 0$, then dz is zero, and we have

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \frac{\partial f}{\partial y} \neq 0 \tag{13}$$

The derivative found in this way is, of course, identical with the result of the method of Art. 27.

Example 1. If $x^2 + 3xy^2 + y^3 - 1 = 0$, find $\frac{dy}{dx}$. Here

$$f(x,y) = x^2 + 3xy^2 + y^3 - 1$$

and
$$\frac{\partial f}{\partial x} = 2x + 3y^2 \qquad \frac{\partial f}{\partial y} = 6xy + 3y^2$$

Therefore
$$\frac{dy}{dx} = -\frac{2x + 3y^2}{6xy + 3y^2}$$

Suppose, now, that z is given as an implicit function of x and y by the equation

$$F(x,y,z) = 0 \tag{14}$$

This means, of course, that $z = \Phi(x,y)$ defined by equation (14). To find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we may proceed as before. Temporarily, we set $u = F(x,y,z)$. Then, whether x, y, z are independent variables or not,

$$du = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz$$

If we now choose z so that equation (14) is satisfied, then $du = 0$, and

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0 \tag{15}$$

But, under these circumstances, z is a function of the independent variables x and y by virtue of (14); therefore,

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Substituting this into (15) and rearranging terms, we obtain

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right)dy = 0$$

Since x and y are independent variables, we may hold y fixed. Then $dy = 0$; and, taking $dx \neq 0$, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \tag{16}$$

from which

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial F}{\partial z} \neq 0 \tag{17}$$

Similarly

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial F}{\partial z} \neq 0 \tag{18}$$

The values for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ given by (17) and (18) are, of course, the same as those obtained by the method of Art. 134.

An argument of the same character will establish the fact that, if u is given as an implicit function of several variables by an equation

$$F(u,x,y,z, \; . \; . \; . \; ,t) = 0,$$

then the partial derivative of u with respect to any one of the variables can be obtained from equations like (16). For example,

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t} = -\frac{\dfrac{\partial F}{\partial t}}{\dfrac{\partial F}{\partial u}} \qquad \frac{\partial F}{\partial u} \neq 0$$

Example 2. If $F(x,y,z) = 4x^5 + 3y^3z^2 + yz^4 - z^5 = 0$, then

$$\frac{\partial F}{\partial x} = 20x^4 \qquad \frac{\partial F}{\partial y} = 9y^2z^2 + z^4 \qquad \frac{\partial F}{\partial z} = 6y^3z + 4yz^3 - 5z^4$$

and

$$\frac{\partial z}{\partial x} = -\frac{20x^4}{6y^3z + 4yz^3 - 5z^4} \qquad \frac{\partial z}{\partial y} = -\frac{9y^2z^2 + z^4}{6y^3z + 4yz^3 - 5z^4} = -\frac{9y^2z + z^3}{6y^3 + 4yz^2 - 5z^3}$$

EXERCISES

Use the methods of Art. 142 to find the indicated derivatives:

1. Find $\dfrac{dy}{dx}$ if $x^3 + 4xy^2 + 3y^4 = 23$.
2. Find $\dfrac{dy}{dx}$ if $x^{2/3} + y^{2/3} = a^{2/3}$.
3. Find $\dfrac{dy}{dx}$ if $x^{1/2} + y^{1/2} = a^{1/2}$.
4. Find $\dfrac{dy}{dx}$ if $x^2 + xy + y^2 = 1$.
5. Find $\dfrac{dy}{dx}$ if $x^4y^2 - 3x^5y + x^6 + 8 = 0$.

6. Find $\dfrac{dy}{dx}$ if $4x^2y^3 - 5x^3y + 2x^4 + y^4 = 1$.

7. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^2 + 4y^2 + 9z^2 = 36$.

8. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

9. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^4 - 3xyz^2 + 5x^2yz + y^2z + z^4 = 1$.

10. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $3x^3yz^2 + xy^4z + y^2z^4 + z^6 = 0$.

11. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $xyz + x^2y + xy^2 + zx^2 + z^3 = 100$.

12. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $\dfrac{y}{x} + \dfrac{xz}{y^2} + \dfrac{yz^3}{x^4} = 1$.

13. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial z}{\partial t}$ if $x^2 + 4y^2 + 9z^2 + 16t^2 = 144$.

14. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial z}{\partial t}$ if $xyz + x^2t^3 + xy^2z^3t^2 + z^4t^5 = 1$.

15. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial z}{\partial t}$ if $\dfrac{x}{y} + \dfrac{y^2}{xz} + \dfrac{z^2}{ty} + \dfrac{t^4}{x^4} = 0$.

143. Tangent Plane and Normal Line. The tangent plane to a surface $z = f(x,y)$ at a point of the surface contains all the tangent lines at the point and is determined by any two of them. In Fig. 190, let P be the point whose coordinates are (x_1,y_1,z_1), and let the plane $PLKM$ be the tangent plane at P. This plane is cut by the plane $y = y_1$ in the line PL, tangent to the curve PR. The slope of the line PL is simply the value of $\dfrac{\partial z}{\partial x}$ at point P, and we denote this for convenience by the symbol $\left[\dfrac{\partial z}{\partial x}\right]_1$. Similarly, the tangent plane is cut by the plane $x = x_1$ in the line PM whose slope is $\left[\dfrac{\partial z}{\partial y}\right]_1$. The equation of any plane through P can be written in the form

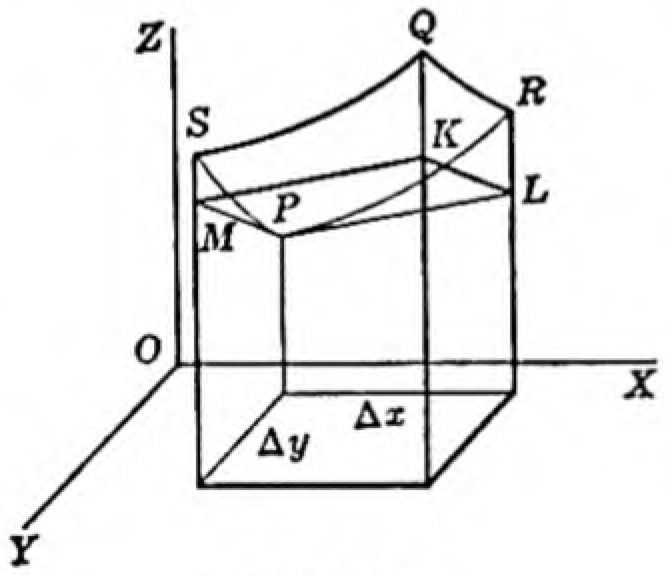


FIG. 190.

$$z - z_1 = A(x - x_1) + B(y - y_1) \tag{19}$$

We shall determine A and B so that lines PL and PM are lines of the plane and, therefore, so that (19) is the equation of the tangent plane.

The intersection of the plane of equation (19) with the plane $y = y_1$ is

$$z - z_1 = A(x - x_1) \qquad y = y_1$$

If $A = \left[\dfrac{\partial z}{\partial x}\right]_1$, these are the equations of line PL.

The intersection of the plane of equation (19) with the plane $x = x_1$ is

$$z - z_1 = B(y - y_1) \qquad x = x_1$$

If $B = \left[\frac{\partial z}{\partial y}\right]_1$, these are the equations of line PM. Therefore, the equation of the tangent plane is

$$z - z_1 = \left[\frac{\partial z}{\partial x}\right]_1 (x - x_1) + \left[\frac{\partial z}{\partial y}\right]_1 (y - y_1) \tag{20}$$

Equation (20) can be written in the form

$$\left[\frac{\partial z}{\partial x}\right]_1 (x - x_1) + \left[\frac{\partial z}{\partial y}\right]_1 (y - y_1) - (z - z_1) = 0$$

The direction cosines of a normal to this plane are proportional to $\left[\frac{\partial z}{\partial x}\right]_1, \left[\frac{\partial z}{\partial y}\right]_1, -1$. The line through P perpendicular to the tangent plane is called the *normal to the surface at* P. Its equations are, therefore,

$$\frac{x - x_1}{\left[\frac{\partial z}{\partial x}\right]_1} = \frac{y - y_1}{\left[\frac{\partial z}{\partial y}\right]_1} = \frac{z - z_1}{-1} \tag{21}$$

If z is given as an implicit function of x and y by the equation

$$F(x,y,z) = 0$$

equations (20) and (21) of the tangent plane and normal line at point $P(x_1,y_1,z_1)$ can be put into very convenient forms. For we have

$$\left[\frac{\partial z}{\partial x}\right]_1 = -\frac{\left[\frac{\partial F}{\partial x}\right]_1}{\left[\frac{\partial F}{\partial z}\right]_1} \qquad \left[\frac{\partial z}{\partial y}\right]_1 = -\frac{\left[\frac{\partial F}{\partial y}\right]_1}{\left[\frac{\partial F}{\partial z}\right]_1}$$

and the equations reduce to

$$\left[\frac{\partial F}{\partial x}\right]_1 (x - x_1) + \left[\frac{\partial F}{\partial y}\right]_1 (y - y_1) + \left[\frac{\partial F}{\partial z}\right]_1 (z - z_1) = 0 \tag{22}$$

$$\frac{x - x_1}{\left[\frac{\partial F}{\partial x}\right]_1} = \frac{y - y_1}{\left[\frac{\partial F}{\partial y}\right]_1} = \frac{z - z_1}{\left[\frac{\partial F}{\partial z}\right]_1} \tag{23}$$

for the tangent plane and normal line, respectively. Equation (23) keeps its sense also if some (but not all) of the denominators are zero, provided that we interpret it to mean proportionality between numerators and denominators.

Examp e. Find the equations of the tangent plane and normal line to the ellipsoid $x^2 + 4y^2 + 9z^2 - 250 = 0$ at the point (3,2,5). Here, we have

$$F(x,y,z) = x^2 + 4y^2 + 9z^2 - 250$$

then

$$\left[\frac{\partial F}{\partial x}\right]_1 = \Big[2x\Big]_{x=3} = 6 \qquad \left[\frac{\partial F}{\partial y}\right]_1 = \Big[8y\Big]_{y=2} = 16 \qquad \left[\frac{\partial F}{\partial z}\right]_1 = \Big[18z\Big]_{z=5} = 90$$

Using (22) and (23), we obtain

Tangent plane $\quad 6(x - 3) + 16(y - 2) + 90(z - 5) = 0$

or $\quad 3x + 8y + 45z - 250 = 0$

Normal line $\quad \dfrac{x-3}{6} = \dfrac{y-2}{16} = \dfrac{z-5}{90}$

or $\quad \dfrac{x-3}{3} = \dfrac{y-2}{8} = \dfrac{z-5}{45}$

EXERCISES

Find the equations of the tangent plane and normal line to the following surfaces at the points indicated (Ex. 1 to 12):

1. The sphere $x^2 + y^2 + z^2 = 14$ at $(-2,1,3)$
2. The ellipsoid $x^2 + 2y^2 + 4z^2 = 26$ at $(2,-3,-1)$
3. The ellipsoid $x^2 + y^2 + 9z^2 = 56$ at $(4,2,-2)$
4. The paraboloid $x^2 + 4y^2 = 2z$ at $(2,1,4)$
5. The paraboloid $x + y^2 + 4z^2 - 8 = 0$ at $(-5,3,1)$
6. The hyperboloid $x^2 - y^2 + z^2 = 6$ at $(3,2,-1)$
7. The hyperboloid $x^2 + 2y^2 - 4z^2 = 15$ at $(-1,5,3)$
8. The cone $x^2 + 5y^2 - z^2 = 0$ at $(4,2,6)$. Sketch.
9. The cylinder $x^2 + y^2 = 25$ at $(3,4,6)$. Sketch.
10. The cylinder $x^2 = 4ay$ at $(2a,a,a)$. Sketch.
11. The surface $x^{2/3} + y^{2/3} + z^{2/3} = 14$ at $(1,8,-27)$
12. The surface $x^{1/2} + y^{1/2} + z^{1/2} = 9$ at $(4,9,16)$

13. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at (x_1,y_1,z_1) is $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} + \dfrac{z_1z}{c^2} = 1$.

14. Find the equation of the tangent plane to the paraboloid $y = xz$ at the point (3,6,2).

15. Find the equation of the tangent plane to the surface

$$x^2y^2 + z^2 = 20 \text{ at } (-1,2,4)$$

16. Find the equation of the tangent plane to the surface

$$xy + yz + zx = 1 \text{ at } (2,3,-1)$$

17. Find the equation of the tangent plane to the hyperboloid

$$x^2 + 3y^2 - 4z^2 + 3x - 2y + 10z - 42 = 0 \text{ at } (4,2,1)$$

18. Find the equation of the tangent plane to the ellipsoid

$$4x^2 + 2y^2 + z^2 - 16x + 8y - 4z - 26 = 0 \text{ at } (-1,1,2)$$

19. Two surfaces are said to be tangent to one another at a point if they have the same tangent plane at this point. Show that the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $yz = a^2$ are tangent at the point $(0,a,a)$.

20. Show that the sum of the squares of the intercepts of the tangent plane to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is constant.

144. Maximum and Minimum Values of a Function. It is frequently important to find any maximum or minimum values of a function of two or more variables. Although a complete treatment of this problem is beyond the scope of this book, an outline of the method of solution will be given.

We start with $z = f(x,y)$, a function of two independent variables, and suppose it to have continuous partial derivatives with respect to x and y. If z has a maximum value z_1 for $x = x_1$, $y = y_1$, this may be interpreted geometrically as the ordinate of a point on the surface $z = f(x,y)$ that is higher than any other nearby point on that surface. Hence, the tangent plane at (x_1,y_1,z_1) must be parallel to the xy plane, and the normal must be parallel to the z axis. Consequently, the direction cosines of the normal are 0, 0, ± 1. But in Art. 143 we found the direction cosines of the normal to be $\left[\frac{\partial z}{\partial x}\right]_1$, $\left[\frac{\partial z}{\partial y}\right]_1$, -1. Therefore, if $z = f(x,y)$ has a maximum value for $x = x_1$, $y = y_1$, we must have $\left[\frac{\partial z}{\partial x}\right]_1 = 0$ and $\left[\frac{\partial z}{\partial y}\right]_1 = 0$.

Similarly, if $f(x,y)$ has a minimum value at $x = x_1$, $y = y_1$, the tangent plane to the surface at (x_1,y_1,z_1) must be parallel to the xy plane, and again $\left[\frac{\partial z}{\partial x}\right]_1 = 0$ and $\left[\frac{\partial z}{\partial y}\right]_1 = 0$.

In general, therefore, *a necessary condition that $z = f(x,y)$ have a maximum or minimum value at a point is that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ evaluated at that point be simultaneously equal to zero.*

Example 1. Find the shape of the covered rectangular box having a given volume V and a minimum surface area. If the box has dimensions x by y by z, then the volume is $V = xyz$, and the surface area is $S = 2(xy + yz + zx)$. Since $z = V/xy$,

$$S = 2\left(xy + \frac{V}{x} + \frac{V}{y}\right)$$

Hence $$\frac{\partial S}{\partial x} = 2\left(y - \frac{V}{x^2}\right) \quad \text{and} \quad \frac{\partial S}{\partial y} = 2\left(x - \frac{V}{y^2}\right)$$

Setting these derivatives simultaneously equal to zero, we have

$$y - \frac{V}{x^2} = 0 \qquad x - \frac{V}{y^2} = 0$$

Therefore $$y = \frac{V}{x^2} \quad \text{and} \quad x - \frac{V}{V^2/x^4} = 0$$

whence $$x - \frac{x^4}{V} = 0$$

Since x cannot be zero, we must have $1 - \frac{x^3}{V} = 0$, and $x = V^{1/3}$. From this, we get

$$y = \frac{V}{x^2} = \frac{V}{V^{2/3}} = V^{1/3} \quad \text{and} \quad z = \frac{V}{xy} = \frac{V}{V^{1/3} \cdot V^{1/3}} = V^{1/3}$$

Since $x = y = z = V^{1/3}$, the box must be a cube. Clearly, the surface area of the box can be made as large as we please by taking z small enough; there is no maximum surface area. Consequently, the cube gives a minimum surface area.

As in the case of functions of a single variable, it is often convenient to express the quantity whose maximum or minimum is to be found in terms of more variables than necessary. For instance, in Example 1, we used three variables x, y, z. But these variables were connected by a relation $V = xyz$, so that there were, in reality, only two *independent* variables. Care must be taken to decide which variables are to be regarded as independent and to write down all relations before differentiating.

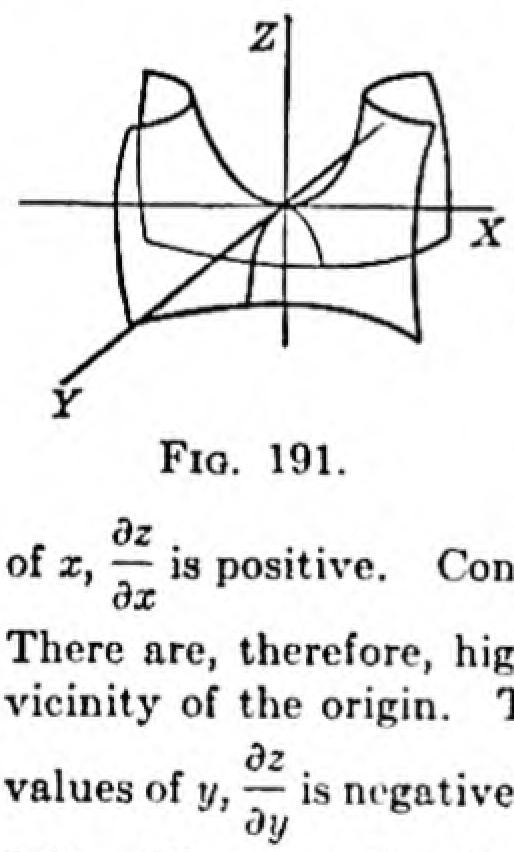


FIG. 191.

Example 2. Find any maximum or minimum values of z if

$$z = x^2 - y^2$$

(Fig. 191). Here $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$, and these are both zero if $x = y = 0$. Hence, the tangent plane is horizontal at the origin. But note that, for small positive values of x, $\frac{\partial z}{\partial x}$ is positive. Consequently, the xz plane cuts the surface in a curve that *rises*. There are, therefore, higher points than the origin on the surface in the immediate vicinity of the origin. The origin is not a maximum point. Also, for small positive values of y, $\frac{\partial z}{\partial y}$ is negative. Consequently, the yz plane cuts the surface in a curve that *falls*. There are, therefore, lower points than the origin on the surface in the immediate vicinity of the origin. The origin is not a minimum point. The situation is indicated in Fig. 191. Since there are no other points at which $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero, there are no maximum or minimum points on the surface.

We see from these two examples that the vanishing of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ does not guarantee a maximum or a minimum. In many cases (as in these two examples), it is not difficult to determine whether there is a maximum, a minimum, or neither. It is, however, possible to state a rule that will enable us to determine in most cases the character of a point at

which the tangent plane is horizontal. This rule will be stated without proof.

Suppose $z = f(x,y)$. *Find values of* x *and* y, *say* x_1 *and* y_1, *for which* $\dfrac{\partial z}{\partial x} = 0$ *and* $\dfrac{\partial z}{\partial y} = 0$. *Calculate*

$$Q = \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} \qquad \text{for } x = x_1, \text{ and } y = y_1$$

Then, $z_1 = f(x_1,y_1)$ *will be*

1. *A maximum value if* $Q < 0$ *and* $\dfrac{\partial^2 z}{\partial x^2}$ *is negative.*
2. *A minimum value if* $Q < 0$ *and* $\dfrac{\partial^2 z}{\partial x^2}$ *is positive.*
3. *Neither a maximum nor a minimum if* $Q > 0$.

If $Q = 0$, *this rule does not suffice to determine the character of* z; *this may be a maximum, a minimum, or neither.*

Example 3. We apply this rule to Example 1 above. Here $\dfrac{\partial^2 S}{\partial x^2} = \dfrac{4V}{x^3}$ has the value 4 for $x = V^{1/3}$. Similarly $\dfrac{\partial^2 S}{\partial y^2} = \dfrac{4V}{y^3}$ has the value 4 for $y = V^{1/3}$. Also

$$\frac{\partial^2 S}{\partial x\, \partial y} = 2$$

Hence $Q = 2^2 - 4 \cdot 4 = -12 < 0$. Since $\dfrac{\partial^2 S}{\partial x^2}$ is positive $(= 4)$, S has a minimum value.

Example 4. We apply this rule to Example 2 above. Here $\dfrac{\partial^2 z}{\partial x^2} = 2$, $\dfrac{\partial^2 z}{\partial y^2} = -2$, $\dfrac{\partial^2 z}{\partial x\, \partial y} = 0$. Hence $Q = 0 - (2)(-2) = 4 > 0$, and z has neither a maximum nor a minimum value.

Example 5. Find the dimensions and volume of the parallelepiped with sides parallel to the coordinate axes inscribed in the ellipsoid

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and having maximum volume. One-eighth of the volume is shown in Fig. 192. The volume that is to be a maximum is, therefore, $V = 8xyz$ where $P(x,y,z)$ is a point of the ellipsoid lying in the first octant. But x, y, z are not all independent variables since they are connected by relation (1). Suppose we choose x and y as independent variables. Then, from (1)

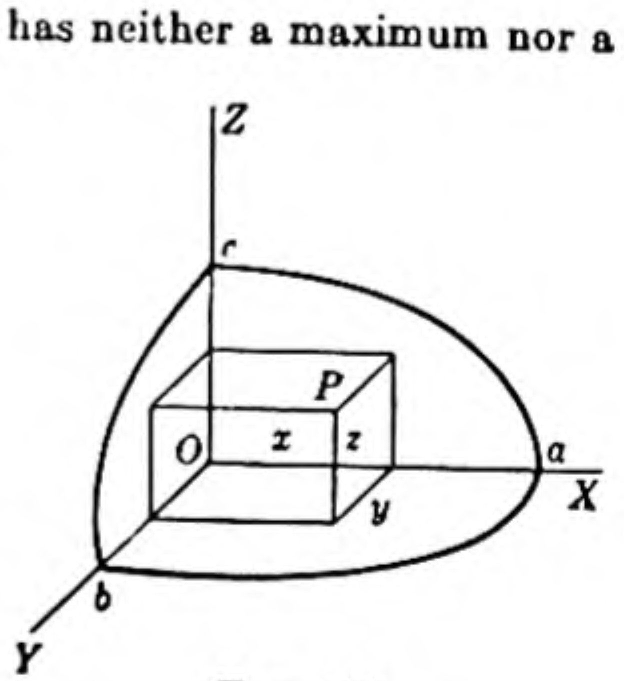


FIG. 192.

(2) $$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

where we take the positive square root to correspond to the point P in the first octant. Hence

$$V = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

From this, we find

$$\frac{\partial V}{\partial x} = 8c\left(y\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} - \frac{x^2 y}{a^2\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\right)$$

$$= \frac{8cy}{a^2\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\left[a^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - x^2\right]$$

Similarly $$\frac{\partial V}{\partial y} = \frac{8cx}{b^2\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\left[b^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - y^2\right]$$

We must set $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ equal to zero and solve simultaneously for x and y. Observe that $x = 0$ or $y = 0$ give zero for the volume and, therefore, obviously do not give a maximum. Hence

$$a^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - x^2 = 0 \qquad b^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - y^2 = 0$$

from which we get

$$2x^2 + \frac{a^2}{b^2}y^2 = a^2 \qquad \frac{b^2}{a^2}x^2 + 2y^2 = b^2$$

Solving simultaneously gives $x = a/\sqrt{3}$, $y = b/\sqrt{3}$. We take the positive square roots to correspond to point P in the first octant. Using (2), we compute z, obtaining $z = c/\sqrt{3}$. This gives $V = 8abc/3\sqrt{3}$. That this actually is a maximum volume can be shown by computing Q as in Examples 3 and 4, or we may rely upon geometrical intuition.

Alternative Method. Since $V = 8xyz$, we have

$$\frac{\partial V}{\partial x} = 8y\left(x\frac{\partial z}{\partial x} + z\right) \qquad \frac{\partial V}{\partial y} = 8x\left(y\frac{\partial z}{\partial y} + z\right)$$

Therefore, since $x = 0$ and $y = 0$ obviously do not give a maximum V,

(3) $$x\frac{\partial z}{\partial x} + z = 0 \qquad y\frac{\partial z}{\partial y} + z = 0$$

Now, since $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, we have $\dfrac{2x}{a^2} + \dfrac{2z}{c^2}\dfrac{\partial z}{\partial x} = 0$, so that

$$\frac{\partial z}{\partial x} = -\frac{c^2x}{a^2z}$$

Similarly $$\frac{\partial z}{\partial y} = -\frac{c^2y}{b^2z}$$

Substituting these values of the derivatives into (3) gives

$$x\left(-\frac{c^2x}{a^2z}\right) + z = 0 \quad \text{or} \quad \frac{z^2}{c^2} = \frac{x^2}{a^2} \quad \text{and} \quad \frac{z^2}{c^2} = \frac{y^2}{b^2}$$

Hence

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Consequently, the point P must have coordinates satisfying these equations and also the equation of the ellipsoid. Substituting x^2/a^2 for y^2/b^2 and z^2/c^2 in the equation of the ellipsoid gives $3x^2/a^2 = 1$ or $x = a/\sqrt{3}$. Therefore, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$, and $V = 8abc/3\sqrt{3}$.

If $u = f(x,y,z, \ldots ,t)$ is a function of more than two independent variables with continuous partial derivatives, the geometrical argument becomes inconvenient. However, if u has a maximum value for $x = x_1$, $y = y_1$, $z = z_1$, $\ldots$ $t = t_1$, then $\dfrac{\partial u}{\partial x}$ must be zero for these values of the variables. For if $y, z, \ldots, t$ are fixed at $y_1, z_1, \ldots, t_1$, then $f(x,y_1,z_1, \ldots ,t_1)$ is a function of x alone. For this function to have a maximum, its derivative with respect to x must be zero. Hence $\dfrac{\partial u}{\partial x} = 0$. Similarly $\dfrac{\partial u}{\partial y} = 0$, $\dfrac{\partial u}{\partial z} = 0, \ldots,$ $\dfrac{\partial u}{\partial t} = 0$ simultaneously at $(x_1,y_1,z_1, \ldots t_1)$. In the same way, these partial derivatives must vanish if the function is to have a minimum value. It must be noted that this is only a *necessary* condition. The presence of a maximum or minimum is not guaranteed, but a discussion of sufficient conditions will not be attempted here.

EXERCISES

Find any maximum and minimum values of the following functions Ex. 1 to 6:

1. $z = x^2 + 4y^2 - 2x + 8y - 1$
2. $z = x^2 - y^2 + 6x - 10y + 2$
3. $z = xy$
4. $z = 9 + 4x - y - 2x^2 - 3y^2$
5. $z = r^2 + 4rs + s^2 - 6s + 1$
6. $z = r^2 - rs + 2s^2 - 5r + 6s - 9$

7. Find any maximum or minimum points of the surface $z = e^{-(x^2+y^2)}$.

8. A covered rectangular box has a fixed surface area. What shape will it have if the volume is a maximum?

9. Formerly, for a package to go by parcel post, the sum of its length and girth could not exceed 100 in. Find the dimensions of the rectangular package of greatest volume that could be sent.

10. Show that the rectangular parallelepiped of maximum volume the sum of whose length and girth is fixed has a square base (compare Exercise 9).

11. An open rectangular box has a fixed volume. What shape will make the surface area a minimum?

12. The base of a rectangular box costs half as much per square foot as the top and sides. Find the most economical proportions.

13. Find by the use of derivatives the shortest distance from the origin to the plane $x + y + z = a$.

14. The cross section of a trough is an isosceles trapezoid. If the trough is made by bending up the sides of a strip of metal a in. wide, what should be the angle of inclination of the sides and the width across the bottom if the cross-sectional area is to be a maximum?

15. A plane passing through the point (1,2,1) cuts off a minimum volume from the first octant. Find its equation.

16. Solve Exercise 15 if the plane is to pass through the point $(k,2k,3k)$ instead of (1,2,1).

17. The volume of an ellipsoid is $\frac{4}{3}\pi abc$ where a,b,c are the semiaxes. If the sum $a + b + c$ is fixed, show that the ellipsoid of maximum volume is a sphere.

18. Find the point the sum of the squares of whose distances from (1,4), (5,2), (3, −2) is a minimum.

19. Find the point the sum of the squares of whose distances from (4,3), (2,4), (3, −7), (−2, −1), (−3, −5) is a minimum.

20. Find the dimensions and volume of the parallelepiped with sides parallel to the coordinate axes and maximum volume inscribed in the sphere $x^2 + y^2 + z^2 = a^2$.

145. Exact Differentials. We have seen that, if $z = f(x,y)$, then the differential of z is defined by

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

The question arises, under what circumstances an expression

$$M\,dx + N\,dy \tag{24}$$

where M and N are continuous functions of x and y with continuous partial derivatives, is the differential of some function $z = f(x,y)$. If it is the differential of some function, we call it an *exact differential.*

It can be proved that a *necessary and sufficient condition that*

$$M\,dx + N\,dy$$

be an exact differential is that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}\cdot$ We shall prove that this is a *necessary* condition; that is to say, if $M\,dx + N\,dy$ is an exact differential, then necessarily $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}\cdot$ For, if $dz = M\,dx + N\,dy$, where $z = f(x,y)$, then $M = \dfrac{\partial z}{\partial x}$, and $N = \dfrac{\partial z}{\partial y}\cdot$ Hence

$$\frac{\partial M}{\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

But these second derivatives are equal; therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To prove that the condition is *sufficient,* that is, sufficient to ensure that $M\,dx + N\,dy$ be an exact differential, requires proving that, if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$,

then there is a function $z = f(x,y)$ such that $\dfrac{\partial z}{\partial x} = M$ and $\dfrac{\partial z}{\partial y} = N$. This proof will be found in Art. 180 (page 476).

A method for finding the function of which $M\,dx + N\,dy$ is the differential, that is, for finding the *integral* of this differential, will be illustrated by an example.

Example. Show that $(2xy^3 + 16x^3y)\,dx + (3x^2y^2 + 4x^4 + 5y^4)\,dy$ is an exact differential, and find its integral. First, since

$$M = 2xy^3 + 16x^3y \qquad \frac{\partial M}{\partial y} = 6xy^2 + 16x^3$$

and since

$$N = 3x^2y^2 + 4x^4 + 5y^4 \qquad \frac{\partial N}{\partial x} = 6xy^2 + 16x^3$$

This is, therefore, an exact differential.

To find the integral, we note that, whatever z may be, we must have

$$\frac{\partial z}{\partial x} = M = 2xy^3 + 16x^3y$$

If we integrate this with regard to x, holding y constant, we obtain

$$z = x^2y^3 + 4x^4y + \varphi(y) \tag{25}$$

where $\varphi(y)$ is any function of y free of x. But if this is z, then its partial derivative with respect to y must be N. This consideration will enable us to determine $\varphi(y)$, for

$$\frac{\partial z}{\partial y} = 3x^2y^2 + 4x^4 + \varphi'(y)$$

This must equal N. Hence

$$3x^2y^2 + 4x^4 + \varphi'(y) = 3x^2y^2 + 4x^4 + 5y^4$$

Therefore

$$\varphi'(y) = 5y^4$$

$$\varphi(y) = y^5 + C \tag{26}$$

Consequently, from (25) and (26),

$$z = x^2y^3 + 4x^4y + y^5 + C$$

The reader can obtain the same z by first integrating N with respect to y holding x constant, then differentiating the result to get $\dfrac{\partial z}{\partial x}$. Setting $\dfrac{\partial z}{\partial x} = M$ will give an equation analogous to (26).

EXERCISES

Determine which of the following differentials are exact, and then find their integrals. Check by differentiating results.

1. $(2x + 3y)\,dx + (3x + 2y)\,dy$
2. $(3x^2y^2 + 2y^4 + 15x^2)\,dx + (2x^3y + 8xy^3)\,dy$
3. $\left(3x^2y^2 + \dfrac{1}{x}\right)dx + \left(\dfrac{2x^3y^2 - 1}{y}\right)dy$ 4. $(x^2 + y^2)\,dx + 2xy\,dy$

5. $x^2 \sin y\, dx + x^3 \cos y\, dy$

6. $2xe^{x^2} \sin y\, dx + e^{x^2} \cos y\, dy$

7. $(x + \sin x \tan y)\, dx + (y + \tan x \sin y)\, dy$

8. $x \cos y^2(x \cos x + 2 \sin x)\, dx - 2x^2 y \sin x \cos y^2\, dy$

9. $\dfrac{x\, dx}{\sqrt{x^2 + y^2}} + \left(\dfrac{y}{\sqrt{x^2 + y^2}} - 1\right) dy$

10. $\dfrac{y\, dx - x\, dy}{x^2 + y^2}$

11. $\dfrac{x\, dx}{\sqrt{x^2 + y^2}} + \sqrt{x^2 + y^2}\, dy$

12. $2x \ln y\, dx + \dfrac{x^2}{y}\, dy$

13. $\left(3x^2 \ln \dfrac{y}{x} - x^2\right) dx + \dfrac{3^{-2}}{y}\, dy$

14. $e^{\frac{y}{x}}\left[\sinh x - \dfrac{y}{x^2} \cosh x\right] dx + \dfrac{1}{x} e^{\frac{y}{x}} \cosh x\, dy$

15. $-\dfrac{y\, dx}{x\sqrt{x^2 - y^2}} + \dfrac{dy}{\sqrt{x^2 - y^2}}$

MISCELLANEOUS EXERCISES

1. Does $u = \dfrac{x^2 + y^2 + z^2 + 2y}{z}$ satisfy the equation

$$(x^2 - y^2 - z^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} + 2xz\frac{\partial u}{\partial z} = 0?$$

2. Verify that $u = xy^2$ satisfies the equation $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

3. Verify that $z = \dfrac{xy}{x + y}$ satisfies $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

4. Verify that $u = 3w^2 - x^2 - y^2 - z^2$ satisfies the equation

$$xyz\frac{\partial u}{\partial w} + yzw\frac{\partial u}{\partial x} + zwx\frac{\partial u}{\partial y} + wxy\frac{\partial u}{\partial z} = 0$$

5. Find $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial x\, \partial y}, \dfrac{\partial^2 z}{\partial y^2}$ if $z = e^x \tan y$.

6. Same as Exercise 5 if $z = x^2 \cosh y$

7. Verify that $6\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\, \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = 5x^2 - 8xy + 13y^2$.

8. Verify that $\dfrac{\partial^4 z}{\partial x^4} - 2\dfrac{\partial^4 z}{\partial x^2\, \partial y^2} + \dfrac{\partial^4 z}{\partial y^4} = 0$ if

$$z = \sin(x + y) + \cos(x - y)$$

9. Verify that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} + u = 2e^{-x}$ if

$$u = e^{-x}[x^2 + 2x + 2y + 2\tan(x + y)]$$

10. Find the differential of the function $z = \dfrac{x^2}{x + y}$.

11. Find dz if $z = \cot(x/y)$.

12. Find dw if $w = e^{x \sin y}$.

13. Find du if $u = \ln (x \sec y)$. **14.** Find du if $u = \tanh (r/s)$.

15. The dimensions of a rectangular box are found to be 30, 24, and 18 in. with possible errors of 0.1 in. in each. Find approximately the greatest error in the computed volume; also, find the greatest percentage error.

16. The radius of the base of a right circular cone is found to be 6 in., with a possible error of 0.05 in. The altitude is found to be 10 in., with a possible error of 0.02 in. Find approximately the greatest error in the computed volume; also, find the greatest percentage error.

17. The inaccessible distance between two points A and B is found by taking a convenient point C and measuring $AC = 75$ ft., $BC = 110$ ft., $\angle ACB = 60°$. If AC and BC have possible errors of 1 ft., find approximately the maximum error in AB.

18. Solve Exercise 17 if the angle ACB can be in error by as much as 30 minutes, with possible errors of 1 ft. in AC and BC.

19. Find the directional derivative of the function $z = x^2 - 4y^2$ at (4,1) in the direction making an angle of 45 deg. with the x axis.

20. In what direction from the point (x_1,y_1) does the function $z = x^2 + y^2$ change the most rapidly?

21. Find $\dfrac{dz}{d\theta}$ if $z = \dfrac{y}{x}$ and $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

22. Find $\dfrac{dz}{dt}$ by use of partial derivatives; check the results by expressing z in terms of t and differentiating: $z = \ln (x^2 - y^2)$ where $x = \cos t$, $y = \sin t$. Evaluate for $t = \pi/6$.

23. Same as Exercise 22 for $z = 4x^2 + 9y^2$ where $x = e^t$, $y = e^{-t}$. Evaluate for $t = \ln 2$.

24. Same as Exercise 22 for $z = e^{\frac{y}{x}}$ where $x = \cos t$, $y = \sin t$. Evaluate for $t = 0$.

25. Find $\dfrac{dz}{dx}$ by two methods: $z = xe^{xy}$ where $y = \sin x$.

26. Find $\dfrac{dz}{dx}$ by two methods: $z = \sqrt{x^2 + 9y^2}$ where $y = x^2$.

27. Find $\dfrac{dz}{dx}$ by two methods: $z = \ln \dfrac{\sin x}{y}$ where $y = x^2$.

28. Find $\dfrac{dz}{dx}$ by two methods: $z = x^2 \arctan y$ where $y = \ln x$.

29. Use the methods of Art. 141 to find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ if $z = x^2 + xy$ where $x = r \cos \theta$, $y = r \sin \theta$.

30. Use the methods of Art. 141 to find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ if $z = \sin u + ve^u$ where $u = x^2 + 3y^2$, $v = x \ln y$.

31. Use the methods of Art. 141 to find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ if $z = u \ln v + v^2 w$ and $u = x/y$, $v = x^2 - y$, $w = y \tan x$.

32. Use the methods of Art. 141 to find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$, $\dfrac{\partial z}{\partial t}$ if $z = x^2 \ln \dfrac{y}{x}$ where $x = rt \cos \theta$, $y = rt \sin \theta$.

33. If $u = f\left(\dfrac{y}{x}\right)$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

34. If $F(x,y,z) = 0$, show that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1$ where $\left(\frac{\partial z}{\partial x}\right)_y$ means the partial derivative of z with y held fixed and the other symbols have similar meanings.

35. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = a^2$ at (x_1,y_1,z_1) on the sphere.

36. The tangent plane to the surface $xyz = a^3$ at any point of the surface forms with the coordinate planes a tetrahedron with three mutually perpendicular faces. Show that the volume of this tetrahedron is a constant, namely, $\frac{9}{2}a^3$.

37. Show that the sum of the intercepts of the tangent plane to the surface

$$x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$$

is a and therefore independent of the position of the point of tangency.

38. Show that the normal to a sphere at any point of the sphere passes through the center.

39. Find any maximum or minimum points on the surface

$$z = 16 - 2x + 10y - x^2 + 2xy - 2y^2$$

40. Find any maximum or minimum points on the surface $z = xy + x^2$.

41. Find any maximum or minimum points of the surface $z = e^{-(x^2+xy+y^2)}$.

42. An open rectangular box has fixed surface area. What shape will make the volume a maximum?

43. Show that the triangle with given perimeter and maximum area is equilateral. [*Hint:* $A = \sqrt{s(s - x)(s - y)(s - z)}$ if x, y, z are the sides. Make A^2 a maximum.]

44. Show that the point the sum of squares of whose distances from the points (x_1,y_1), (x_2,y_2), . . . , (x_n,y_n) is a minimum has coordinates $\frac{1}{n}(x_1 + x_2 + \cdots + x_n)$, $\frac{1}{n}(y_1 + y_2 + \cdots + y_n)$.

CHAPTER 18

MULTIPLE INTEGRALS

146. Volumes by Iterated Integration. In Art. 113, we calculated certain volumes by integration using the *lamina method*. The success of the method depended upon the possibility of expressing the area of a plane cross section of the volume as a function of a single variable. We now proceed to consider this method in a somewhat more general form. We shall still rely upon our intuition for the meaning of the expression "volume bounded by curved surfaces" and postpone a definition to Art. 151.

Consider the volume capped by the surface $z = f(x,y)$, enclosed by a cylinder whose elements are parallel to the z axis, and whose base is a given region S of the xy plane. Suppose that $z = f(x,y)$ is positive (or zero) at every point within and upon the boundary C of the region S (Fig. 193). Furthermore, suppose C to be a curve that is cut by any line parallel to the y axis in not more than two points, as shown in the figure. Let cross sections of this volume be made by n planes parallel to the yz plane and at a distance Δx apart. Suppose the plane $x = x_i$ cuts the surface $z = f(x,y)$ in the curve HK, the cylinder in the lines MH and LK, and the region S of the xy plane in the line LM. The area $HKLM$ multiplied by Δx is the volume of a typical element of volume. The sum of the n such elements is approximately the required volume; and, as in Art. 113, the limit of this sum (as Δx approaches zero and n increases indefinitely) is equal to the required volume.

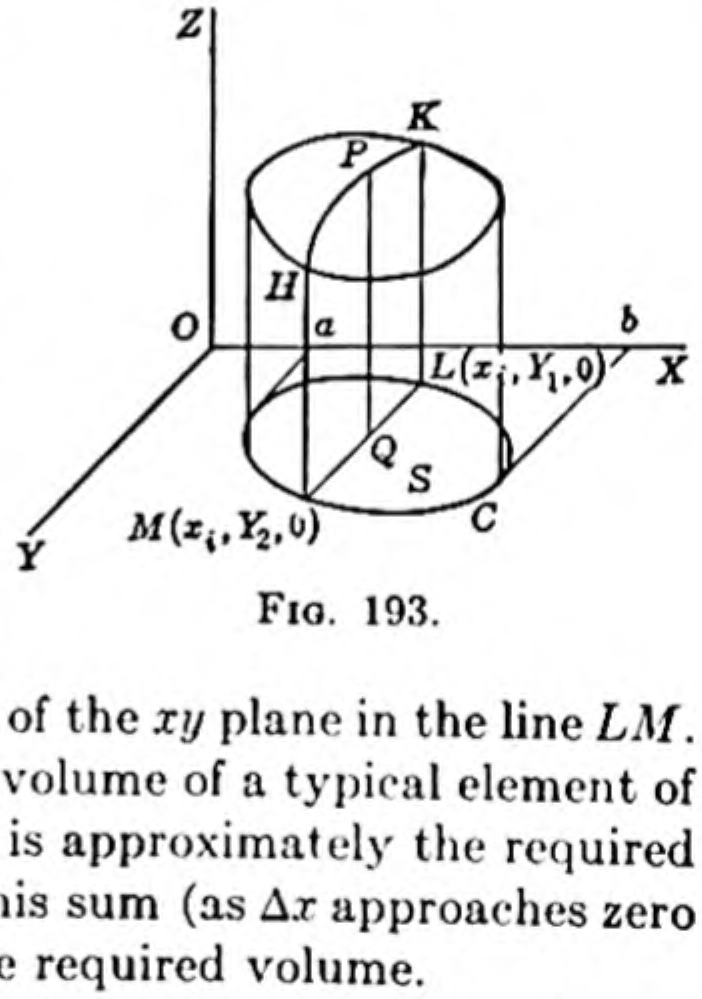


FIG. 193.

We may calculate the area $HKLM = A_i$ as follows: The ordinate $QP = z$ of any point x_i, y, z on the curve HK is obtained from the equation of the surface by setting $x = x_i$. Thus, $QP = z = f(x_i,y)$. Note that, for any *fixed* x_i, this is a function of y alone. The area $HKLM = A_i$ is, therefore, given by the integral $\int_{Y_1}^{Y_2} f(x_i,y)\,dy$ where Y_1 and Y_2 are the y coordinates of L and M, respectively. Let the reader note particularly that Y_1 and Y_2 depend upon x_i. In fact, they may be found by replacing

x by x_i in the equation of the curve C and solving for y. To emphasize the fact that Y_1 and Y_2 are functions of x_i, we shall write $Y_1(x_i)$ and $Y_2(x_i)$. The volume of the ith element of volume (the typical element) is, therefore

$$A_i\,\Delta x = \Delta x \int_{Y_1(x_i)}^{Y_2(x_i)} f(x_i,y)\,dy$$

Note that, in making this integration, x_i is regarded as a *fixed constant.*

As in Art. 113, the total volume is $\int_a^b A\,dx$ where a and b are the x coordinates of the extreme left- and right-hand points of the curve C. Hence

$$V = \int_a^b \left[\int_{Y_1(x)}^{Y_2(x)} f(x,y)\,dy \right] dx$$

The integral enclosed by brackets is to be evaluated on the basis of x as a fixed constant; the result is a function of x alone. This becomes the integrand of the final integration. The subscript i, having served its purpose, need not be written.

This expression for V is called an *iterated* or a *repeated integral* (not a *double integral*) since it is an integral *of* an integral. It is usually written without the brackets as

$$\int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,dy\,dx \tag{1}$$

where the "inside integral" is integrated with respect to y, x being held constant. Note that, in writing this symbol for the iterated integral, the inside integral sign with the adjoined limits refers to the variable designated by the inside differential dy, whereas the outside integral sign and limits refer to the variable designated by the outside differential dx.

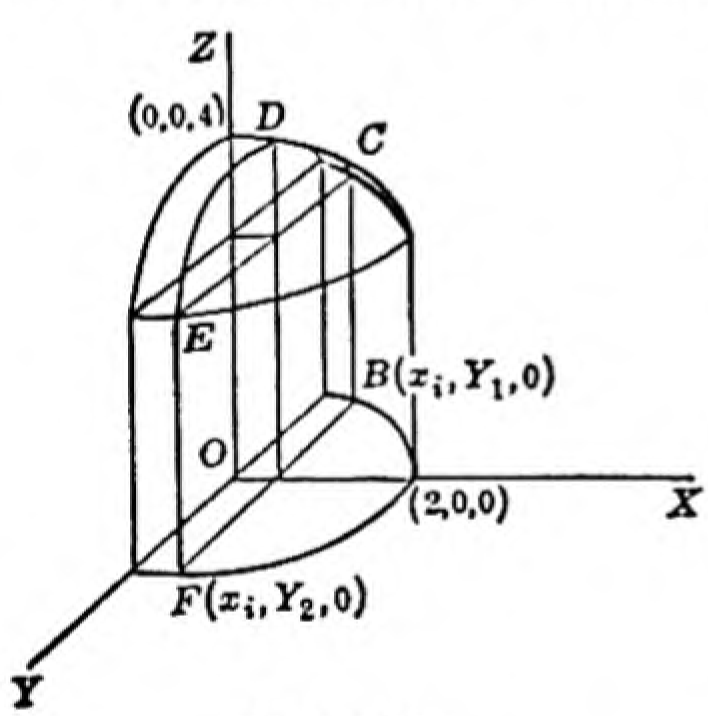


FIG. 194.

Observe that the limits for the first integration are, in general, variables, whereas those for the second integration are constants. Another symbol used for this iterated integral is

$$\int_a^b dx \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,dy^*$$

Example. Find the volume underneath the surface

$$x^2 + y^2 + 4z - 16 = 0$$

above the xy plane and inside the cylinder $x^2 + y^2 = 4$. One-half of the required volume is shown in Fig. 194. Cut this volume by planes parallel to the yz plane at a distance Δx apart. Let the section $BCDLF$ be the one made by the plane $x = x_i$. Its area is

$$A_i = \int_{Y_1(x_i)}^{Y_2(x_i)} f(x_i,y)\,dy$$

* The symbol $\int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,dx\,dy$, occasionally used, should be avoided.

Now, $z = f(x_i,y) = \frac{1}{4}(16 - x_i^2 - y^2)$. Also, the points B and F lie upon the circle $x^2 + y^2 = 4$ in the xy plane. The y coordinate of B, namely, $Y_1(x_i)$, is, therefore, $Y_1(x_i) = -\sqrt{4 - x_i^2}$. Similarly, $Y_2(x_i) = \sqrt{4 - x_i^2}$. Consequently, the area of the ith cross section is

$$A_i = \int_{-\sqrt{4-x_i^2}}^{\sqrt{4-x_i^2}} \tfrac{1}{4}(16 - x_i^2 - y^2)\,dy$$

where x is held constantly equal to x_i during the integration.

If we multiply A_i by Δx, we get the volume of the ith element of volume. The least value of x on the circle $x^2 + y^2 = 4$ is -2, and the greatest value is 2. The total volume is, therefore

$$V = \int_{-2}^{2} A\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \tfrac{1}{4}(16 - x^2 - y^2)\,dy\,dx$$

Carrying out the first integration with regard to y, holding x fixed, we get

$$V = \tfrac{1}{4}\int_{-2}^{2}\left[(16 - x^2)y - \tfrac{1}{3}y^3\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

Substituting the limits $\sqrt{4 - x^2}$ and $-\sqrt{4 - x^2}$ for y gives

$$\begin{aligned}V &= \tfrac{1}{2}\int_{-2}^{2}[(16 - x^2)\sqrt{4 - x^2} - \tfrac{1}{3}(4 - x^2)^{3/2}]\,dx\\ &= \tfrac{1}{6}\int_{-2}^{2}\sqrt{4 - x^2}\,(44 - 2x^2)\,dx = \tfrac{22}{3}\int_{-2}^{2}\sqrt{4 - x^2}\,dx - \tfrac{1}{3}\int_{-2}^{2} x^2\sqrt{4 - x^2}\,dx\end{aligned}$$

These integrals may now be evaluated by the trigonometric substitution $x = 2 \sin \theta$ which gives $V = 14\pi$.

We might have used considerations of symmetry and noted that the volume V is four times the volume appearing in the first octant. The effect is to change the limits of integration and the constant factor. Thus

$$V = \int_0^2\int_0^{\sqrt{4-x^2}} (16 - x^2 - y^2)\,dy\,dx$$

The student should compare this method with that of Art. 113 and observe the similarity. There is no essential difference. In the earlier case, we were able to find A_i by inspection for volumes whose plane cross section was already known. In this case, we find this typical cross-sectional area by means of an integral.

We have supposed $f(x,y)$ to be positive (or zero) for all points of the region S. If, however, $f(x,y)$ is negative for a portion of S, then the volume bounded by $z = f(x,y)$ lies below the xy plane in this part of S and is counted as negative. Clearly, the iterated integral (1) gives the algebraic sum of the positive and negative portions of the volume V. We also supposed the curve C to be cut by a line parallel to the y axis in not more than two points. If, however, this is not the case and the curve C is as shown in Fig. 195, we may think of S as composed of a number of parts such as (i), (ii), and (iii) and find the total volume V by adding together the volumes standing on these several parts of S.

We might have made cross sections of the volume shown in Fig. 193 by n planes parallel to the xz plane and at a distance Δy apart. The area A_i of the cross section $HKLM$ (Fig. 196) made by the plane $y = y_i$ is then

$$A_i = \int_{X_1(y_i)}^{X_2(y_i)} f(x,y_i)\,dx$$

where $f(x,y_i) = z = QP$ is the z coordinate of point P. Note that, for y held constant $(=y_i)$, QP is a function of x only. Points M and L have

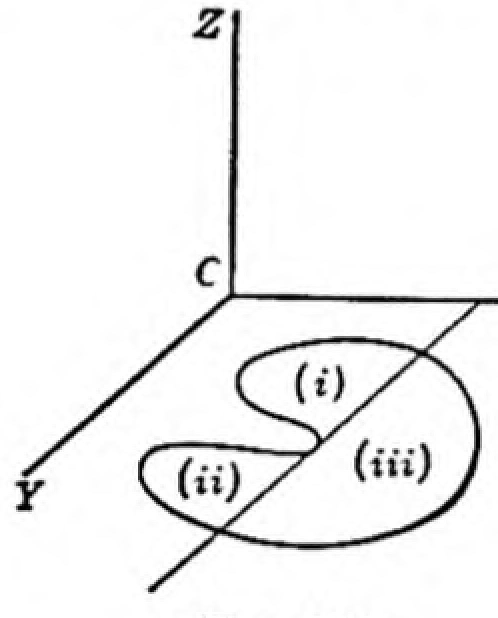


FIG. 195.

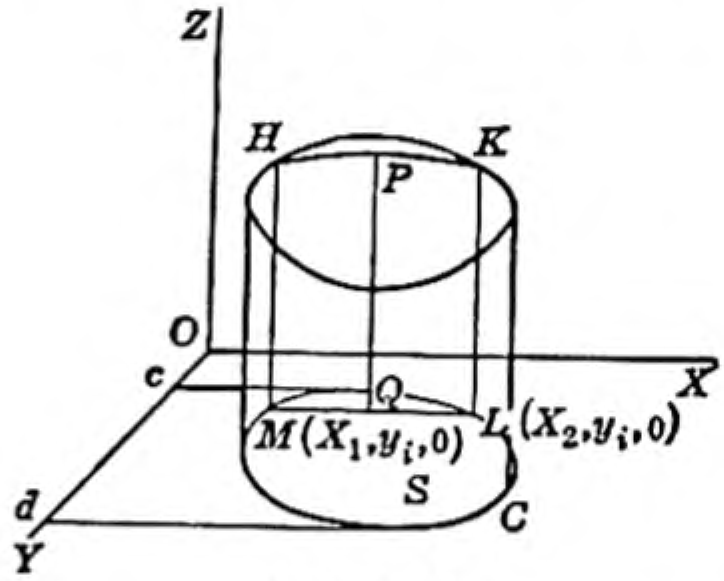


FIG. 196.

x coordinates X_1 and X_2 which can be found by setting $y = y_i$ in the equation of the curve C in the xy plane. The volume of the typical element is $A_i\,\Delta y$; and the required volume is, therefore, equal to $\int_c^d A\,dy$ where c and d are the extreme values of y on the curve C. Thus

$$V = \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\,dx\,dy \tag{2}$$

Here, of course, y is to be held constant during the first integration.

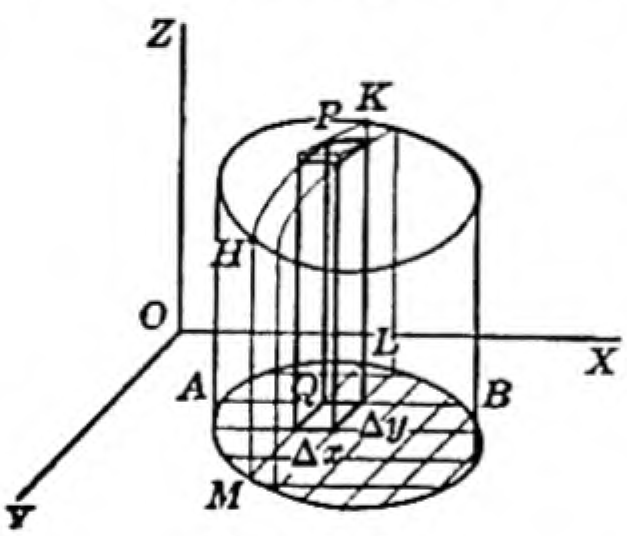


FIG. 197.

The two integrals (1) and (2) are equal to one another since both are equal to V. It is sometimes more convenient to use (2) than (1), or the second iterated integral may be calculated to serve as a check on the work. If we replace (1) by (2), we speak of *inverting the order of integration.*

147. Iterated Integrals, Continued. A somewhat different interpretation may be given to the integral (1). Imagine the volume V cut by n planes parallel to the yz plane at a distance Δx apart and by m planes parallel to the xz plane at a distance Δy apart. These planes cut the region S by lines parallel to the y and x axes, respectively (Fig. 197). Let L be the point $(x_i, Y_1, 0)$ and M the point $(x_i, Y_2, 0)$, as before. Now, imagine a vertical rectangular column of height QP and cross-sectional area $\Delta x\,\Delta y$ as shown in the figure. The volume of this column is $f(x_i,y)\,\Delta x\,\Delta y$ where $(x_i,y,0)$ are the coordinates of Q and $QP = z = f(x_i,y)$.

The two systems of parallel planes form columns of this type, the sum of whose volumes is approximately the required volume under the surface. If we add together all the columns along the (fixed) line LM, we get approximately the volume of the lamina $HKLM$. This sum may be expressed by the symbol $\left[\sum_y f(x_i,y)\,\Delta y\right]\Delta x$ where the sum is extended for y along the line LM. Holding x and Δx fixed and taking the limit of the sum in brackets as Δy is made to approach zero, we get, by a familiar argument,

$$\lim_{\Delta y\to 0}\left[\sum_y f(x_i,y)\,\Delta y\right]\Delta x = \left[\int_{Y_1(x_i)}^{Y_2(x_i)} f(x_i,y)\,dy\right]\Delta x$$

But this is just the volume of the lamina $HKLM$. We now form the sum of all such laminae, take the limit of the sum as Δx is made to approach zero, and obtain the required volume,

$$V = \int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\,dy\,dx$$

We see, therefore, that the integral (1) (Art. 146) may be formed as follows: Consider the volume $\Delta V = f(x,y)\,\Delta y\,\Delta x$ of an elementary rectangular column of cross section $\Delta y\,\Delta x$ and height $z = f(x,y)$. First, add the columns along a fixed line parallel to the y axis; that is, hold x and Δx fixed. The limit of this sum gives the volume of a lamina, or slice. Next, add all these slices, and take the limit of their sum. The result is the integral (1) which gives the required volume.

Clearly, the columns could first be added along a line AB where y and Δy are held fixed. Taking the limit of the sum as Δx is made to approach zero, the volume of a lamina, or slice, whose faces are parallel to the xz plane is

$$\Delta y \int_{X_1(y)}^{X_2(y)} f(x,y)\,dx$$

Here, $X_1(y)$ and $X_2(y)$ are the x coordinates of A and B. Now, add all these slices, and take the limit of their sum to obtain the required volume,

$$V = \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\,dx\,dy$$

It is particularly convenient in setting up integrals to regard an iterated integral from the point of view adopted in this section, namely, as the limit of a sum of elements added first in one direction, then in another.

Example 1. Recall the example of the preceding section. Cut the area of the circle $x^2 + y^2 = 4$ in the xy plane by lines parallel to the x and y axes, forming elements of area $\Delta y\,\Delta x$. Let MN (Fig. 198) be a typical rectangular column whose cross-sectional area is $\Delta y\,\Delta x$ and altitude $f(x,y) = \frac{1}{4}(16 - x^2 - y^2)$. This column is a typical element of volume, $\Delta V = \frac{1}{4}(16 - x^2 - y^2)\,\Delta y\,\Delta x$. The required volume

is, therefore, given by the integral,

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \tfrac{1}{4}(16 - x^2 - y^2)\, dy\, dx$$

To find the inside limits of integration, we note that we first add columns from B to F, with x held fixed to obtain a slice parallel to the yz plane. The y coordinates of B and F are, respectively, $-\sqrt{4 - x^2}$ and $\sqrt{4 - x^2}$. We then add the slices from $x = -2$ to $x = 2$. This integral has already been found to be equal to 14π.

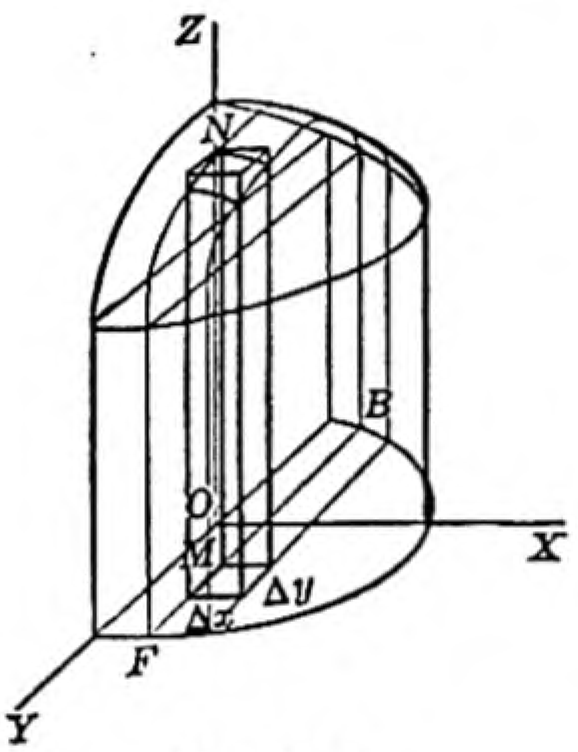


FIG. 198.

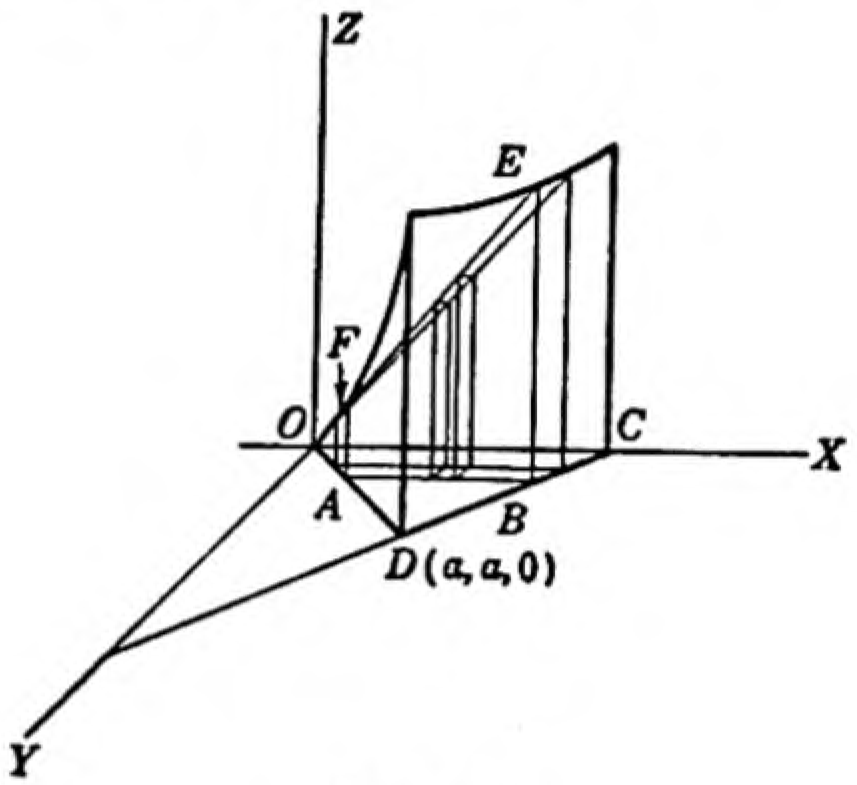


FIG. 199.

Example 2. Find the volume bounded by the surfaces $(a > 0)$ $az = y^2 + ax$, $y = x$, $x + y = 2a$, $y = 0$, $z = 0$. The volume is shown in Fig. 199. Cut the area of the triangle OCD by lines parallel to the x and y axes, forming elements of area $\Delta x\, \Delta y$. Consider the rectangular columns standing on these elements. A typical column is shown in the figure. Its volume is

$$z\, \Delta x\, \Delta y = \frac{1}{a}(y^2 + ax)\, \Delta x\, \Delta y$$

If we first hold y and Δy fixed, add all such columns along a line AB, and then take the limit of this sum, we shall have the volume of a slice of cross section $ABEF$ and thickness Δy. This volume is

$$\left[\int_{y}^{2a-y} \frac{1}{a}(y^2 + ax)\, dx\right] \Delta y$$

where the limits of integration are the x coordinates of points A and B, each expressed in terms of y. We next add all the slices from the point C to the point D and take the limit of the sum, obtaining for the entire volume

$$\frac{1}{a}\int_0^a \int_y^{2a-y} (y^2 + ax)\, dx\, dy = \frac{1}{a}\int_0^a [y^2x + \tfrac{1}{2}ax^2]_y^{2a-y}\, dy$$
$$= \frac{2}{a}\int_0^a (-y^3 + ay^2 - a^2y + a^3)\, dy = \tfrac{7}{6}a^3$$

Observe that it would be inconvenient to reverse the order of integration by first adding columns along a line parallel to the y axis and holding x fixed.

EXERCISES

Evaluate the following integrals (Ex. 1 to 14):

1. $\int_0^2 \int_0^4 (1 + x)\, dy\, dx$

2. $\int_{-1}^1 \int_0^2 (x + y)\, dy\, dx$

3. $\int_1^2 \int_0^y xy\, dx\, dy$

4. $\int_0^1 \int_{y^2/2}^{\sqrt{y}} x\, dx\, dy$

5. $\int_1^2 \int_1^x \frac{x^2}{y^2}\, dy\, dx$

6. $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

7. $\int_0^{\sqrt{\pi/2}} \int_0^{x^2} x \cos y\, dy\, dx$

8. $\int_0^{\pi/2} \int_0^{2a \cos\theta} r\, dr\, d\theta$

9. $\int_0^{\pi} \int_0^{a(1 + \cos\theta)} r^2 \sin\theta\, dr\, d\theta$

10. $\int_0^{\pi/4} \int_0^{a \sec\theta} r^3\, dr\, d\theta$

11. $\int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin^2\theta\, dr\, d\theta$

12. $\int_0^{\pi/2} \int_0^{\cos\theta} r^2(\cos\theta - r)\, dr\, d\theta$

13. $\int_0^{a\sqrt{3}} \int_0^{\sqrt{y^2+a^2}} \frac{y\, dx\, dy}{x^2 + y^2 + a^2}$

14. $\int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2 - y^2)^{3/2}\, dx\, dy$

Find the following volumes by use of iterated integrals (Ex. 15 to 32):

15. The volume bounded by the paraboloid $x^2 + y^2 = z$ and the plane $z = 4$

16. The volume in the first octant bounded by the surfaces $y^2 = 4 - x$, $y = 2x$, $x = 0$, $z = 0$

17. The volume in the first octant bounded by the surfaces $y^2 = 4x$, $2x + y = 4$, $z = y$, and inside the first cylinder

18. The volume in the first octant bounded by the planes $x + y + z = 9$,

$$2x + 3y = 18$$

and $x + 3y = 9$

19. The volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x + y$

20. The volume bounded by the cylinder $x^2 = 4ay$ and the planes $x + 2y = z$, $2y = x$, and $z = 0$

21. The volume bounded by the paraboloid $4x^2 + y^2 = 4z$ and the plane $z = 2$

22. The volume in the first octant bounded by the cylinder $y^2 = 4ax$ and the plane $x + z = a$

23. The volume in the first octant between the cylinders $y^2 = ax$ and $x^2 = ay$ and under the plane $x + y = z$

24. The volume above the xy plane and bounded by the cylinders $x^2 + y^2 = a^2$ and $x^2 = 4a^2 - az$

25. The volume in the first octant bounded by the cylinder $ay = a^2 - x^2$ and the plane $x + y = z$

26. The volume in the first octant bounded by the surface $x^2 + y = z$ and the planes $y = x$, $x = 1$

27. The volume cut from the paraboloid $x^2 + y^2 = z$ by the plane $z = y$

28. The volume bounded by the paraboloid $x^2 + y^2 = z$ and the planes

$$3x + 4y - 18 = 0$$

$2y = 3x$, $y = 0$, $z = 0$

29. The volume bounded by the cylinder $x^{1/2} + y^{1/2} = a^{1/2}$ and the planes $y + z = a$, $x = 0$, $y = 0$, $z = 0$

30. The volume inside the cylinder $x^2 + y^2 - 2ax = 0$, outside the paraboloid $x^2 + y^2 = az$, and above the xy plane

31. The volume of a sphere of radius a

32. The volume in the first octant bounded by the surface

$$x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$$

33. Show that the area under the curve $y = f(x)$ in the xy plane from $x = a$ to $x = b$ can be calculated by using the iterated integral

$$\int_a^b \int_0^{f(x)} dy\, dx$$

Using the results of Exercise 33, calculate the following areas by use of iterated integrals (Ex. 34 to 40):

34. Bounded by the curve $y^2 = x^3$ and the line $y = x$

35. Bounded by the curves $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$

36. The area of a circle of radius a

37. Bounded by $x^2 - y^2 = a^2$ and the line $x = 2a$

38. Bounded by $y^2 = 4x$ and $2x + y - 4 = 0$

39. Bounded by $x^{1/2} + y^{1/2} = a^{1/2}$ and $x + y = a$

40. Bounded by $x^{2/3} + y^{2/3} = a^{2/3}$ and $x + y = a$

148. Cylindrical Coordinates. The reader will recall from his study of analytic geometry a coordinate system known as *cylindrical coordinates*. Instead of specifying the rectangular coordinates (x,y,z) of a point in space, we replace x and y by the usual polar coordinates r and θ. In other words, to locate point P (Fig. 200), we first locate Q, the foot of the perpendicular from P to the xy plane, by giving its polar coordinates. The distance QP is then the usual z coordinate of P. The relations between the rectangular coordinates (x,y,z) and the cylindrical coordinates (r,θ,z) of point P are, clearly,

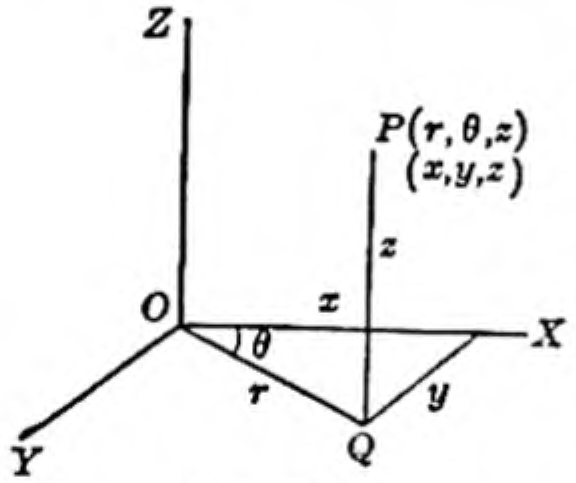


FIG. 200.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x} \qquad z = z$$

149. Volumes by Iterated Integration; Cylindrical Coordinates. Let S be a region in the xy plane with a boundary C whose equation is given in polar coordinates. Consider the volume V topped by the surface $z = \varphi(r,\theta)$ and enclosed by the cylinder, with elements parallel to the z axis, standing on the curve C. Suppose that C is cut by any line through O in not more than two points. Let the radius vectors OL and OM (Fig. 201) make angles α and β, respectively, with OX, and suppose these to be the smallest and largest values of θ for points on the curve C.

We shall modify the "column method" of Art. 147 to apply to this case. Divide the angle LOM into n equal angles each of measure $\Delta\theta$. Now draw m circular arcs with O as center, with radii differing by Δr, and cutting the area S bounded by the curve C. The planes through the z axis standing upon the sides of the n angles and the cylinders with axes on the z axis standing upon the m circular arcs will form vertical

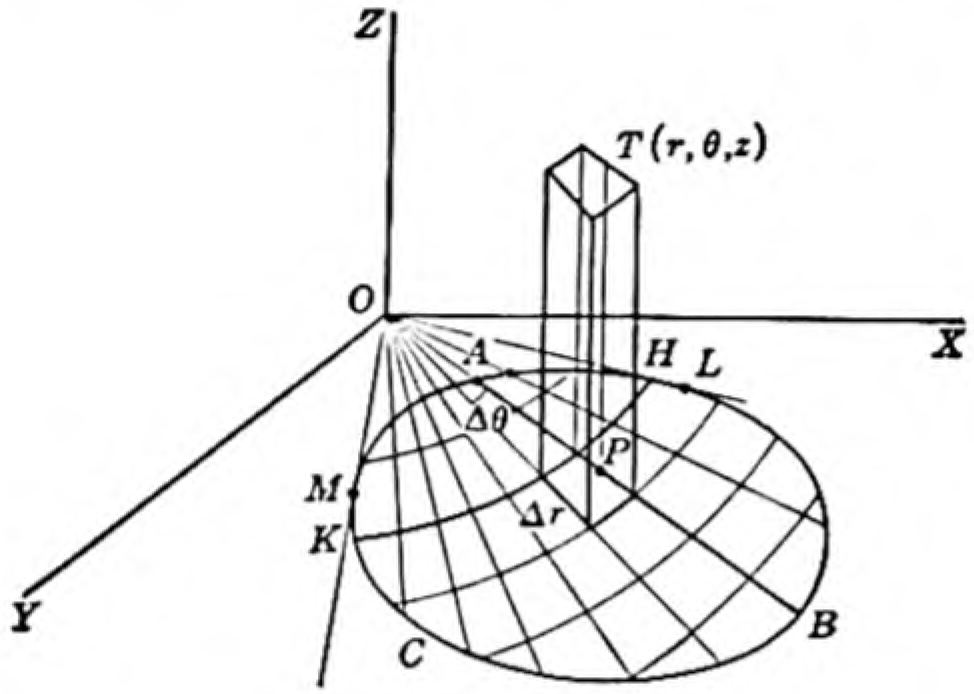


FIG. 201.

columns. A typical column is shown in Fig. 201. If T is a point of the surface $z = \varphi(r,\theta)$ with coordinates r, θ, z, then the volume of this typical column (with horizontal plane top) is its altitude $\varphi(r,\theta)$ multiplied by the area of the base.

The area of the base can be shown to be $r\,\Delta r\,\Delta\theta$ as follows: Let P be a point in the xy plane with polar coordinates r, θ (Fig. 202). Consider points A and B with the same vectorial angle but with radius vectors $r - \frac{1}{2}\Delta r$ and $r + \frac{1}{2}\Delta r$, respectively. Rotate the line OB about O through an angle $\Delta\theta$, forming an area $ABCD = \Delta S$ as indicated in the figure. This area can be calculated, for it is simply the area of the circular sector OBC minus the area of the sector OAD. Thus

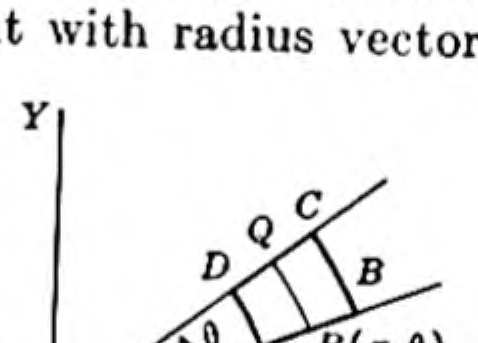


FIG. 202.

$$\Delta S = \tfrac{1}{2}(r + \tfrac{1}{2}\Delta r)^2\,\Delta\theta - \tfrac{1}{2}(r - \tfrac{1}{2}\Delta r)^2\,\Delta\theta = r\,\Delta r\,\Delta\theta$$

Now, multiplying the area $r\,\Delta r\,\Delta\theta$ by $\varphi(r,\theta)$, we obtain

$$\Delta V = \varphi(r,\theta) r\,\Delta r\,\Delta\theta \tag{3}$$

Holding $\theta = \theta_i$ and $\Delta\theta$ fixed and adding such columns along the fixed line AB, we get approximately the volume of a wedge-shaped element. If we take the limit of this sum as Δr is made to approach zero, we get exactly the volume of this wedge-shaped element, namely,

$$\left[\int_{R_1(\theta_i)}^{R_2(\theta_i)} \varphi(r,\theta_i) r\,dr\right]\Delta\theta$$

where $R_1(\theta_i)$ and $R_2(\theta_i)$ are the r coordinates of A and B, respectively. Now, adding all these wedge-shaped elements and taking the limit of the sum, we obtain

$$V = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} \varphi(r,\theta)r\,dr\,d\theta \tag{4}$$

Example 1. Consider the volume cut from the sphere $r^2 + z^2 = 4a^2$ by the cylinder $r = a$. By cutting it by planes through the z axis and by cylinders with axes on OZ as just described, elements of volume, namely, columns, are formed. The half of a typical element appearing above the xy plane is shown in Fig. 203. Its volume is $zr\,\Delta r\,\Delta\theta = \sqrt{4a^2 - r^2}\,r\,\Delta r\,\Delta\theta$ where r, θ, z are coordinates of a point T on the sphere $r^2 + z^2 = 4a^2$. Hence, the whole column has volume $2\sqrt{4a^2 - r^2}\,r\,\Delta r\,\Delta\theta$. Adding the columns along the line OC from $r = 0$ to $r = a$ gives the volume of a wedge-shaped slice

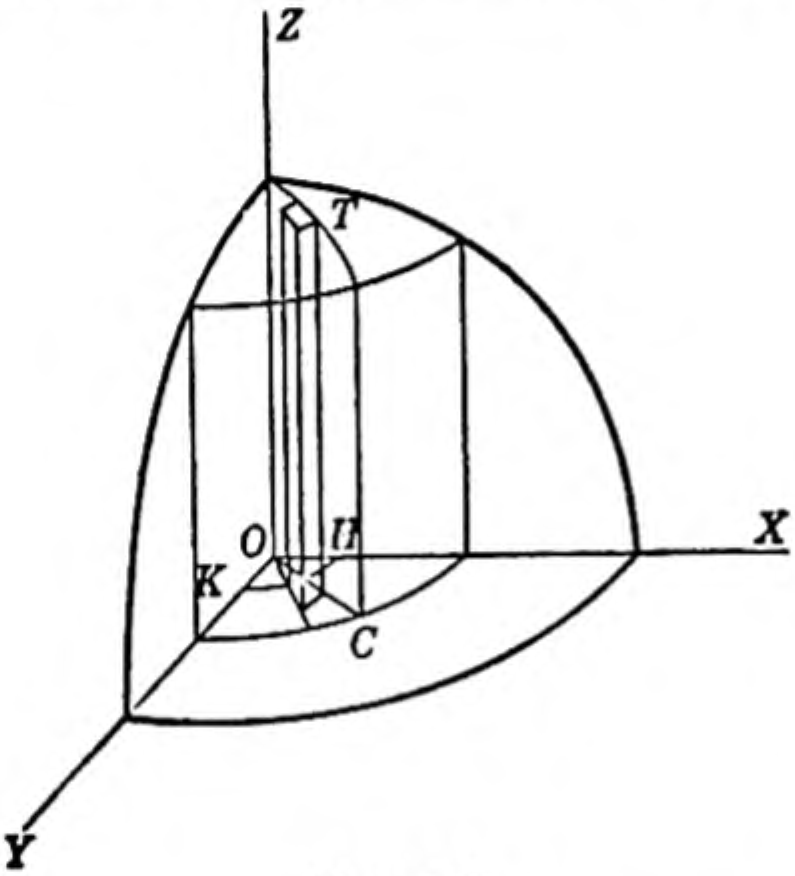


FIG. 203.

$$\left[2\int_0^a \sqrt{4a^2 - r^2}\,r\,dr\right]\Delta\theta$$

Adding all these slices from $\theta = 0$ to $\theta = 2\pi$ gives

$$V = 2\int_0^{2\pi}\int_0^a \sqrt{4a^2 - r^2}\,r\,dr\,d\theta = \tfrac{4}{3}\pi(8 - 3\sqrt{3})a^3$$

The order of integration could be inverted. That is, we could first add the columns along the arc HK (Fig. 201) for which $r = r_j$ and Δr are held fixed. The sum would be approximately the volume of the cylindrical shell of thickness Δr standing on the arc HK. The limit of this sum as $\Delta\theta$ is made to approach zero is the volume $\Delta_j V$ of such a shell,

$$\Delta_j V = \left[\int_{\theta_1(r_j)}^{\theta_2(r_j)} \varphi(r_j,\theta)r_j\,d\theta\right]\Delta r$$

where $\theta_1(r_j)$ and $\theta_2(r_j)$ are the vectorial angles of the points H and K. Now, adding these shells from the innermost point on curve C to the outermost (that is, from the smallest to the largest value of r) and taking the limit of the sum as Δr is made to approach zero, we obtain

$$V = \int_a^b \int_{\theta_1(r)}^{\theta_2(r)} \varphi(r,\theta)r\,d\theta\,dr$$

where a and b are the extreme values of r.

Example 2. Let us invert the order of integration in Example 1. The volume of the typical element has been seen to be $2\sqrt{4a^2 - r^2}\,r\,dr\,d\theta$. If we hold r fixed and integrate first for θ, we get a cylindrical shell whose volume is

$$\Delta V = \left[2\int_0^{2\pi} \sqrt{4a^2 - r^2}\,r\,d\theta\right]\Delta r$$

Now, adding all these shells and taking the limit of the sum, we obtain

$$\begin{aligned} V &= 2 \int_0^a \int_0^{2\pi} \sqrt{4a^2 - r^2}\, r\, d\theta\, dr \\ &= 2 \int_0^a 2\pi \sqrt{4a^2 - r^2}\, r\, dr \\ &= \tfrac{4}{3}\pi(8 - 3\sqrt{3})a^3 \end{aligned}$$

EXERCISES

Find the indicated volumes by use of iterated integrals and cylindrical coordinates (Ex. 1 to 18).

1. The volume above the xy plane, under the paraboloid, $r^2 = 2z$, and inside the cylinder $r = 2$

2. The volume of a right circular cone of altitude h and radius of base a

3. The volume of a sphere of radius a

4. The volume cut from a hemisphere of radius a by a cone with vertical angle 90 deg., with vertex at the center of the hemisphere

5. The volume under the cone $r = z$, inside the cylinder $r = 2a \cos \theta$, and above the xy plane. Transform these equations to rectangular coordinates, and set up the integral for the volume. Observe the advantage of cylindrical coordinates.

6. The volume cut from a sphere of radius $2a$ by a cylinder of radius a if the center of the sphere lies on the surface of the cylinder

7. The volume under the cone $z = r$, inside the cylinder $r = a(1 + \cos \theta)$, and above the xy plane

8. The volume inside the cylinder $r = a(1 + \cos \theta)$, above the plane $z = 0$, and under the plane $z = r \sin \theta$

9. The volume under the paraboloid $x^2 + y^2 = 4z$, inside the cylinder $x^2 + y^2 = 4x$, and above the xy plane (first transform the equations to cylindrical coordinates)

10. The volume between the cone $x^2 + y^2 = z^2$ and the paraboloid $x^2 + y^2 = z$ (first transform the equations to cylindrical coordinates)

11. The volume above the plane $z = 0$, inside a cylinder standing on one loop of the curve $r = a \cos 2\theta$, and under the spherical surface $r^2 + z^2 = a^2$

12. The volume above the plane $z = 0$, inside the cylinder $r = a$, and below the paraboloid $az = 4a^2 - r^2$

13. The volume under the plane $x + z = a$, inside the cylinder $x^2 + y^2 = a^2$, and above the plane $z = 0$

14. The volume inside the cylinder $x^2 + y^2 = 4a^2$ and outside the hyperboloid $x^2 + y^2 - z^2 = a^2$

15. Set up the integral for the volume in Exercise 1 without first transforming the equations to cylindrical coordinates. Note the advantage of cylindrical coordinates.

16. The volume under the paraboloid $r^2 = az$, inside the cylinder $r = 2a \sin \theta$, and above the plane $z = 0$

17. The volume under the paraboloid $r^2 = 4a(a - z)$, inside the cylinder

$$r = a(1 + \cos \theta)$$

and above the plane $z = 0$

18. The volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

19. Show that the area bounded by the polar curve $r = f(\theta)$ and the radius vectors $\theta = \alpha$ and $\theta = \beta$ can be found from the iterated integral $\int_\alpha^\beta \int_0^{f(\theta)} r\, dr\, d\theta$.

Using the results of Exercise 19, calculate the following areas (Ex. 20 to 22):

20. Inside the cardioid $r = a(1 + \sin\theta)$ and outside the circle $r = 2a \sin\theta$

21. Between the circles $r = 2a \cos\theta$ and $r = 2a \sin\theta$

22. Bounded by the spiral $r\theta = k$ and any two of its radius vectors r_1 and r_2

150. The Double Integral. In Chap. 14 (Art. 104), we introduced the idea of the limit of a sum of the type $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta_i x$, where $f(x)$ is a function of a single variable and where the limit is taken as the greatest interval $\Delta_i x$ is made to approach zero. We shall extend the notion of the limit of such a sum to the case of a function of two independent variables. Instead of subdividing a given *interval* from a to b on the x axis, we shall subdivide a given *region* in the xy plane; instead of dealing with values of a function $f(x)$ for points in an interval, we shall deal with values of a function $f(x,y)$ for points of this region.

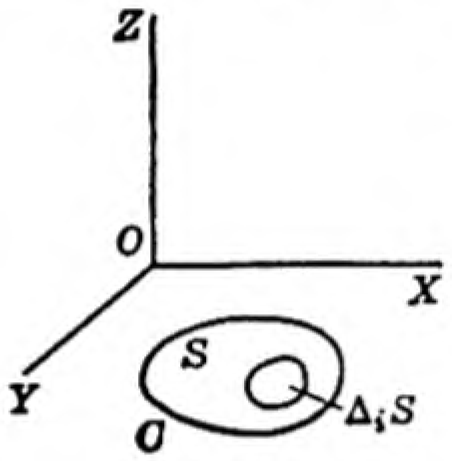


FIG. 204.

Let $z = f(x,y)$ be a function of two independent variables that is continuous throughout a finite region S of the xy plane. Let C be the curve bounding the region (Fig. 204). Divide the region into n subregions $\Delta_1 S, \Delta_2 S, \ldots, \Delta_n S$ of any desired shapes. Let $P_i(x_i,y_i)$ be any point in, or on the boundary of, the subregion $\Delta_i S$. Then $f(x_i,y_i)$ is the value of the function at this point. Form the sum

$$f(x_1,y_1)\,\Delta_1 S + f(x_2,y_2)\,\Delta_2 S + \cdots + f(x_n,y_n)\,\Delta_n S = \sum_{i=1}^{n} f(x_i,y_i)\,\Delta_i S \quad (5)$$

Now, let the greatest linear dimension of each of these subregions approach zero; consequently, let n increase indefinitely. The limit of the sum (5) is called the *double integral* of $f(x,y)$ over the region S, and we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\,\Delta_i S = \iint\limits_{(S)} f(x,y)\,dS$$

Note particularly that the shapes of the subregions $\Delta_i S$ play no part in the definition. Some may be circles, others rectangles, others of any arbitrary shape. All that is important is that the maximum linear dimension of each should approach zero. The proof of the existence of this limit is left for a more advanced course.

151. Geometrical Interpretation of the Double Integral. In Art. 105 the definite integral of a function of a single variable was interpreted as an area under a curve. We shall now see that a double integral may be

interpreted as a *volume under a surface.* Suppose $f(x,y)$ positive (or zero) throughout S; and consider the portion of space under this surface and enclosed by the cylinder with elements parallel to the z axis and standing upon the curve C. The ith term of the sum (5), namely, $f(x_i,y_i)\ \Delta_i S$ is the product of the area $\Delta_i S$ by the altitude $f(x_i,y_i)$ of a slender cylindrical column with horizontal plane top. We define this product to be the *measure of volume of the column.* Hence the sum (5) of the preceding section, namely,

$$\sum_{i=1}^{n} f(x_i,y_i)\ \Delta_i S$$

is the sum of measures of volume of all such columns (see Fig. 205). The limit of this sum, taken as the maximum linear dimension of each $\Delta_i S$ is made to approach zero, is defined to be the *measure of the volume* V bounded by the surface $z = f(x,y)$, the cylinder standing upon C, and the xy plane. That is

$$V = \iint_{(S)} f(x,y)\ dS$$

We have now assigned a meaning to the expression "volume bounded by curved surfaces."

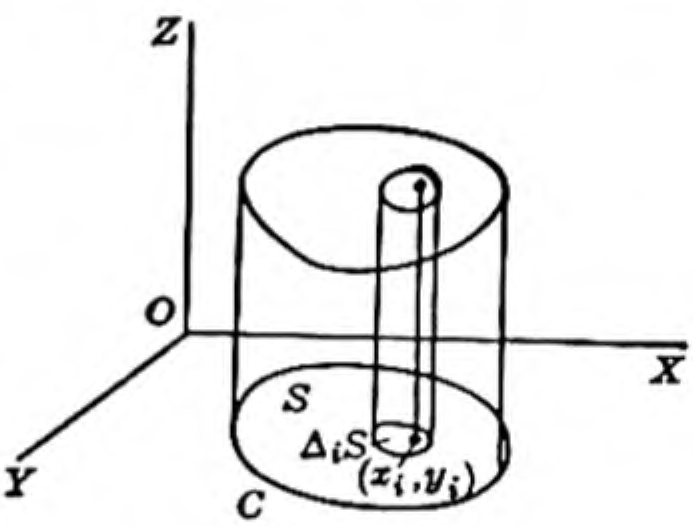


FIG. 205.

Though we have supposed $f(x,y)$ *positive* throughout S, this is an unnecessary restriction. If it is negative, the volume lies below the xy plane, and each term of the sum (5) is negative. Hence, the double integral, though numerically equal to the measure of volume, will be negative. If $f(x,y)$ is positive for some parts of S but negative for others, the double integral clearly gives the algebraic sum of the volumes above and below the xy plane.

152. The Fundamental Theorem. In Art. 105, we made use of a geometrical argument to show the connection between the limit of the sum $\sum_{i=1}^{n} f(x_i)\ \Delta_i x$ and a function whose derivative is $f(x)$, that is, between a definite integral and an antiderivative. We again make use of a geometrical argument to achieve an analogous result in the case of a function of two variables. Observe that the volume of measure V of the preceding section is simply the volume calculated by means of the iterated integrals of Art. 146 and 147. Consequently, the value of the *double integral*

$$\iint_{(S)} f(x,y)\ dS$$

can be found by calculating either of the *iterated integrals*

$$\int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\, dy\, dx \qquad \text{or} \qquad \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\, dx\, dy$$

This result may be stated as follows, as the **fundamental theorem of integral calculus for double integrals:**

Let $f(x,y)$ be a function of x and y, continuous throughout the region S of the xy plane. Subdivide S into n subregions of area $\Delta_1 S, \Delta_2 S, \ldots, \Delta_n S$, and let (x_i,y_i) be any point of the ith subregion. Form the sum

$$\sum_{i=1}^{n} f(x_i,y_i)\, \Delta_i S$$

Let the maximum linear dimension of each subregion approach zero; consequently, let n increase indefinitely. The sum will approach a limit, namely, the double integral

$$\iint_{(S)} f(x,y)\, dS$$

and the value of this limit is given by the iterated integral

$$\int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\, dy\, dx \qquad \text{or} \qquad \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\, dx\, dy$$

That is,

$$\iint_{(S)} f(x,y)\, dS = \int_a^b \int_{Y_1(x)}^{Y_2(x)} f(x,y)\, dy\, dx = \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\, dx\, dy$$

If $z = f(x,y)$ is transformed into cylindrical coordinates, the above double integral may be computed by either of the iterated integrals

$$\int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} \varphi(r,\theta) r\, dr\, d\theta = \int_a^b \int_{\theta_1(r)}^{\theta_2(r)} \varphi(r,\theta) r\, d\theta\, dr$$

Note that the argument is actually independent of the physical or geometrical meaning of $f(x,y)$. Any continuous function of two independent variables can be represented by a surface; consequently, if in any problem an arbitrarily close approximation to the required quantity can be found by adding terms of the type $f(x,y)\, \Delta S$ over a region in the xy plane, then that quantity is given exactly by the double integral $\iint_{(S)} f(x,y)\, dS$. This is called the *double integral of the function over the region S*. In other words, the problem is *formulated* in terms of a double integral which can then be *computed* by means of a convenient iterated integral.

EXERCISES

Find the values of the following double integrals:

1. The function x^2y^3 over the rectangle whose vertices are (0,0), (3,0), (3,2), (0,2)
2. The function x/y over the rectangle whose vertices are (0,1), (3,1), (3,2), (0,2)

3. The function xy^2 over the right-hand semicircle of $x^2 + y^2 = a^2$

4. The function xy over the area in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5. The function x^2y over the area bounded by the x axis and the first arch of the curve $y = \sin x$

6. The function $(x^2 + y^2)$ over the area bounded by the parabolas $y^2 = ax$ and $x^2 = ay$

7. The function r^2 over the area bounded by the circle $r = 2a \sin \theta$

8. The function $r^2 - ra$ over one loop of the curve $r = a \cos 2\theta$

9. The function r over the area inside the cardioid $r = a(1 + \sin \theta)$ and outside the circle $r = a$

10. The function r^3 over the area bounded by the spiral $r = a\theta$ and the radius vectors $\theta = 0$ and $\theta = \pi$

153. Volumes of Revolution, Polar Coordinates. Suppose that an area S bounded by curves whose equations are given in polar coordinates rotates about the polar axis. If this area is divided into elements of area $\Delta_1 S, \Delta_2 S, \ldots, \Delta_n S$, then when the area S rotates about the polar axis each such element will generate a ring-shaped element of volume whose cross-sectional area is $\Delta_i S$. The volume of the ring will be $2\pi R_i \Delta_i S$, where R_i is the distance from the polar axis to some point r_i, θ_i. The limit of the sum of all such elements is the volume of revolution. It can, therefore, be expressed by the double integral

$$\iint\limits_{(S)} 2\pi R\, dS$$

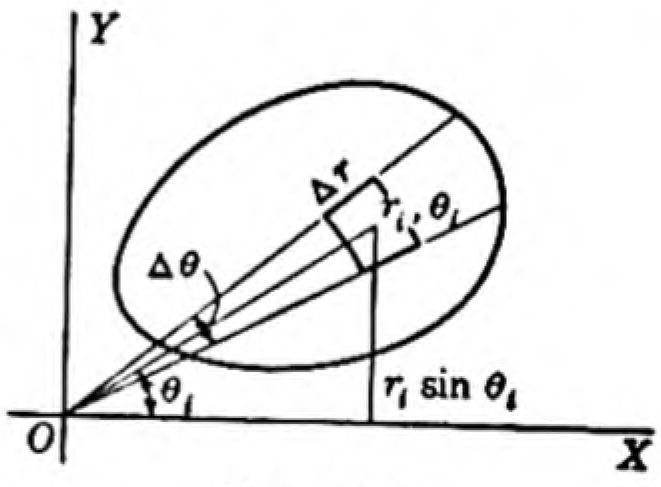


FIG. 206.

To evaluate this double integral, we set up an iterated integral as follows: Form elements of area by drawing lines through O making an angle $\Delta\theta$ with one another, and circles with centers at O and radii differing by a distance Δr. If r_i, θ_i is a suitable point of a typical element of area (Fig. 206), then, when the element rotates about the polar axis, we shall have $R_i = r_i \sin \theta_i$. The area of the cross section of the ring-shaped element formed by this rotation is $r_i\, \Delta r\, \Delta\theta$ (Art. 149), and its volume is

$$2\pi r_i \sin \theta_i\, r_i\, \Delta r\, \Delta\theta$$

Adding all such elements together and taking the limit of their sum, we have

$$V = \iint\limits_{(S)} 2\pi R\, dS = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} 2\pi r^2 \sin \theta\, dr\, d\theta$$

The limits $R_1(\theta)$ and $R_2(\theta)$ are the values of r for a fixed θ upon the bounding curves, and α and β are the smallest and largest values of θ

for points on the boundary. If more convenient, the order of integration may be reversed.

Clearly, if the area is rotated about the line $\theta = \pi/2$, $R_i = r_i \cos \theta_i$, and the iterated integral would be

$$V = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} 2\pi r^2 \cos \theta \, dr \, d\theta$$

If rotation is about some other line of the plane, a suitable adjustment must be made in expressing R.

Example. The upper half of the cardioid $r = a(1 + \cos \theta)$ rotates about the polar axis. Find the volume generated. Here, we may use the iterated integral

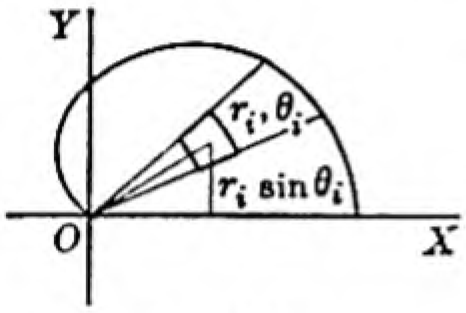


FIG. 207.

$$V = 2\pi \int_0^\pi \int_0^{a(1+\cos\theta)} r^2 \sin \theta \, dr \, d\theta$$

to calculate the volume. The limits of integration may be found from the following considerations: When the element of area $r_i \, \Delta r \, \Delta\theta$ rotates about the polar axis, it generates a ring-shaped element of volume. For a fixed θ, we may add all such elements by letting r increase from 0 to its value $r = a(1 + \cos \theta)$ on the cardioid (Fig. 207) along the line θ = constant. This forms an element of volume something like a conical shell whose thickness decreases as we approach the vertex. Now, add all such shells from $\theta = 0$ to $\theta = \pi$ to include the entire volume. This gives

$$\begin{aligned} V &= \frac{2\pi}{3} \int_0^\pi r^3 \sin \theta \Big]_0^{a(1+\cos\theta)} d\theta = \frac{2\pi}{3} a^3 \int_0^\pi (1 + \cos \theta)^3 \sin \theta \, d\theta \\ &= \frac{2\pi a^3}{3} \left[-\frac{(1 + \cos \theta)^4}{4} \right]_0^\pi = \frac{8}{3} \pi a^3 \end{aligned}$$

EXERCISES

Find by use of double integrals the volumes of revolution described in the following:

1. The volume of a sphere, using the equation $r = a$; the equation $r = 2a \sin \theta$

2. The area in the first and fourth quadrants bounded by the cardioid

$$r = a(1 + \cos \theta)$$

rotated about the line $\theta = \pi/2$

3. The volume of a torus, using the equations $r = a$ and $r = b \sec \theta$

4. The area bounded by the right-hand half of the curve, $r^2 = a^2 \sin \theta$, rotated about the line $\theta = \pi/2$

5. The area bounded by the part in the first quadrant of the curve $r = a \cos^2 \theta$, rotated about the polar axis

6. The area bounded by the upper half of the first loop of the curve $r = a \cos 2\theta$, rotated about the polar axis

7. The area in the first quadrant inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$, rotated about the polar axis

8. The area inside the circle $r = 2a \cos \theta$, rotated about the line $\theta = \pi/2$

9. The area of the upper half of the limaçon $r = 2 + \cos \theta$, rotated about the polar axis

10. The area bounded by $r = a \tan \theta$, $r = (a/\sqrt{2}) \sec \theta$, and $\theta = 0$, rotated about the line $\theta = \pi/2$

154. Area of a Curved Surface. We have seen how to calculate the area of a surface of revolution (Art. 115) and of a cylindrical surface (Art. 116) by means of simple integrals. We shall now see how to find the area of a curved surface of less restricted type and, incidentally, shall provide ourselves with a definition of such an "area." Let $z = f(x,y)$ be the equation of a surface, and let S' be a portion of the surface bounded by a curve C'. Let S and C be the projections of S' and C' upon the xy plane, as indicated in Fig. 208. Suppose, further, that $f(x,y) > 0$ throughout S. Divide the region S into n subregions $\Delta_1 S$, $\Delta_2 S$, . . . , $\Delta_n S$. Construct cylinders with elements parallel to the z axis upon these n subregions, as indicated in Fig. 208. These cylinders cut the region S' into n subregions $\Delta_1 S'$, $\Delta_2 S'$, . . . , $\Delta_n S'$. Let P_i be any point in the subregion $\Delta_i S'$, and let the angle between the normal to the surface at P_i and the direction of the z axis be γ_i. Observe that, since we shall suppose that the tangent plane is not vertical, this angle is not a right angle. Consider the tangent plane to the surface at P_i. The cylinder standing on $\Delta_i S$ cuts an area $\Delta_i T$ from this tangent plane of which $\Delta_i S$ is the projection upon the xy plane. The areas $\Delta_i T$ and $\Delta_i S$ are connected by the relation

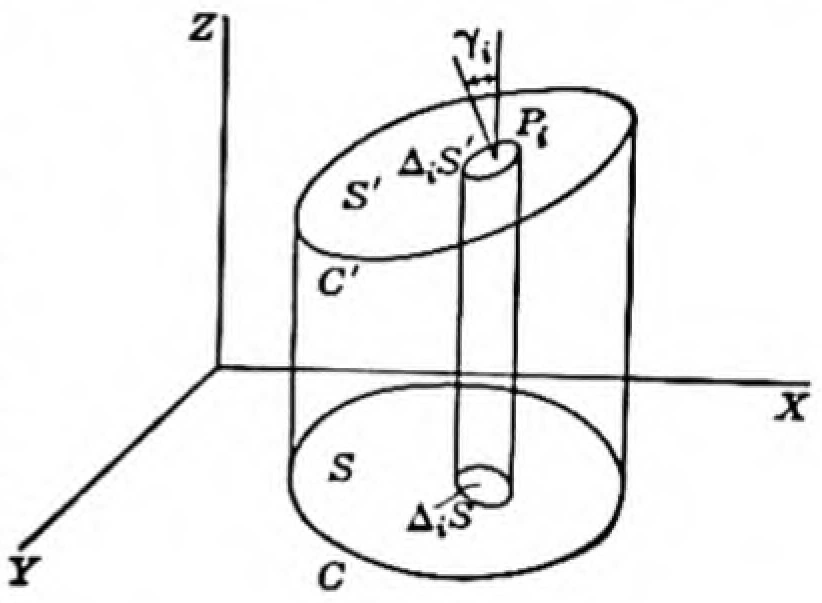


FIG. 208.

$$\Delta_i S = \Delta_i T \cos \gamma_i$$

Hence

$$\Delta_i T = \Delta_i S \sec \gamma_i$$

If we now add together all such areas $\Delta_i T$, we obtain a sum

$$\sum_{i=1}^{n} \Delta_i T = \sum_{i=1}^{n} \Delta_i S \sec \gamma_i$$

whose limit, taken as the maximum linear dimension of the greatest $\Delta_i S$ is made to approach zero, is the double integral

$$\iint\limits_{(S)} \sec \gamma \, dS$$

Since we have had, as yet, no definition of the "area" of a curved surface, this supplies us with such a **definition**: *The area is defined as the limit of the sum of elementary plane areas.*

In order to express the double integral in more convenient form, we shall find an expression for $\sec \gamma$. We recall (Art. 143) that, if $P(x,y,z)$

is a point on the surface $z = f(x,y)$, then the direction cosines of the normal at that point are proportional to $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ where the partial derivatives are evaluated at P. The student will remember from his study of analytic geometry that, if the direction cosines of a line are proportional to a, b, c, then the cosine of the angle between the line and the direction of the z axis is numerically equal to $|c|/\sqrt{a^2 + b^2 + c^2}$. Hence, in the present problem,

$$\cos \gamma = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

If the equation of the surface is given in the form

$$F(x,y,z) = 0$$

the direction cosines of the normal at P are proportional to $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$, these partial derivatives being evaluated at P.

Hence
$$\cos \gamma = \frac{\left|\frac{\partial F}{\partial z}\right|}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

Therefore

$$S' = \iint\limits_{(S)} \sec \gamma \, dS = \iint\limits_{(S)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dS$$

$$= \iint\limits_{(S)} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}{\left|\frac{\partial F}{\partial z}\right|}\, dS$$

To evaluate the double integral, we use an iterated integral, replacing dS by $dy\,dx$ and choosing limits of integration to include the region S of the xy plane. The method is best made clear by examples.

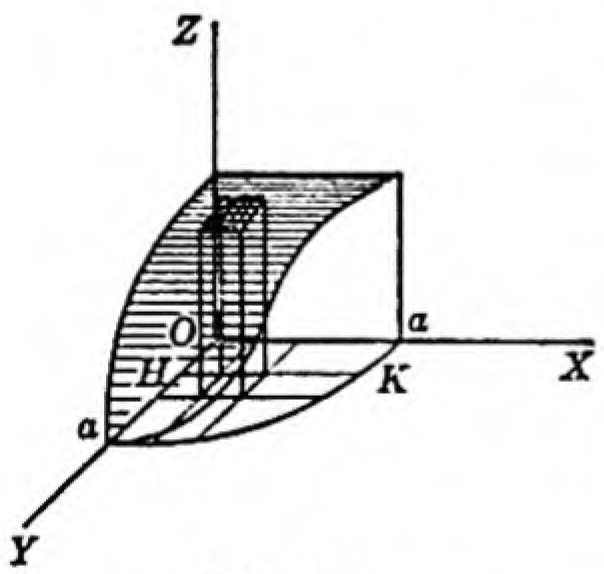


FIG. 209.

Example 1. Find the area A of the surface cut from the cylinder $y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = a^2$. The shaded area in Fig. 209 is that part of the required area which appears in the first octant. It is evident from symmetry that the total area is eight times the area shown. Since a portion of the surface

$$F(x,y,z) = y^2 + z^2 - a^2 = 0$$

forms the required area, we may set up the double integral as follows:

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 2y \qquad \frac{\partial F}{\partial z} = 2z$$

Therefore

$$\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2} = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2 + z^2} = 2a$$

and

$$\frac{1}{8}A = \iint\limits_{(S)} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}{\left|\frac{\partial F}{\partial z}\right|}\, dS = \iint\limits_{(S)} \frac{2a}{2z}\, dS = a \iint\limits_{(S)} \frac{1}{\sqrt{a^2 - y^2}}\, dS$$

where the region S is the first quadrant of the circle $x^2 + y^2 = a^2$ in which the xy plane cuts the cylinder $x^2 + y^2 = a^2$. We may calculate this double integral by use of an iterated integral. Form rectangular elements of area as indicated in the figure. We then have

$$\frac{1}{8}A = a \iint\limits_{(S)} \frac{1}{\sqrt{a^2 - y^2}}\, dS = \int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{1}{\sqrt{a^2 - y^2}}\, dx\, dy$$

The limits of integration for the inside integral are the x coordinates of points H and K (Fig. 209) for a fixed y. Since K is a point of the circle $x^2 + y^2 = a^2$, $z = 0$, we have $x = \sqrt{a^2 - y^2}$ for its abscissa. Integrating, we obtain

$$\frac{1}{8}A = a \int_0^a \frac{x}{\sqrt{a^2 - y^2}}\bigg]_0^{\sqrt{a^2-y^2}} dy = a \int_0^a dy = a^2$$

Therefore $A = 8a^2$. Note the convenience of this order of integration. If we reverse the order of integration, we get

$$\frac{1}{8}A = a \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2 - y^2}}\, dy\, dx = a \int_0^a \arcsin \frac{\sqrt{a^2 - x^2}}{a}\, dx$$
$$= a \int_0^a \arccos \frac{x}{a}\, dx$$

since $\arcsin \frac{\sqrt{a^2 - x^2}}{a} = \arccos \frac{x}{a}$. This can be integrated by parts to obtain

$$A = 8a^2$$

Example 2. Find the area cut from a sphere of radius $2a$ by a circular cylinder of radius a one of whose elements lies along a diameter of the sphere. We shall take the sphere with center at the origin and the cylinder with elements parallel to the z axis (Fig. 210). The equations of the sphere and cylinder are, respectively,

$$x^2 + y^2 + z^2 = 4a^2 \qquad \text{and} \qquad x^2 + y^2 - 2ax = 0$$

To find the area cut from the sphere, we have

$$F(x,y,z) = x^2 + y^2 + z^2 - 4a^2 = 0$$

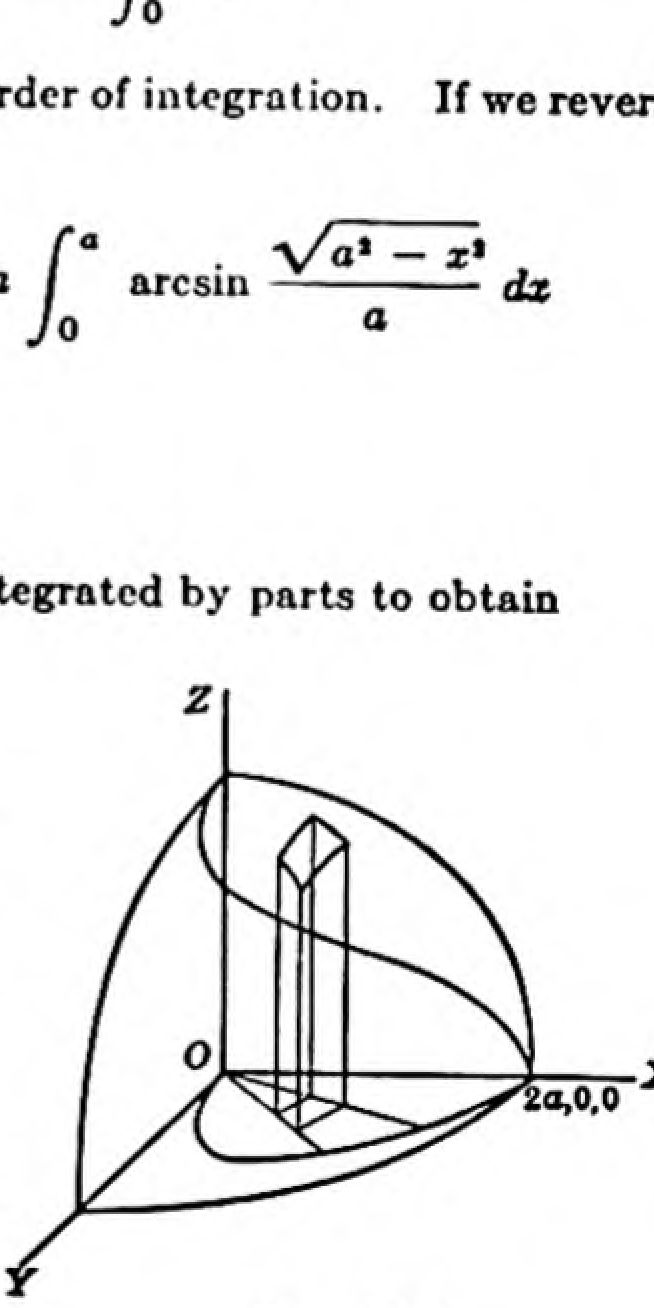


FIG. 210.

and
$$\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2} = \sqrt{4x^2 + 4y^2 + 4z^2} = 4a$$

Therefore
$$A = 4\iint_{(S)} \frac{4a}{2z}\,dS = 8a\iint_{(S)} \frac{1}{z}\,dS = 8a\iint_{(S)} \frac{dS}{\sqrt{4a^2 - (x^2 + y^2)}}$$

where we take for our region S the area in the first quadrant of the xy plane, bounded by the circle $x^2 + y^2 - 2ax = 0$. The form of the integrand suggests a change to cylindrical coordinates since the combination $x^2 + y^2 = r^2$ appears. Note that the equation of the sphere becomes $r^2 + z^2 = 4a^2$ and the equation of the cylinder becomes $r = 2a\cos\theta$. Since we now use $dS = r\,dr\,d\theta$, the iterated integral which gives the required area is

$$\begin{aligned} A &= \iint_{(S)} \frac{dS}{\sqrt{4a^2 - r^2}} = 8a\int_0^{\pi/2}\int_0^{2a\cos\theta} \frac{r\,dr\,d\theta}{\sqrt{4a^2 - r^2}} \\ &= -8a\int_0^{\pi/2} \sqrt{4a^2 - r^2}\,\Big]_0^{2a\cos\theta} d\theta \\ &= -8a\int_0^{\pi/2} (2a\sin\theta - 2a)\,d\theta = 8a^2(\pi - 2) \end{aligned}$$

EXERCISES

1. Find the area of the surface of a sphere of radius a using the method of Art. 154.
2. Find the area of the surface of a cone of altitude h and radius of base a.
3. Find the area cut from the plane $z = mx$ by the cylinder $x^2 + y^2 = a^2$.
4. Find the area cut from the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ by the coordinate planes.
5. Find the area cut from the surface of the upper half of the cone $x^2 + y^2 = z^2$ by the cylinder $x^2 + y^2 - 2ax = 0$.
6. Find the area cut from the surface of the upper half of the cone $x^2 + y^2 = z^2$ by the cylinder standing on the cardioid $r = a(1 + \cos\theta)$.
7. Find the area above the plane $z = 0$, cut from the surface of the cone
$$x^2 + y^2 = z^2$$
by the cylinder standing on one loop of the curve $r = a\cos 2\theta$.
8. Find the area cut from the surface of the upper hemisphere $x^2 + y^2 + z^2 = a^2$ by the cylinder standing on one loop of the curve $r = a\cos 2\theta$.
9. Find the area cut from the surface of the upper hemisphere $x^2 + y^2 + z^2 = a^2$ by the cylinder standing on one loop of the curve $r = a\cos n\theta$.
10. Find the area of that portion of the surface of the sphere $x^2 + y^2 + z^2 = 2ax$ lying within the paraboloid $y^2 + z^2 = bx$.
11. Find the area above the plane $z = 0$ of the surface cut from the cylinder $y^2 + z^2 = a^2$ by the planes $x = a/2$, $y = x$, $y = 0$.
12. Find the area of that part of the surface of the paraboloid $y^2 + z^2 = 4ax$ which is inside the parabolic cylinder $y^2 = ax$ and cut off by the plane $x = 8a$.
13. Modify the argument of Art. 154 to cover the case in which the portion S' of the surface is projected upon the xz plane rather than upon the xy plane. Similarly for S' projected upon the yz plane.

14. Find the area cut from the surface of a cylinder of radius a by a sphere of radius $2a$ whose center is on the surface of the cylinder (compare Example 2, Art. 154).

15. Find the area above the plane $z = 0$ cut from the cylinder $x^2 + y^2 = a^2$ by the cylinder $x^2 + az = a^2$.

155. The Triple Integral. Consider a region V of the space of three dimensions, and suppose that $f(x,y,z)$ is a function which is continuous throughout this region. Divide V into n small parts $\Delta_1 V, \Delta_2 V, \ldots, \Delta_n V$ in any desired manner. If x_i, y_i, z_i is any point in or upon the boundary of the ith subregion $\Delta_i V$, then $f(x_i,y_i,z_i)$ is the value of the function at this point, and we may form the sum

$$\sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta_i V$$

It can be shown, although the proof will not be given here, that when the maximum dimension of the subregions is made to approach zero, and therefore when n increases indefinitely, this sum approaches a limit. This limit is called the *triple integral* of $f(x,y,z)$ throughout the region V, and we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta_i V = \iiint\limits_{(V)} f(x,y,z)\,dV$$

The meaning of such an integral may be clarified by an example. Suppose a nonhomogeneous mass occupies the volume V, and suppose we know the density at every point of V. That is, the density at any point is a function $f(x,y,z)$ of the coordinates of that point. Now, imagine the volume V divided into n small parts $\Delta_1 V, \Delta_2 V, \ldots, \Delta_n V$, and let the element of mass contained in the ith element of volume be $\Delta_i M$. Now, $\Delta_i M = f(x_i,y_i,z_i)\,\Delta_i V$. The total mass is, therefore,

$$M = \sum_{i=1}^{n} \Delta_i M = \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta_i V$$

To find M, we take the limit of this sum,

$$M = \int_{(V)} dM = \iiint\limits_{(V)} f(x,y,z)\,dV$$

In other words, to find the mass M, we take the triple integral of the density function throughout the volume V.

156. Iterated Integral, Rectangular Coordinates. In Art. 152, we were able to show that a double integral can be evaluated by use of an iterated integral. The method depended upon showing that both could be represented by the same volume. In order to use this method to show that a triple integral can be evaluated by use of an iterated integral, we

should require a space of four dimensions. We shall, therefore, modify our method of attack and proceed as follows: Let V be cut by a system of planes parallel to the yz plane and at a distance Δx apart. Cut V, similarly, by a system of planes parallel to the xz plane at a distance Δy apart and by a system of planes parallel to the xy plane at a distance Δz apart. These systems of planes form rectangular box-shaped elements of volume each of which has volume $\Delta V = \Delta z\,\Delta y\,\Delta x$. All these elements together approximately fill the volume V (Fig. 211). If $f(x,y,z)$ is the function whose triple integral is to be found throughout V, then we form the sum

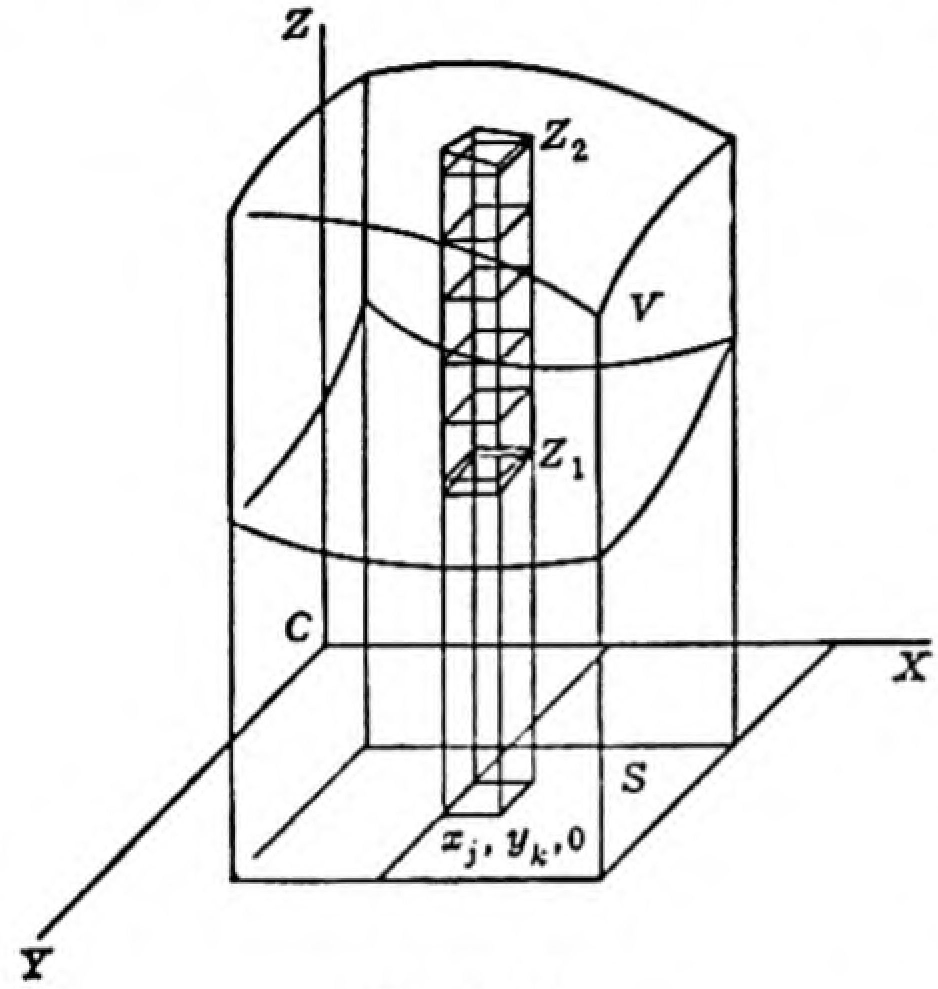


FIG. 211.

$$\sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta z\,\Delta y\,\Delta x \qquad (6)$$

the summation being taken to cover all the elements of volume included in V. The point x_i, y_i, z_i is any point of the ith element of volume. Let us select from this sum certain terms, namely, those which include elements of volume situated in a column parallel to the z axis, as indicated in the figure. This involves holding x, y, Δx, Δy fixed. Suppose we take $x = x_j$, $y = y_k$ and add the corresponding terms in the sum (6):

$$\left[\sum_{l=1}^{m} f(x_j,y_k,z_l)\,\Delta z\right]\Delta y\,\Delta x$$

if there are m elements in this column. Now, let Δz approach zero, and let m increase indefinitely, at the same time keeping x, y, Δx, Δy *fixed*. The function $f(x_j,y_k,z_l)$ is now a function of z *alone*, say $\varphi(z)$, and the sum $\sum_{l=1}^{m} \varphi(z_l)\,\Delta z$ approaches a limit, namely,

$$\int_{Z_1}^{Z_2} \varphi(z)\,dz = \int_{Z_1(x_j,y_k)}^{Z_2(x_j,y_k)} f(x_j,y_k,z)\,dz$$

where $Z_1(x_j,y_k)$ and $Z_2(x_j,y_k)$ are coordinates of the points in which the line $x = x_j$, $y = y_k$ cuts the bounding surfaces of V. For simplicity, we shall assume that any line parallel to a coordinate axis cuts the boundary of V in only two points. Hence

$$\left[\int_{Z_1(x_j,y_k)}^{Z_2(x_j,y_k)} f(x_j,y_k,z)\,dz\right]\Delta y\,\Delta x = \Phi(x_j,y_k)\,\Delta y\,\Delta x$$

is the integral of $f(x,y,z)$ over the *column* Z_1Z_2 shown in Fig. 211. If we now hold $x = x_j$ and Δx fixed and add all such columns together along the plane $x = x_j$, we shall obtain the sum

$$\left[\sum_{k=1}^{r} \Phi(x_j,y_k)\,\Delta y\right]\Delta x$$

if there are r columns in the slice cut from V by the planes $x = x_j$ and $x = x_j + \Delta x$. If we now let Δy approach zero and r increase indefinitely, the sum in square brackets approaches a limit, namely,

$$\int_{Y_1(x_j)}^{Y_2(x_j)} \Phi(x_j,y)\,dy$$

Hence

$$\left[\int_{Y_1(x_j)}^{Y_2(x_j)} \Phi(x_j,y)\,dy\right]\Delta x$$

[where $Y_1(x_j)$ and $Y_2(x_j)$ are the extreme values of y in the curve of intersection made by the plane $x = x_j$ with the boundary of V] is the value of the integral of $f(x,y,z)$ throughout the *slice* suggested in Fig. 211. Recalling the meaning of $\Phi(x_j,y)$, we may write

$$\left\{\int_{Y_1(x_j)}^{Y_2(x_j)}\left[\int_{Z_1(x_j,y)}^{Z_2(x_j,y)} f(x_j,y,z)\,dz\right]dy\right\}\Delta x = \psi(x_j)\,\Delta x$$

where y is held fixed throughout the first integration. Observe that the expression in braces is a function of x_j only. We now form the sum $\sum_{j=1}^{s} \psi(x_j)\,\Delta x$, if there are s slices cut from V by the planes parallel to the yz plane. If we let Δx approach zero and s increase indefinitely, the limit of this sum is $\int_a^b \psi(x)\,dx$ where a and b are the x coordinates of the extreme left- and right-hand points of V. Recalling the meaning of $\Psi(x)$, we have

$$\int_a^b\left\{\int_{Y_1(x)}^{Y_2(x)}\left[\int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\,dz\right]dy\right\}dx$$

This is equal to the triple integral over the volume V, and we have the **fundamental theorem of integral calculus for triple integrals:**

$$\iiint\limits_{(V)} f(x,y,z)\,dV = \int_a^b \int_{Y_1(x)}^{Y_2(x)} \int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\,dz\,dy\,dx \tag{7}$$

Here, x and y are held fixed during the first integration, and x is held fixed during the second.

Another notation frequently used for this iterated integral is

$$\int_a^b dx \int_{Y_1(x)}^{Y_2(x)} dy \int_{Z_1(x,y)}^{Z_2(x,y)} f(x,y,z)\,dz$$

The order of integration may be changed if more convenient, modifications in the limits of integration being properly made.

Example. Find the mass of a tetrahedron with three mutually perpendicular faces if the density at any point is proportional to the square of its distance from one of the three perpendicular edges. Let the three mutually perpendicular faces lie in the coordinate planes, and let the edges formed by these planes be of lengths a, b, c. The equation of the fourth plane surface ABC of this tetrahedron is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

If (x,y,z) is any point within or upon the boundaries of the tetrahedron, the density at that point is proportional to the square of its distance from, say the z axis. Thus, $\delta = k(x^2 + y^2)$. If the mass is divided into elements of mass $\Delta_i M$ each of volume $\Delta_i V$, then the mass of an element of mass is $k(x_i^2 + y_i^2)\,\Delta_i V$, and the total mass is given by the triple integral

$$M = k \iiint\limits_{(V)} (x^2 + y^2)\, dV$$

To evaluate this triple integral, we employ an iterated integral. Form rectangular elements of volume $\Delta z\, \Delta y\, \Delta x$ as shown in Fig. 212. The mass of a typical element is $k(x_i^2 + y_i^2)\,\Delta z\, \Delta y\, \Delta x$. The total mass is

$$M = k \int_0^a \int_0^{\frac{b}{a}(a-x)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} (x^2 + y^2)\, dz\, dy\, dx$$

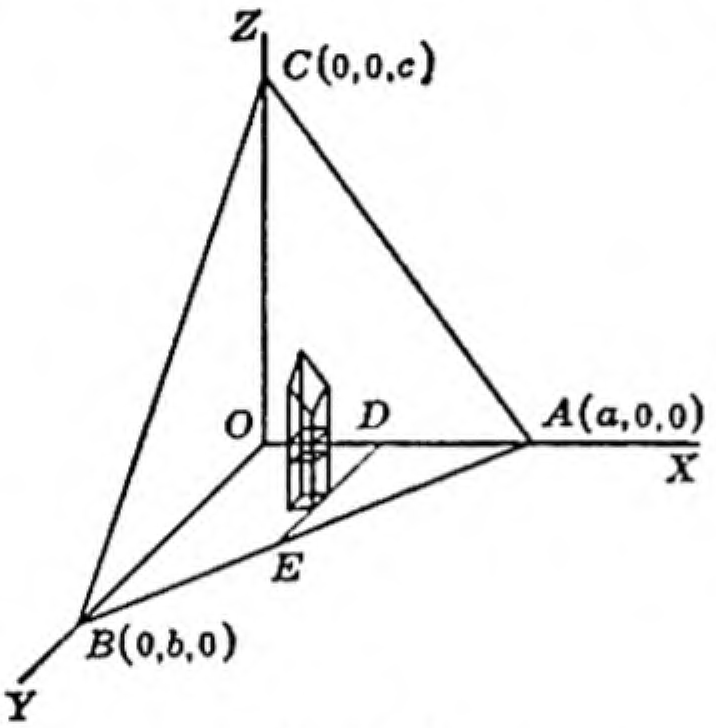


FIG. 212.

The limits of integration are obtained as follows: In the first integration for z, x and y are held fixed, and we pile the small boxlike elements up from the xy plane to the plane ABC; hence, z varies from zero to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$. This gives the mass of a rectangular column, as indicated in the figure. These columns are now added from back to front along a line DE parallel to the y axis (that is, x is kept fixed). On this line, y varies from zero to its value on the line AB. The equations of AB are $\frac{x}{a} + \frac{y}{b} = 1$, $z = 0$. Hence, y varies from zero to $\frac{b}{a}(a - x)$. This gives the mass of a slice parallel to the yz plane. These slices are now added from left to right, x varying from zero to a to include the entire volume. As indicated in this example, limits should be found by considering carefully the geometrical and physical meaning of the successive integrations. Proceeding to the evaluation of this integral, we have

$$\begin{aligned}
M &= k \int_0^a \int_0^{\frac{b}{a}(a-x)} (x^2 + y^2) z \Big]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\, dx \\
&= kc \int_0^a \int_0^{\frac{b}{a}(a-x)} (x^2 + y^2)\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx \\
&= kc \int_0^a \int_0^{\frac{b}{a}(a-x)} \left[x^2\left(1 - \frac{x}{a}\right) - \frac{x^2}{b} y + \left(1 - \frac{x}{a}\right) y^2 - \frac{y^3}{b}\right] dy\, dx
\end{aligned}$$

$$= kc \int_0^a \left[x^2 \left(1 - \frac{x}{a}\right) y - \frac{x^2}{b} \cdot \frac{y^2}{2} + \left(1 - \frac{x}{a}\right) \frac{y^3}{3} - \frac{y^4}{4b} \right]_0^{\frac{b}{a}(a-x)} dx$$
$$= kc \int_0^a \left\{ \frac{(a-x)^2}{a^2} \left[\frac{b}{2} x^2 + \frac{b^3}{12a^2} (a-x)^2 \right] \right\} dx$$
$$= \frac{k}{60} abc(a^2 + b^2)$$

157. Iterated Integral, Cylindrical Coordinates. A triple integral may frequently be evaluated readily by use of cylindrical coordinates. If the region V is cut by planes through the z axis that make equal angles $\Delta\theta$ with one another, by circular cylinders whose axes are on the z axis and whose radii differ by Δr, and by planes parallel to the xy plane ($r\theta$ plane) at a distance Δz apart, an element of volume (Fig. 213) is formed whose cross-sectional area is $r_i \Delta r \Delta\theta$ and whose altitude is Δz. Its volume is, therefore, $r_i \Delta r \Delta\theta \Delta z$. If $f(r,\theta,z)$ is some function of r, θ, z, form the sum

$$\sum_{i=1}^{n} f(r_i,\theta_i,z_i) r_i \Delta z \Delta r \Delta\theta$$

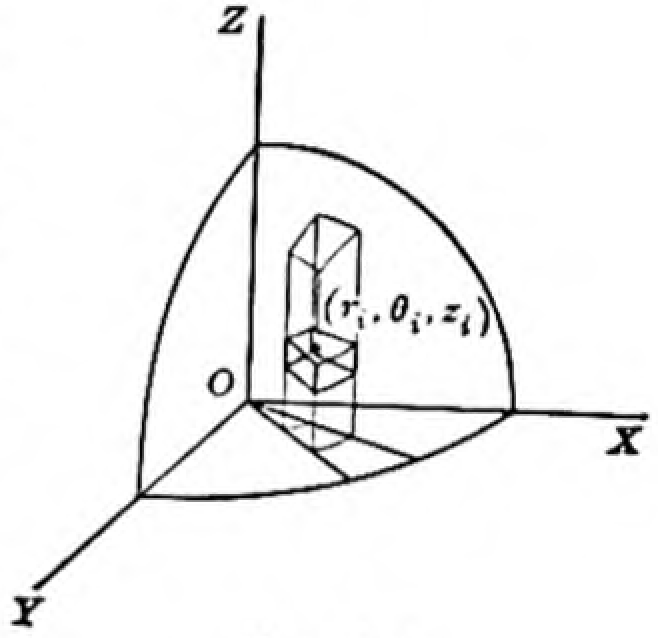


FIG. 213.

The limit of this sum is expressed by the iterated integral

$$\int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} \int_{Z_1(r,\theta)}^{Z_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta$$

The first integration adds the boxlike elements to form a column. The next integration adds the columns to form a slice. The third integration adds the slices to include the whole volume.

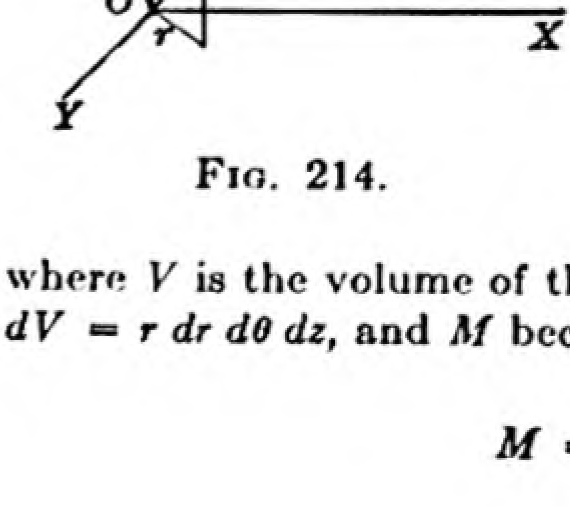


FIG. 214.

Example. Find the mass of a right circular cone of altitude 4 and radius of base 2 if the density at any point is proportional to the square of its distance from the vertex. We take the vertex at the origin and axis on the z axis (Fig. 214). The equation of the cone is $z = 2r$. If $P(r,\theta,z)$ is some point within the element of mass, then the distance of this point from the vertex of the cone is $OP = \sqrt{r^2 + z^2}$. Hence

$$M = k \iiint_{(V)} (r^2 + z^2) \, dV$$

where V is the volume of the cone. To express this as an iterated integral, we have $dV = r \, dr \, d\theta \, dz$, and M becomes

$$M = k \int_0^{2\pi} \int_0^2 \int_{2r}^4 (r^2 + z^2) r \, dz \, dr \, d\theta$$

Limits are chosen as follows: The first integration keeps r and θ fixed and adds the masses of the boxlike elements along a vertical line from the surface of the cone to the plane $z = 4$. Hence z varies from $2r$ to 4. This gives the mass of a column. The next integration keeps θ fixed and adds the columns from the z axis out to the circle AB. This gives the mass of a slice. All such slices are now added from $\theta = 0$ to $\theta = 2\pi$. Calculation of the integral gives $M = \frac{288}{5}\pi k$, as the student may readily verify.

The order of integration may be changed if desired, suitable modifications being made in the limits of integration.

EXERCISES

Evaluate the integrals (Ex. 1 to 9):

1. $\displaystyle\int_0^1 \int_0^x \int_0^{x+y} (2x + y - 1)\, dz\, dy\, dx$

2. $\displaystyle\int_0^2 \int_{-1}^{x^2} \int_1^y xyz\, dz\, dy\, dx$

3. $\displaystyle\int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} \frac{(x^2 + y^2)^{3/2}\, dz\, dy\, dx}{x^2 + y^2 + z^2}$

4. $\displaystyle\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z}\, dz\, dy\, dx$

5. $\displaystyle\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{r\cos\theta} r^2\, dz\, dr\, d\theta$

6. $\displaystyle\int_0^{\pi} \int_0^{\sin\theta} \int_0^{r\sin\theta} r\cos^2\theta\, dz\, dr\, d\theta$

7. $\displaystyle\int_0^1 \int_0^{y^2} \int_0^{x+y} x\, dz\, dx\, dy$

8. $\displaystyle\int_1^2 \int_y^{y^2} \int_0^{\ln(y+z)} e^x\, dx\, dz\, dy$

9. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{a-z} z\, dx\, dy\, dz$

10. Find the mass of a cube of edge a if the density at any point is proportional to the square of the distance from one edge.

11. Find the mass of a cube of edge a if the density at any point is proportional to the square of the distance from one corner.

12. Find the mass of a right circular cone of altitude h and radius of base a if the density varies as the distance from the axis (use cylindrical coordinates).

13. Set up the integral for Exercise 12, using rectangular coordinates, and compare with the work in cylindrical coordinates.

14. Find the mass of a right circular cone of altitude h and radius of base a if the density varies as the distance from the base.

15. Find the mass of the cone of Exercise 14 if the density at any point is proportional to the square of the distance from the vertex.

16. Find the mass of a sphere of radius a if the density at any point is proportional to the distance from a fixed diametral plane.

17. Find the mass of a sphere of radius a if the density at any point is proportional to the distance from a fixed diameter. Set up with both cylindrical and rectangular coordinates.

18. Find the mass of a sphere of radius a if the density at any point is proportional to the square of the distance from the center.

19. Show that a volume can be calculated from either of the iterated integrals $\int\int\int dz\,dy\,dx$ or $\int\int\int r\,dz\,dr\,d\theta$, provided that limits are chosen to extend the integration over the entire volume.

Using the results of Exercise 19, find the following (Ex. 20 and 21):

20. The volume bounded by the paraboloid $x^2 + y^2 = z$ and the plane $z = 4$ (Exercise 15, page 397)

21. The volume under the cone $z = r$, inside the cylinder $r = a(1 + \cos\theta)$, and above the plane $z = 0$ (Exercise 7, page 401)

158. Iterated Integral, Spherical Coordinates. The reader will recall from analytic geometry that a point may be located by spherical coordinates as indicated in Fig. 215. The distance of the point from the origin is $\rho = OP$, its *radius vector*. The plane containing OZ and OP cuts the xy plane in a line OQ, making an angle θ with the positive half of the x axis, while the line OP makes an angle φ with the positive half of the z axis. The numbers ρ, θ, φ are called the *spherical coordinates* of P. If P is regarded as a point on the surface of a sphere of radius ρ, then θ is the *longitude* of the point, and φ is its *colatitude*.* In the figure, it is clear that, if P has rectangular coordinates x, y, z, then these are related to the spherical and cylindrical coordinates in the following way:

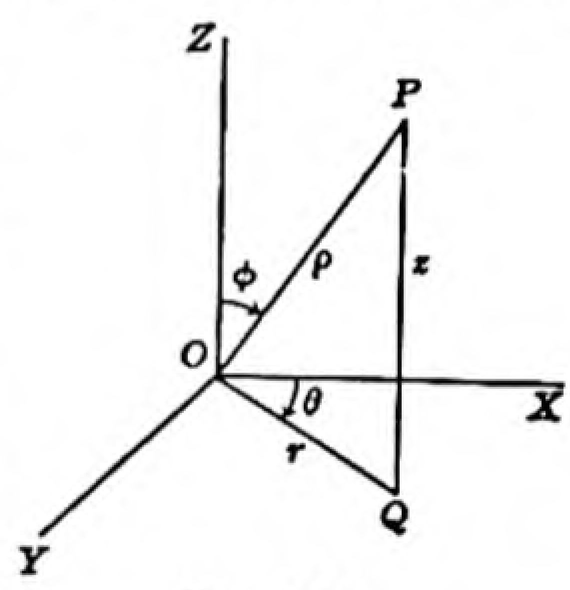


FIG. 215.

$$
\begin{aligned}
r &= \rho \sin\varphi \\
x &= r\cos\theta = \rho\sin\varphi\cos\theta \\
y &= r\sin\theta = \rho\sin\varphi\sin\theta \\
z &= \rho\cos\varphi
\end{aligned}
$$

If we wish to evaluate a triple integral over a volume V by use of an iterated integral in spherical coordinates, we proceed as follows: First, we consider a point $P_i(\rho_i,\theta_i,\varphi_i)$ and construct a set of spheres with centers at O and with radii differing by length $\Delta\rho$. One such sphere of radius ρ_i passes through P_i and is indicated in Fig. 216. Next, we construct a set of planes through OZ, making equal angles $\Delta\theta$ with one another. That plane passing through P_i has for its equation $\theta = \theta_i$. Lastly, we construct a set of cones with vertices at O and axes on OZ and with the semivertical angles of successive cones differing by $\Delta\varphi$. The cone passing through P_i has for its equation $\varphi = \varphi_i$. The radius of the circle cut by

* It is not uncommon to interchange the designations for the angles θ and φ.

this cone from the sphere $\rho = \rho_i$ is $QP_i = \rho_i \sin \varphi_i$ (Fig. 216). An element of volume is shown in the figure; it is bounded by portions of two consecutive spheres, two consecutive planes, and two consecutive cones. Three edges are, evidently, of lengths as indicated: a straight-line segment of length $\Delta\rho$ formed by the intersection of a plane and a cone, a circular arc of length $\rho_i\,\Delta\varphi$ formed by a plane and a sphere, and a circular arc of length $QP_i\,\Delta\theta = \rho_i \sin \varphi_i\,\Delta\theta$ formed by a cone and a sphere. This element of volume $\Delta_i V$ can be shown to differ *very little* from a rectangular

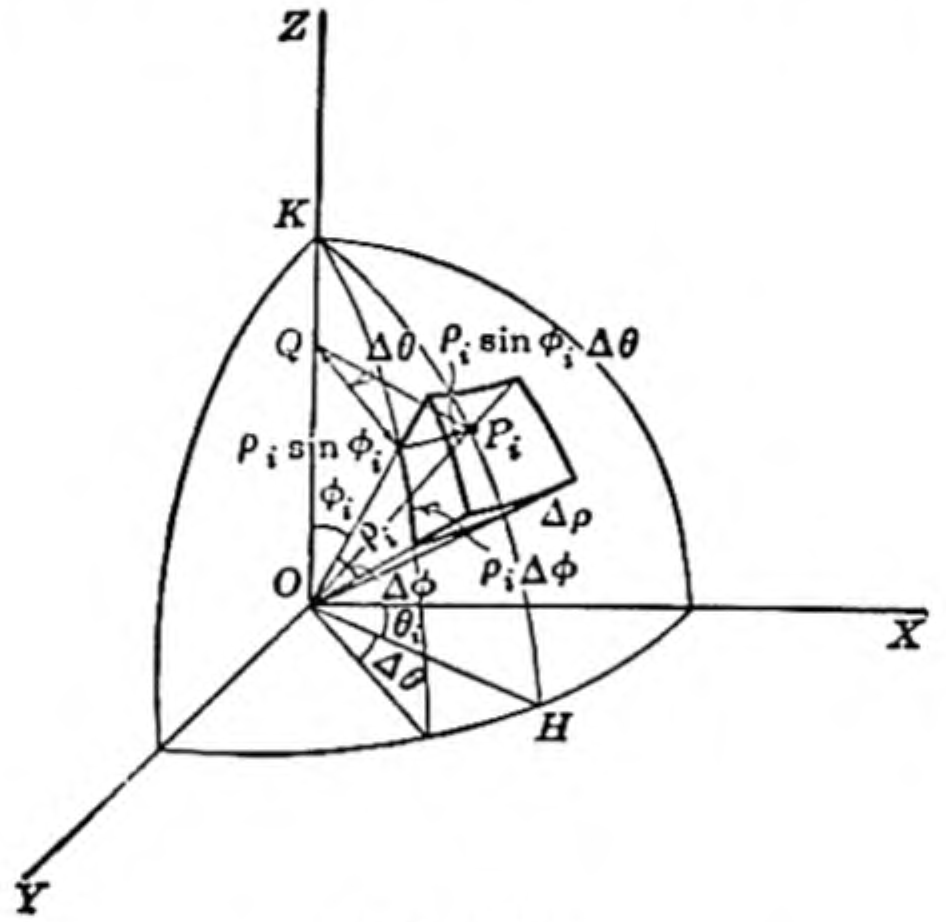


FIG. 216.

parallelepiped with these three lengths for edges. Furthermore, it can be shown that, in taking the limit of the sum of such elements of volume as the maximum linear dimension of each element approaches zero, these small differences have a sum whose limit is zero. Although we shall not establish these facts here, we shall use for the element of volume

$$\begin{aligned}\Delta_i V &= \Delta\rho \cdot \rho_i\,\Delta\varphi \cdot \rho_i \sin \varphi_i\,\Delta\theta \\ &= \rho_i^2 \sin \varphi_i\,\Delta\rho\,\Delta\varphi\,\Delta\theta\end{aligned}$$

Now, if $F(\rho,\theta,\varphi)$ is a continuous function throughout the volume V, we evidently have

$$\iiint\limits_{(V)} F(\rho,\theta,\varphi)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} F(\rho_i,\theta_i,\varphi_i)\rho_i^2 \sin \varphi_i\,\Delta\rho\,\Delta\varphi\,\Delta\theta$$

there being n elements of volume and the limit being taken for the maximum linear dimension of each element approaching zero and n increasing indefinitely. This limit is expressed by the iterated integral

$$\int_{\theta_1}^{\theta_2}\int_{\Phi_1}^{\Phi_2}\int_{R_1}^{R_2} F(\rho,\theta,\varphi)\rho^2 \sin \varphi\,d\rho\,d\varphi\,d\theta$$

The first integration holding φ and θ fixed adds all elements along a radius vector to form a column which tapers toward the origin. The second integration holding θ fixed adds all such columns along a circle HK (Fig. 216) to form a slice much like a section of an orange. The third integration adds all such slices to form the integral of $F(\rho,\theta,\varphi)$ over the volume V. The choice of limits is best made clear by an example.

Example. A solid sphere of radius a has the density at any point proportional to the distance from the center. Find the mass of the sphere. Let the center of the sphere be the origin. Its equation is then $\rho = a$. Divide the sphere into elements of mass $\Delta_1 M, \Delta_2 M, \ldots, \Delta_n M$ with volumes $\Delta_1 V, \Delta_2 V, \ldots, \Delta_n V$, respectively. Let $\rho_i, \theta_i, \varphi_i$ be an appropriate point of the ith element. We then have $\Delta_i M = k\rho_i \Delta_i V$. Hence, the mass is given by the triple integral $M = k \iiint\limits_{(V)} \rho\, dV$. This may be expressed by an iterated integral with

$$dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

Thus $$M = k \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

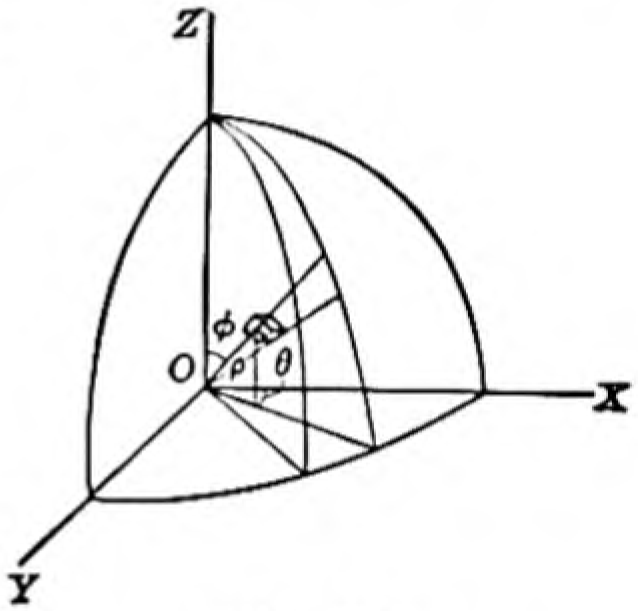


FIG. 217.

Observe the choice of limits. For the first integration with φ and θ fixed, we add elements along a radius vector from the origin out to the sphere, that is, from $\rho = 0$ to $\rho = a$ (Fig. 217). This gives the mass of a tapering column. We next add all such columns with θ held fixed, from the upward vertical position to the downward vertical position, that is, from $\varphi = 0$ to $\varphi = \pi$. This gives the mass of a slice, as indicated in the figure. All such slices are then added together, that is, from $\theta = 0$ to $\theta = 2\pi$. The calculation is as follows:

$$M = k \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta = k \int_0^{2\pi} \int_0^{\pi} \frac{a^4}{4} \sin \varphi \, d\varphi \, d\theta$$
$$= \frac{ka^4}{4} \int_0^{2\pi} \left[-\cos \varphi \right]_0^{\pi} d\theta = \frac{ka^4}{2} \int_0^{2\pi} d\theta = k\pi a^4 \qquad \text{units of mass}$$

In choosing limits, the student should always visualize clearly the geometrical situation.

159. Change of Coordinate System.

If a triple integral

$$I = \iiint\limits_{(V)} f(x,y,z) \, dV$$

is to be evaluated, it may be convenient to use an iterated integral in the rectangular coordinate system, $I = \int\int\int f(x,y,z) \, dz \, dy \, dx$, where the limits are chosen to extend the integration over the volume V. If it is desired to use cylindrical coordinates, we may use $x = r \cos \theta$, $y = r \sin \theta$, and write

$$I = \int\int\int f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$

where limits are chosen to extend the integral over the volume V. If spherical coordinates are more convenient, the triple integral may be expressed by the iterated integral

$$I = \iiint f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)\rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

limits again being chosen to include the entire volume.

Example. It is instructive to set up the integrals for the mass of the sphere discussed in the example of the preceding section. Using rectangular coordinates, we have the density at the point x,y,z given by $\delta = k\sqrt{x^2 + y^2 + z^2}$. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$. The element of mass is, therefore,

$$dM = k\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

and the total mass is

$$M = 8k \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

Note that the limits are chosen to cover the first octant of the sphere and that the result is multiplied by 8. The difficulty of carrying out the integration indicates the advantage in using spherical coordinates.

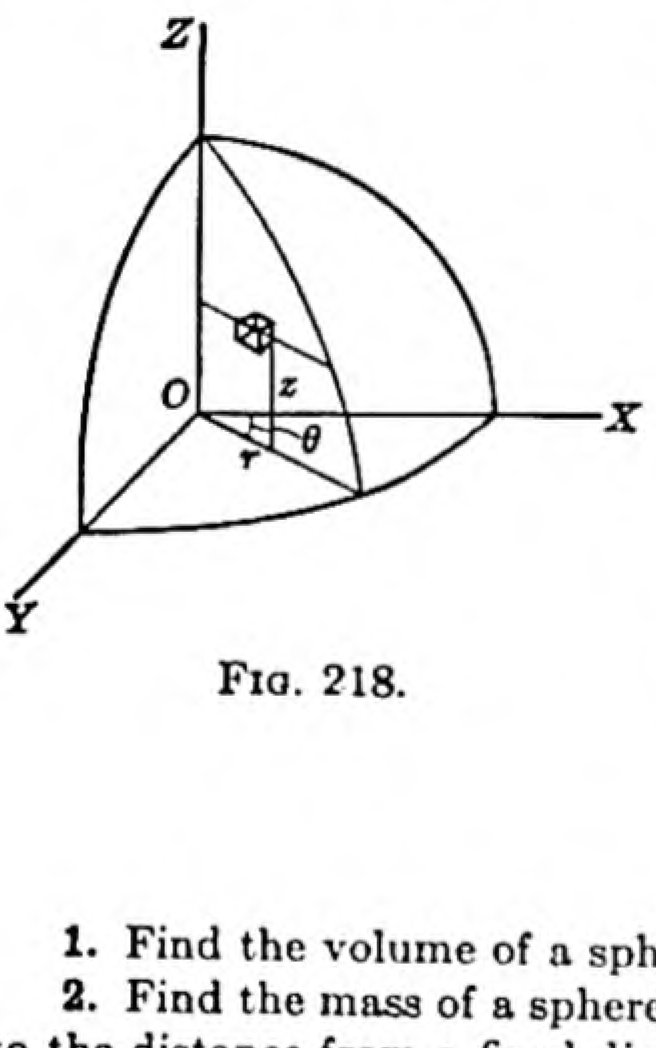


FIG. 218.

Using cylindrical coordinates, the equation of the sphere is $r^2 + z^2 = a^2$, and the density at any point is $\delta = k\sqrt{r^2 + z^2}$. The element of mass is

$$dM = k\sqrt{r^2 + z^2}\, r\, dr\, dz\, d\theta$$

and

$$M = 8k \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-z^2}} \sqrt{r^2 + z^2}\, r\, dr\, dz\, d\theta$$

Note that limits have been chosen to cover the first octant and that the result is multiplied by 8 (see Fig. 218).

EXERCISES

1. Find the volume of a sphere of radius a, using spherical coordinates.
2. Find the mass of a sphere of radius a if the density at any point is proportional to the distance from a fixed diameter. Use spherical coordinates.
3. Find the mass of a sphere of radius a if the density at any point is proportional to the distance from a fixed diametral plane. Use spherical coordinates.
4. Find the mass of a spherical shell of inside radius a and outside radius b if the density at any point is proportional to the distance from the center.
5. Find the mass of a spherical shell of inside radius a and outside radius b if the density at any point is inversely proportional to the distance from the center.
6. Find the volume inside a sphere of radius $2a$ and outside a cylinder of radius a whose axis is a diameter of the sphere. Use spherical coordinates.
7. Find the mass of a solid occupying the volume of Exercise 6 if the density at any point is proportional to the square of the distance from the center of the sphere.

8. Find the mass in Exercise 7 if the density at any point is inversely proportional to the square of the distance from the center of the sphere.

9. Find the triple integral of the function $x^2 + y^2$ over a cube of edge a three of whose faces are in the coordinate planes.

10. Find the triple integral of xyz over the cube of Exercise 9.

11. Find $\iiint\limits_{(V)} x^2\,dV$ where V is the volume of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$. (*Hint:* Use cylindrical coordinates.)

12. Find $\iiint\limits_{(V)} xy\,dV$ where V is the volume of Exercise 11.

13. Find $\iiint\limits_{(V)} z\,dV$ where V is the volume above the plane $z = 0$, inside the cylinder $x^2 + y^2 = 2ax$, and under the sphere $x^2 + y^2 + z^2 = 4a^2$.

14. Find $\iiint\limits_{(V)} yz\,dV$ where V is the volume in the first octant, inside the cylinder $x^2 + y^2 = 2ax$, and under the sphere $x^2 + y^2 + z^2 = 4a^2$.

15. Find $\iiint\limits_{(V)} (x^2 + y^2)\,dV$ where V is the volume of the sphere

$$x^2 + y^2 + z^2 = a^2$$

16. Find $\iiint\limits_{(V)} \sqrt{x^2 + y^2}\,dV$ where V is the volume of Exercise 15.

160. Centroids; Moments of Inertia. The methods for finding the centroid of a mass (Art. 121) and the moment of inertia of the mass with respect to some axis (Art. 129) can readily be generalized to include cases in which the moment of the mass is best expressed by a double or triple integral. The formulas required, namely,

$$M = \int dM \qquad M\bar{x} = \int X\,dM \qquad I = \int R^2\,dM$$

are easily expressed by double or triple integrals. It is only necessary to express dM, X, R in terms of the coordinate system involved. The method is best made clear by examples.

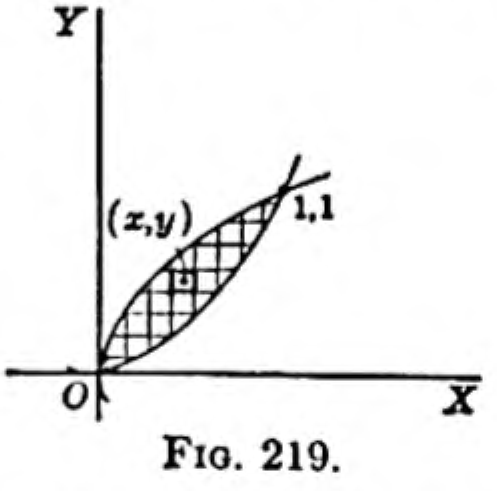


FIG. 219.

Example 1. Find the centroid of a thin plate of constant density $\delta = k$ g. per square centimeter that covers the area bounded by the curves $y = x^2$ and $y^2 = x$ (Fig. 219). We note first of all that, because of symmetry, $\bar{x} = \bar{y}$. Now, divide the area into elements of area dS. Then

$$dM = k\,dS \qquad \text{and} \qquad M = \int dM = k \iint\limits_{(S)} dS$$

Also, if x, y are the coordinates of the centroid of a typical element of mass, then $X = x$, and we have

$$M\bar{x} = \int X\,dM = k \iint\limits_{(S)} x\,dS$$

Expressing these double integrals as iterated integrals, we have

$$M = k\int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx = \tfrac{1}{3}k$$

$$M\bar{x} = k\int_0^1\int_{x^2}^{\sqrt{x}} x\,dy\,dx = k\int_0^1 (x^{3/2} - x^3)\,dx = \tfrac{3}{20}k$$

$$\bar{x} = \frac{3}{20}k\cdot\frac{3}{k} = \frac{9}{20}$$

Example 2. Find the moment of inertia of a circular sector (area) of radius a and central angle α (see Example 4, Art. 129) with respect to a line through the center perpendicular to the plane of the sector (polar moment of inertia). Using polar coordinates, we have $r = a$ for the equation of the circle. Dividing the sector into convenient elements of area and letting (r,θ) be a suitable point of a typical element, we have $R = r$ and $dM = dS$. Hence $I = \iint_{(S)} r^2\,dS$. This double integral may be evaluated by an iterated integral in which $dS = r\,dr\,d\theta$ (see Fig. 220). Thus

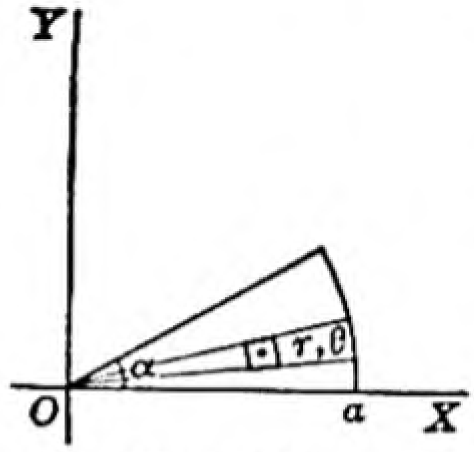


FIG. 220.

$$I = \iint_{(S)} r^2\,dS = \int_0^\alpha\int_0^a r^3\,dr\,d\theta = \tfrac{1}{4}a^4\alpha$$

Since $M = \dfrac{1}{2}a^2\alpha$, we may write $I = \dfrac{1}{4}a^4\alpha\cdot\dfrac{2M}{a^2\alpha} = \dfrac{1}{2}a^2M$.

Example 3. Find the centroid of a right circular solid cone with radius of base a and altitude h if the density at any point is proportional to the distance from the axis of the cone. If we take the vertex at O and use cylindrical coordinates, the equation of the conical surface is $z = (h/a)r$. Considerations of symmetry give $\bar{x} = \bar{y} = 0$. Divide the volume into elements of volume, and let r, θ, z be a point of a typical element as indicated in Fig. 221. We then have $dM = kr\,dV$, $Z = z$, and

$$M = \iiint_{(V)} kr\,dV \qquad M\bar{z} = \iiint_{(V)} kzr\,dV$$

To evaluate these triple integrals we use $dV = r\,dr\,d\theta\,dz$ and write iterated integrals as follows:

$$\begin{aligned} M &= k\int_0^{2\pi}\int_0^a\int_{(h/a)r}^h r^2\,dz\,dr\,d\theta \\ &= \frac{hk}{a}\int_0^{2\pi}\int_0^a (r^2a - r^3)\,dr\,d\theta = \tfrac{1}{6}\pi k a^3h \\ M\bar{z} &= k\int_0^{2\pi}\int_0^a\int_{(h/a)r}^h zr^2\,dz\,dr\,d\theta \\ &= \frac{kh^2}{2a^2}\int_0^{2\pi}\int_0^a (a^2 - r^2)r^2\,dr\,d\theta = \tfrac{2}{15}\pi kh^2a^3 \end{aligned}$$

Hence
$$\bar{z} = \frac{2\pi kh^2a^3}{15}\cdot\frac{6}{\pi ka^3h} = \frac{4}{5}h$$

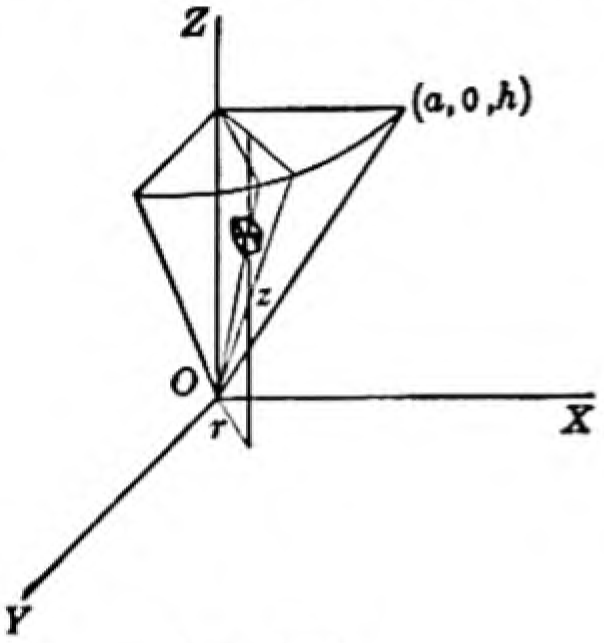


FIG. 221.

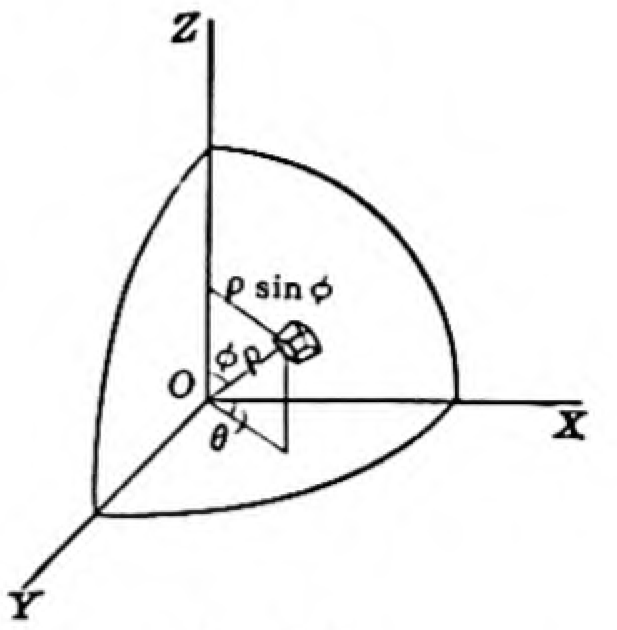


FIG. 222.

Example 4. Find the moment of inertia of a solid sphere of radius a with respect to a diameter if the density at any point is proportional to the distance of that point from the center of the sphere. In spherical coordinates the equation of the spherical surface, if we take the center at O, is $\rho = a$. We divide the mass into elements of mass dM each occupying an element of volume dV. If ρ, θ, φ is a suitable point, the mass of a typical element is $dM = k\rho\, dV$. Furthermore, the radius of gyration of the element of mass is the distance of some point of the element from a diameter, say from the z axis. Hence, $R = \rho \sin \varphi$ (Fig. 222). Therefore

$$I = \int R^2\, dM = \iiint\limits_{(V)} \rho^2 \sin^2 \varphi k\rho\, dV$$

To evaluate this triple integral, we employ an iterated integral, using

$$dV = \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta$$

thus:

$$\begin{aligned} I &= k \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \sin^2 \varphi \cdot \rho \cdot \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta \\ &= k \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^5 \sin^3 \varphi\, d\rho\, d\varphi\, d\theta = \frac{ka^6}{6} \int_0^{2\pi} \int_0^{\pi} \sin^3 \varphi\, d\varphi\, d\theta \\ &= \frac{ka^6}{6} \int_0^{2\pi} \int_0^{\pi} (1 - \cos^2 \varphi) \sin \varphi\, d\varphi\, d\theta \\ &= \frac{ka^6}{6} \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{4}{9} \pi k a^6 \end{aligned}$$

From the example of Art. 158, we have $M = \pi k a^4$. Consequently

$$I = \frac{4}{9} \pi k a^6 \cdot \frac{M}{\pi k a^4} = \frac{4}{9} a^2 M$$

Observe that the radius of gyration of this mass is $\frac{2}{3}a$.

EXERCISES

Find the centroids of the masses as indicated (Ex. 1 to 17).

1. A straight wire AB, l units long, if the density at any point of the wire is proportional to the distance from A.

2. A straight wire AB, l units long, if the density at any point is proportional to the nth power ($n > 0$) of the distance from A.

3. A semicircular plate of radius a and constant density. Use double integrals.

4. The area bounded by the cardioid $r = a(1 + \cos \theta)$. Use double integrals.

5. A semicircular plate if the density at any point is proportional to the distance from the center.

6. A semicircular plate if the density is proportional to the distance from the diameter that forms its straight edge.

7. A thin plate covering the area between the parabolas $x^2 = ay$ and $y^2 = ax$ if the density at any point is proportional to the square of the distance from the origin.

8. A thin circular plate of radius a if the density at any point is proportional to the distance from a point A on the circumference.

9. A thin plate covering the area bounded by the cardioid $r = a(1 + \cos \theta)$ if the density at any point is proportional to the distance from the pole.

10. A thin plate covering the area inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$ if the density at any point is inversely proportional to the distance from the pole.

11. A rectangular plate of base a and altitude b if the density at any point is proportional to the square of the distance from one corner.

12. A cube of edge a if the density at any point is proportional to the distance from one face.

13. A cube of edge a if the density at any point is proportional to the square of the distance from one edge.

14. A cone of altitude h and radius of base a if the density at any point is proportional to the distance from the base.

15. A hemisphere of radius a if the density at any point is proportional to the distance from the plane that forms the base.

16. A hemisphere of radius a if the density at any point is proportional to the distance from the center.

17. The mass cut from a hemisphere of radius a by a right circular cone having the same base, the same axis, and vertex on the hemisphere; the density at any point is proportional to the distance from the common base.

In each of the following, find the moment of inertia with respect to the indicated axis (Ex. 18 to 32):

18. The polar moment of inertia of a circular plate of radius a if the density at any point is proportional to the distance from the center.

19. The plate of Exercise 18 with respect to a diameter.

20. A square plate of side a with respect to one side if the density at any point is proportional to the distance from this side.

21. A square plate of side a with respect to a line through one corner perpendicular to the surface of the plate if the density at any point is proportional to the square of the distance from this corner.

22. The polar moment of inertia of a thin plate covering the area bounded by the circle $r = 2a \cos \theta$ if the density at any point is proportional to the distance from the pole.

23. The polar moment of inertia of a thin plate covering the area bounded by the four-leaved rose $r = a \cos 2\theta$ if the density at any point is proportional to the distance from the pole.

24. The polar moment of inertia of a thin plate covering the area bounded by the cardioid $r = a(1 + \sin \theta)$ if the density at any point is proportional to the distance from the pole.

25. A right circular cylindrical shell of inner radius a and outer radius b with respect to the axis of the shell if the density at any point is proportional to the distance from the axis of the shell.

26. A solid sphere of radius a with respect to a diameter if the density at any point is proportional to the square of the distance from the center of the sphere.

27. A cube of edge a with respect to one edge if the density at any point is proportional to the square of the distance from this edge.

28. A right circular cone of altitude h and radius of base a with respect to its axis if the density at any point is proportional to the distance from this axis.

29. A sphere of radius a with respect to a diameter if the density at any point is proportional to the distance from this diameter.

30. A solid bounded by the sphere $x^2 + y^2 + z^2 = a^2$ with respect to the z axis if the density at any point is proportional to the distance from the xy plane.

31. The volume of Exercise 6, page 420, with respect to the axis of the cylinder.

32. The solid of Exercise 7, page 420, with respect to the axis of the cylinder.

MISCELLANEOUS EXERCISES

Evaluate the integrals (Ex. 1 to 8):

1. $\displaystyle\int_0^4 \int_0^{\sqrt{3}x} \frac{y\,dy\,dx}{\sqrt{x^2 + y^2}}$ **2.** $\displaystyle\int_0^1 \int_0^{\ln x} e^{x+y}\,dy\,dx$

3. $\displaystyle\int_0^{\pi/2} \int_1^y x \sin y\,dx\,dy$ **4.** $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy$

5. $\displaystyle\iiint\limits_{(V)} z\,dV$ where V is the volume of the hemisphere $x^2 + y^2 + z^2 = a^2$ above $z = 0$

6. $\displaystyle\iiint\limits_{(V)} z\,dV$ where V is the volume inside the shell between the spheres

$$x^2 + y^2 + z^2 = a^2$$

and $x^2 + y^2 + z^2 = b^2$ $(b > a)$ and above $z = 0$

7. $\displaystyle\iiint\limits_{(V)} \sqrt{x^2 + y^2}\,dV$ where V is the volume inside the cylinder $x^2 + y^2 = 2ax$ and between the planes $z = 0$ and $z = h$

8. $\displaystyle\iiint\limits_{(V)} xyz\,dV$ where V is the volume of Exercise 7

9. Find by use of an iterated integral the volume of the solid bounded by the planes $x = a$, $x = 2a$, $y = a$, $y = 2a$, $z = 0$, $z = x$.

10. Find the volume in the first octant bounded by the surfaces $x^2y^2 = z$, $y = 2x$, $y = 2$.

11. Find the volume enclosed by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

12. Find the volume cut from the paraboloid $x^2 + y^2 = 2z$ by the plane $y + z = 1$.

13. Find the volume in the first octant bounded by the surface

$$(x/a)^{1/2} + (y/b)^{1/2} + (z/c)^{1/2} = 1$$

14. Find the volume inside the cylinder $r = a(1 + \cos\theta)$, above the plane $z = 0$, and under the plane $z = r\cos\theta$.

15. Find the volume cut from the paraboloid $z = r^2$ by the plane $z = r\cos\theta$.

16. Find the volume above the plane $z = 0$, under the plane $x + z = 2a$, and between the cylinders $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4a^2$.

17. Find the volume under the spherical surface $r^2 + z^2 = 25a^2$ and above the paraboloid $3r^2 = 16az$.

18. Find the volume generated by revolving the curve $r^2 = a^2\cos\theta$ about the polar axis.

19. Find the volume generated by rotating the right-hand half of the cardioid $r = a(1 + \sin\theta)$ about the line $\theta = \pi/2$.

20. Set up integrals to express the volume of a sphere of radius a, using as many different methods as you can think of.

21. Find the area of the surface of the cylinder $x^{2/3} + z^{2/3} = a^{2/3}$ which is inside the cylinder $x^{2/3} + y^{2/3} = a^{2/3}$.

22. Find the area of the surface of the cylinder $x^{2/3} + y^{2/3} = a^{2/3}$ which is inside the sphere $x^2 + y^2 + z^2 = a^2$.

23. Find the mass of a thin circular plate if the density at any point is proportional to the distance from the center.

24. Find the mass of a thin circular plate if the density at any point is proportional to the distance from a fixed diameter.

25. Find the mass of a right circular cylinder of altitude h and radius of base a if the density at any point is proportional to the distance from the axis.

26. Find the mass of a right circular cone of altitude h and radius of base a if the density at any point is proportional to the square of the distance from the base.

27. Find the mass of a thin circular washer whose edges are concentric circles of radii a and b if the density at any point is inversely proportional to the square of the distance from the center.

28. Find the centroid of a thin semicircular plate if the density at any point is proportional to the square of the distance from the center.

29. Find the centroid of the thin plate whose boundary is the upper semicircle $x^2 + y^2 = a^2$ if the density at any point is proportional to the square of the distance from the y axis.

30. Find the centroid of a thin circular plate if the density at any point is proportional to the square of the distance from a point A on the circumference.

31. Find the centroid of a thin plate covering the area inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$ if the density at any point is inversely proportional to the distance from the pole.

32. Find the centroid of a cube of edge a if the density at any point is proportional to the sum of the distances from three adjacent faces.

33. Find the centroid of a cube of edge a if the density at any point is proportional to the square of the distance from one corner.

34. Find the centroid of a hemisphere of radius a if the density at any point is proportional to the distance from the axis.

35. Show by using double integrals that the polar moment of inertia of an area equals the sum of the moments of inertia of the area with respect to the x axis and the y axis.

36. Find the moment of inertia of the volume under the cone $z = r$, inside the cylinder $r = a(1 + \cos\theta)$, and above the plane $z = 0$ (Exercise 7, page 401), with respect to the z axis.

37. Find the moment of inertia with respect to the axis of a right circular cone of altitude h and radius of base a if the density at any point is proportional to the distance from the base.

38. Find the moment of inertia with respect to the z axis of the mass lying above the xy plane and common to the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder

$$x^2 + y^2 = 2ax$$

if the density at any point is proportional to the distance from the xy plane.

39. A homogeneous spherical shell of density δ has inside radius 3 ft. and outside radius 4 ft. A ring-shaped solid is cut from it by two parallel planes on the same side of the center and at a distance 1 ft. and 2 ft., respectively, from the center. Find the moment of inertia of this ring with respect to its axis.

40. Solve Exercise 39 if the density at any point of the ring is proportional to the distance from a plane through the center of the spherical shell and parallel to the two cutting planes.

41. The value of $A = \int_0^\infty e^{-x^2}\,dx$ can be found as follows:

$$A^2 = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$$

Since the limits of integration do not involve x or y, this product can be expressed as an iterated integral, thus: $A^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. This is equivalent to the double integral $\iint\limits_{(S)} e^{-(x^2+y^2)}\,dS$ where S is the entire first quadrant. Express this double integral as an iterated integral in polar coordinates, evaluate, and thereby show that $A = \frac{1}{2}\sqrt{\pi}$. As previously mentioned, this integral is of importance in the theory of probability and statistics.

CHAPTER 19

INFINITE SERIES

161. Infinite Series. If we have $n + 1$ numbers $a_0, a_1, a_2, \ldots, a_n$ and add them together, the result is called a *series of $n + 1$ terms.* The sum may be denoted by s_n, thus:

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{i=0}^{n} a_i$$

For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n} = s_n$ is familiar as a geometric progression.

If n becomes infinite, the symbol

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=0}^{\infty} a_n$$

is called an *infinite series.* For example,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

is an infinite series. Frequently, Σa_n is written instead of $\sum_{n=0}^{\infty} a_n$ if no ambiguity can result.

The term a_n of our infinite series is called the *general term,* and it expresses the law of formation of the series. If the general term is not given, but the symbol $a_0 + a_1 + a_2 + \cdots$ is written to express the infinite series, we may try to formulate an expression for the general term. For example, if we write $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, it is suggested that this is simply $\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$, so that the general term is $1/2^n$.

It should be noted that it may be more convenient in certain cases to start with $n = 1$ instead of $n = 0$, thus

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Example 1. Given the series $\frac{3}{1 \cdot 2} + \frac{4}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots$. Here the general term is $\frac{n+3}{(n+1)(n+2)}$, and the series may be written $\sum_{n=0}^{\infty} \frac{n+3}{(n+1)(n+2)}$. It is

more convenient, however, to begin with $n = 1$ instead of $n = 0$, in which case the series can be written $\sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}$.

Example 2. Given the series $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \cdots$. Here the general term is $\frac{2n-1}{2n}$, if we start with $n = 1$, and the series is $\sum_{n=1}^{\infty} \frac{2n-1}{2n}$.

EXERCISES

Discover the general term and express each series in the form Σa_n (Ex. 1 to 10).

1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

3. $1 + x + x^2 + x^3 + \cdots$

4. $\frac{\sqrt{x}}{2} + \frac{x}{2 \cdot 3} + \frac{x\sqrt{x}}{3 \cdot 4} + \frac{x^2}{4 \cdot 5} + \cdots$

5. $1 - \frac{y}{1 \cdot 3} + \frac{y^2}{1 \cdot 3 \cdot 5} - \frac{y^3}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

6. $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$

7. $\frac{1}{2} + \frac{3}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \frac{7}{4 \cdot 5} + \cdots$

8. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \cdots$

9. $\frac{2}{1} - \frac{4}{4} + \frac{8}{9} - \frac{16}{16} + \frac{32}{25} - \cdots$

10. $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81} - \cdots$

Write the first five terms of the series whose general terms are given below (Ex. 11 to 18). In each case start with $n = 1$.

11. $\frac{\sqrt{n}}{n^2 + 1}$

12. $\frac{(-1)^{n+1}(2n - 1)}{n!}$

13. $\frac{(-1)^{n+1}x^n}{n^2}$

14. $\frac{(-1)^{n-1}x^{2n-2}}{(2n - 2)!}$

15. $\frac{(-1)^{n-1}x^{2n-1}}{(2n - 1)!}$

16. $\frac{3^{n/2}}{\sqrt{n}}$

17. $\left(\frac{1}{n} - \frac{1}{n+1}\right)$

18. $\frac{k(k-1)(k-2) \cdots (k-n+1)}{n!} x^n$ (k a fixed constant). This is known as the binomial series.

162. Convergence and Divergence. Let us consider the following sums formed from the infinite series

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

and called the successive *partial sums:*

$$
\begin{aligned}
s_0 &= a_0 \\
s_1 &= a_0 + a_1 \\
s_2 &= a_0 + a_1 + a_2 \\
&\cdots\cdots\cdots\cdots \\
s_n &= a_0 + a_1 + a_2 + \cdots + a_n
\end{aligned}
$$

If the series starts with a_1, we may take $s_0 = a_0 = 0$.

If s_n has a limit as n increases indefinitely, the series $\sum_{n=0}^{\infty} a_n$ is said to be convergent. If s_n does not have a limit as n increases indefinitely, the series is said to be divergent.

The limit $\lim_{n\to\infty} s_n = S$ is called the *sum* of the series, or the series is said to *converge to the sum S*. It is important, however, to note that S is not actually a sum, but the *limit* of a sum. For example, in the series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots \qquad s_n = 2 - \frac{1}{2^n}$$

Evidently, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2 - \frac{1}{2^n}\right) = 2$. The series *converges to the sum* 2.

It should be carefully noted that a series is *divergent* if $\lim_{n\to\infty} s_n$ fails to exist for *any* reason. For example, the series $1 + 2 + 3 + \cdots + n + \cdots$ has $s_n = n(n + 1)/2$, and $s_n \to +\infty$ when $n \to \infty$. Such a series is said to *diverge to* $+\infty$. On the other hand, the series

$$1 - 1 + 1 - 1 + \cdots$$

has $s_n = 1$ if n is even but $s_n = 0$ if n is odd. The series *diverges* but is said to *oscillate*.

It will be clear to the student, especially if he tries to find s_n for the series in the exercises following Art. 161, that the determination of convergence or divergence by investigating directly the behavior of s_n is difficult if not impossible in most cases. However, it is often of great importance to know whether or not a given series is convergent. In more advanced texts devoted to the theory of infinite series, many useful tests for convergence are developed in which use is made of the general term a_n, rather than the partial sum s_n. We shall consider a few of the less elaborate of these tests. We first investigate two important special series, then series of positive terms, and then series of both positive and negative terms. Having discussed these various cases of series of constant terms, we turn our attention to *power series* in which each term is a constant times a power of a variable x. It should be remarked that, if a

series is divergent, little use can be made of it in work of an elementary character.

It is important to observe that, in determining the convergence or the divergence of a given series, *we may neglect a finite number of terms*, either at the beginning or scattered through the series. Their sum is simply some definite constant which affects the sum, but not the convergence, of the given series.

163. Geometric and Harmonic Series. Two very important series are the *geometric series*

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

and the *harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ The former is simply an extension of the familiar geometric progression $a + ar + ar^2 + ar^3 + \cdots + ar^n$ with whose sum the student is already acquainted, namely, $\frac{a - ar^{n+1}}{1 - r}$, $r \neq 1$.

We investigate first the properties of the geometric series. To clarify our ideas, we suppose that a is positive. Since

$$s_n = \frac{a - ar^{n+1}}{1 - r} = \frac{a}{1 - r} - \frac{ar^{n+1}}{1 - r}$$

we have at once

1. If $|r| < 1$, $\lim\limits_{n \to \infty} s_n = \frac{a}{1 - r}$, and the series *converges* with sum $\frac{a}{1 - r}$, since $\lim\limits_{n \to \infty} r^{n+1} = 0$.

2. If $|r| > 1$, $\lim\limits_{n \to \infty} r^{n+1}$ does not exist, and therefore $\lim\limits_{n \to \infty} s_n$ does not exist. The series diverges to $+\infty$ if $r > 1$ and oscillates if $r < -1$.

3. If $r = 1$, the series becomes $a + a + a + \cdots$, so that

$$s_n = (n + 1)a$$

Therefore, $s_n \to +\infty$ when $n \to \infty$, and the series diverges to $+\infty$.

4. If $r = -1$, the series becomes $a - a + a - a + \cdots$, so that $s_n = a$ for n even and 0 for n odd. Consequently, s_n oscillates, and the series diverges.

To sum up: The geometric series

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

converges for $|r| < 1$* and *diverges* for $|r| \geqq 1$. This holds also for $a < 0$.

Consider now the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

* Including, of course, the trivial case $r = 0$ for which the series reduces to a.

with terms grouped as follows:

$$1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}) + \cdots$$

In the first parentheses, there is $1 = 2^0$ term, in the second parentheses there are $2 = 2^1$ terms, in the third $4 = 2^2$ terms, in the fourth $8 = 2^3$ terms, and so on. In the kth parentheses there are 2^{k-1} terms. Note that the last term in the kth parentheses is $1/2^k$. The sum in each parentheses (except the first) exceeds $\frac{1}{2}$, for

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{2^1}{2^2} = \frac{1}{2}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{2^2}{2^3} = \frac{1}{2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > \frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k} = \frac{2^{k-1}}{2^k} = \frac{1}{2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Consequently, we can make s_n as large as we please by taking n sufficiently large; for s_n exceeds some multiple of $\frac{1}{2}$, and this multiple can be made as great as we please. Therefore, $s_n \to +\infty$, and the harmonic series *diverges*. It is interesting to point out that the series "diverges very slowly," for the sum of 1 million terms is less than 15.*

164. A Necessary Condition for Convergence. Let us suppose that the series $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ converges to the sum S. We then have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = S$$

But $s_n - s_{n-1} = a_n$. Taking limits,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0$$

Consequently, if a series $\sum_{n=0}^{\infty} a_n$ converges, a_n *must* approach zero as n becomes infinite. If a_n does not approach zero, the series cannot converge. This condition is, therefore, a *necessary* condition for convergence. However, it is obviously not sufficient to ensure convergence; for the general term of the harmonic series $a_n = 1/n$ approaches zero as n increases indefinitely, but the series diverges.

We may now easily prove the following useful theorem: *If k is any*

* $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < 1 + \displaystyle\int_1^n \frac{dx}{x} = 1 + \ln n.$ Hence

$$s_{1,000,000} < 1 + \ln 1{,}000{,}000 = 1 + 6 \ln 10 < 1 + 14 = 15$$

constant (other than zero), then, if $\sum_{n=0}^{\infty} a_n$ *converges to a sum* S, $\sum_{n=0}^{\infty} ka_n$ *converges to the sum* kS. If $\sum_{n=0}^{\infty} a_n$ *diverges, so also does* $\sum_{n=0}^{\infty} ka_n$.

The nth partial sum of the first series is

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n$$

Hence, the partial sum of the second series is

$$s'_n = ka_0 + ka_1 + ka_2 + \cdots + ka_n = ks_n$$

Consequently $\lim_{n\to\infty} s'_n = \lim_{n\to\infty} ks_n = k \lim_{n\to\infty} s_n = kS$ if the first series converges or fails to exist if the first series diverges. This proves the theorem.

Example 1. The series $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots + \frac{1}{2 \cdot 3^n} + \cdots$ converges. For $\frac{1}{2 \cdot 3^n} = \frac{1}{2} \cdot \frac{1}{3^n}$, and $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is the geometric series with $r = \frac{1}{3}$. This sum is $\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{3}{4}$.

Example 2. The series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n}$$

diverges, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

EXERCISES

Determine the convergence or divergence of the following series:

1. $7 + \frac{7}{2} + \frac{7}{3} + \frac{7}{4} + \cdots$
2. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$
3. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$
4. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$
5. $1 + \sqrt{2} + 2 + 2\sqrt{2} + 4 + 4\sqrt{2} + \cdots$
6. $1 - \sqrt{3} + 3 - 3\sqrt{3} + 9 - 9\sqrt{3} + \cdots$
7. $\sqrt{5} + 106 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$
8. $1 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$
9. $\log \frac{1}{2} + \log \frac{1}{4} + \log \frac{1}{8} + \log \frac{1}{16} + \cdots$
10. $100 + \frac{100}{3} + \frac{100}{9} + \frac{100}{27} + \frac{100}{81} + \cdots$
11. $\frac{1}{100} + \frac{3}{100} + \frac{9}{100} + \frac{27}{100} + \cdots$
12. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots$
13. $1 + \frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \cdots$
14. $2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \cdots$
15. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$

A. SERIES OF POSITIVE TERMS

165. Integral Test. A test of convergence due to Maclaurin makes use of integration and often is easily applied.

If $f(x)$ is a positive nonincreasing function in the interval $a \leqq x < \infty$ (a a positive integer) such that $f(x) \to 0$ as $x \to +\infty$, then the series $\sum_{n=1}^{\infty} f(n)$ converges if the integral $\int_a^\infty f(x)\,dx$ converges and diverges if the integral $\int_a^\infty f(x)\,dx$ diverges to $+\infty$.*

We give a proof based upon geometrical considerations. We shall make use of a fundamental principle, the proof of which will not be given here: *If a function steadily increases (or remains unchanged) but never exceeds a definite constant, that function must approach a limit.*

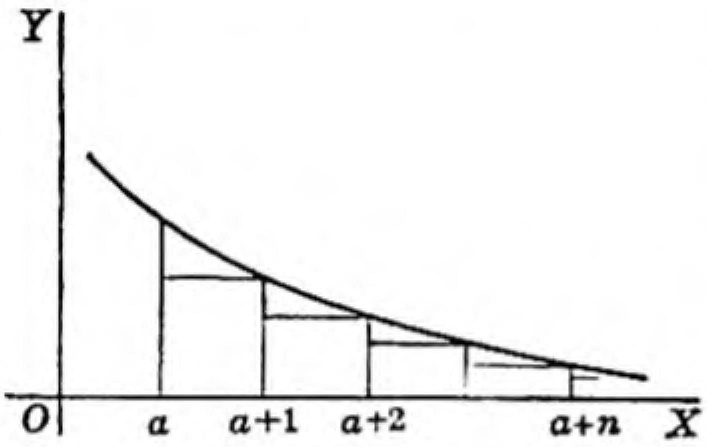


FIG. 223.

The series that we wish to test for convergence is

$$f(1) + f(2) + f(3) + \cdots + f(a) + f(a+1) + \cdots + f(a+n) + \cdots$$

Since convergence is not affected by dropping a finite number of terms, we need consider only the series

$$f(a) + f(a+1) + f(a+2) + \cdots + f(a+n) + \cdots$$

First, suppose that $\int_a^\infty f(x)\,dx$ converges (Art. 108). The graph of $f(x)$ is shown in Fig. 223. Draw ordinates at points a, $a+1$, $a+2$, $\ldots$, $a+n$, $\ldots$, and form rectangles as indicated. The width of each rectangle is unity, and the altitudes are $f(a+1), f(a+2), f(a+3), \ldots, f(a+n), \ldots$. Consequently, the partial sum

$$s_n = f(a) + f(a+1) + f(a+2) + \cdots + f(a+n)$$

is $f(a)$ plus the measure of area of the sum of these rectangles. Hence

$$s_n < f(a) + \int_a^{a+n} f(x)\,dx$$

since $\int_a^{a+n} f(x)\,dx$ is the area under the curve. Now, let n increase; s_n steadily increases but remains less than

$$f(a) + \int_a^{a+n} f(x)\,dx < f(a) + \int_a^\infty f(x)\,dx$$

Because this integral is supposed convergent, s_n steadily increases and

* See Art. 108.

always remains less than a known constant and so must approach a limit. The series

$$f(a) + f(a + 1) + \cdots + f(a + n) + \cdots$$

therefore converges.

Next, suppose that $\int_a^\infty f(x)\,dx$ diverges to $+\infty$. Construct rectangles as shown in Fig. 224. Here we have

$$s_n = f(a) + f(a + 1) + f(a + 2) + \cdots + f(a + n) > \int_a^{a+n} f(x)\,dx$$

But this integral diverges to $+\infty$ as n increases indefinitely; and since s_n is greater than this integral, the series diverges.

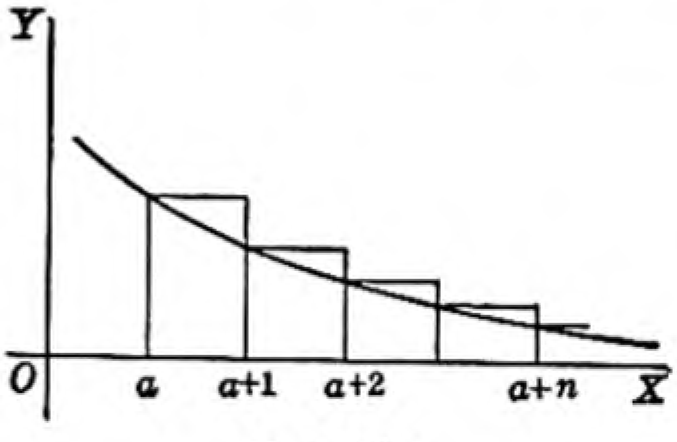


FIG. 224.

Example. This test can be used advantageously to investigate the behavior of the following important series of positive terms:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

where k is any (real) number. Here we take $f(x) = 1/x^k$, and $a = 1$.

Consider first $k \neq 1$. We have

$$\int_1^b f(x)\,dx = \int_1^b \frac{dx}{x^k} = \frac{1}{1-k} \cdot \frac{1}{x^{k-1}}\bigg]_1^b = \frac{1}{1-k}\left(\frac{1}{b^{k-1}} - 1\right)$$

If $k > 1$,

$$\lim_{b\to\infty} \frac{1}{1-k}\left(\frac{1}{b^{k-1}} - 1\right) = \frac{1}{k-1} = \int_1^\infty \frac{dx}{x^k} = \int_1^\infty f(x)\,dx$$

and the series *converges* by the integral test. If $k < 1$,

$$\frac{1}{b^{k-1}} = b^{1-k} \to +\infty \qquad \text{as } b \to +\infty$$

so that $\int_1^\infty \frac{dx}{x^k}$ diverges to $+\infty$, and the series *diverges* by the integral test.

Next consider $k = 1$. The series is now the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

which diverges by Art. 163. Or, using the integral test,

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \ln b = +\infty$$

and the series diverges by the integral test.

Summarizing these results, we have the following: *The series*

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

converges if $k > 1$ *and diverges if* $k \leqq 1$.

EXERCISES

Test for convergence by the integral test:

1. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$
2. $1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + 4}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2 + 4)^2}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 4}}$
7. $1 + \dfrac{1}{2\sqrt{2^2 - 1}} + \dfrac{1}{3\sqrt{3^2 - 1}} + \dfrac{1}{4\sqrt{4^2 - 1}} + \cdots$
8. $1 + \dfrac{1}{1 + 9} + \dfrac{2}{16 + 9} + \dfrac{3}{81 + 9} + \dfrac{4}{256 + 9} + \cdots$
9. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$
10. $a + ar + ar^2 + ar^3 + ar^4 + \cdots$ where a and r are both positive. (Consider $r < 1$, $r = 1$, $r > 1$.)
11. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{2\sqrt{3}} + \dfrac{1}{3\sqrt{4}} + \dfrac{1}{4\sqrt{5}} + \cdots$
12. $\dfrac{1}{e} + \dfrac{2}{e^2} + \dfrac{3}{e^3} + \dfrac{4}{e^4} + \cdots$

166. Comparison Test. We are now acquainted with certain known convergent and certain known divergent series. Convergence or divergence of other series can often be ascertained by comparison with these series of known character.

We have the following test for convergence: *If* $\sum_{n=0}^{\infty} c_n$ *is a convergent series of positive terms and if* $0 \leqq a_n \leqq c_n$ *for every* n, *then* $\sum_{0=n}^{\infty} a_n$ *converges.*

In applying this test and also the succeeding test for divergence, if the relation indicated does not hold for the first few terms of the series, we may omit these terms and consider the two new series in which the relation holds for every n. Convergence is not, of course, affected.

The proof follows at once, for

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n \leqq c_0 + c_1 + c_2 + \cdots + c_n < C$$

where C is the sum to which the series $\sum_{n=0}^{\infty} c_n$ converges. Since we are dealing with series of positive terms only, s_n continually increases as n increases but remains less than the constant C. Consequently, s_n approaches a limit, and $\sum_{n=0}^{\infty} a_n$ converges.

Example 1. The series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent; for we have $\frac{1}{n(n+1)} < \frac{1}{n^2}$ for every n, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the series of Art. 165 which converges for $k = 2$.

Example 2. The series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^n}$ is convergent; for $\frac{1}{n^n} < \frac{1}{2^n}$ (for $n > 2$), and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is the geometric series with $r = \frac{1}{2}$ which converges by Art. 163.

Example 3. The series $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$ converges. We may compare this with the convergent series

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots + \frac{1}{n^4} + \cdots$$

since we have $\frac{1}{(2n+1)^4} < \frac{1}{n^4}$ for all $n \geqq 1$.

We have the following test for divergence: *If $\sum_{n=0}^{\infty} d_n$ is a divergent series of positive terms, and if $a_n \geqq d_n$ for every n, then $\sum_{n=0}^{\infty} a_n$ diverges.*

We have

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n \geqq d_0 + d_1 + d_2 + \cdots + d_n = D_n$$

But since $\sum_{n=0}^{\infty} d_n$ is a divergent series of positive terms, $D_n \to +\infty$ as n increases indefinitely. Hence, $s_n \to +\infty$, and $\sum_{n=0}^{\infty} a_n$ diverges.

Example 4. The series $\dfrac{1}{\sqrt{1\cdot 2}}+\dfrac{1}{\sqrt{2\cdot 3}}+\dfrac{1}{\sqrt{3\cdot 4}}+\cdots=\sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n(n+1)}}$ diverges, for

$$\frac{1}{\sqrt{n(n+1)}}>\frac{1}{\sqrt{(n+1)(n+1)}}=\frac{1}{n+1}$$

and $\sum_{n=1}^{\infty}\dfrac{1}{n+1}$ is the harmonic series which diverges.

Example 5. The series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=\sum_{n=0}^{\infty}\dfrac{1}{2n+1}$ diverges, for

$$\frac{1}{2n+1}>\frac{1}{2n+2}=\frac{1}{2}\cdot\frac{1}{n+1}$$

and $\sum_{n=0}^{\infty}\dfrac{1}{2}\cdot\dfrac{1}{n+1}$ diverges.

The student should note carefully that, to obtain information about a given series from the comparison test, terms of the series must be *less than* or equal to corresponding terms of a *convergent* series or *greater than* or equal to corresponding terms of a *divergent* series. Nothing is gained from knowing that terms are greater than those of a convergent series or less than those of a divergent series. In making the comparison, a finite number of terms may be neglected if desired.

EXERCISES

Test the following series for convergence by use of the comparison tests:

1. $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{17}+\frac{1}{33}+\cdots$
2. $1+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{7}}+\cdots$
3. $\dfrac{1}{1\cdot 2}+\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 6}+\dfrac{1}{7\cdot 8}+\cdots$
4. $\dfrac{1}{2}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 8}+\dfrac{1}{4\cdot 16}+\dfrac{1}{5\cdot 32}+\cdots$
5. $1+\dfrac{2}{5}+\dfrac{2^4}{5^4}+\dfrac{2^9}{5^9}+\dfrac{2^{16}}{5^{16}}+\cdots$
6. $1+\dfrac{1}{1+1}+\dfrac{1}{4+1}+\dfrac{1}{9+1}+\cdots$
7. $2+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\cdots$
8. $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{10}}+\dfrac{1}{\sqrt{17}}+\cdots$

9. $\dfrac{2}{3 \cdot 1} + \dfrac{4}{9 \cdot \sqrt{2}} + \dfrac{8}{27 \cdot \sqrt{3}} + \dfrac{16}{81 \cdot \sqrt{4}} + \cdots$

10. $\dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{2}{3 \cdot 4 \cdot 5} + \dfrac{3}{4 \cdot 5 \cdot 6} + \dfrac{4}{5 \cdot 6 \cdot 7} + \cdots$

11. $\dfrac{1}{2 \cdot 3} + \dfrac{2}{3 \cdot 4} + \dfrac{3}{4 \cdot 5} + \dfrac{4}{5 \cdot 6} + \cdots$

12. $\sqrt{2} + \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{4}}{3} + \dfrac{\sqrt{5}}{4} + \cdots$

13. $\frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \frac{4}{65} + \cdots$

14. $\dfrac{1}{3+1} + \dfrac{1}{9+1} + \dfrac{1}{27+1} + \dfrac{1}{81+1} + \cdots$

15. $\dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + \dfrac{1}{\ln 4} + \dfrac{1}{\ln 5} + \cdots$

16. $\frac{1}{2} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{4}{9} + \frac{3}{4} \cdot \frac{8}{27} + \frac{4}{5} \cdot \frac{16}{81} + \cdots$

167. Ratio Test. A very useful test of convergence is D'Alembert's ratio test* which involves the ratio of any term to the preceding one.

Given the series of positive terms $\sum_{n=0}^{\infty} a_n$. *If* $\lim_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \rho$ *exists, the series converges when* $\rho < 1$ *and diverges when* $\rho > 1$; *if* $\rho = 1$, *the test fails.*

The proof is given in three parts.

1. Suppose $\rho < 1$. Now let r be some number between ρ and 1; thus, $\rho < r < 1$. Since $\lim_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \rho$, the variable quantity $\dfrac{a_{n+1}}{a_n}$ must eventually become and remain less than r; that is, $\dfrac{a_{n+1}}{a_n} < r$ for all $n \geqq N$ where N is some definite number whose value depends upon r. The situation is clear graphically. Let values of $\dfrac{a_{n+1}}{a_n}$ be represented by points on the horizontal axis, as in Fig. 225. Values of this variable must become and remain closer to ρ than any arbitrarily assigned constant. Points representing these values must, therefore, eventually all fall to the left of r; that is, for all n greater than or equal to some definite integer N, these points fall to the left of r. If r is close to ρ, N may be large, since we may have to go quite far along in the series before the ratio of a_{n+1} to a_n becomes and remains less than r. If r is comparatively far from ρ, N may be small; but once r is chosen, N is fixed, and the first N terms may be disregarded in determining convergence.

FIG. 225.

Now let n take successively the values $N, N+1, N+2, \ldots$. We have

* Named for J. D'Alembert (1717–1783).

$$
\begin{array}{llll}
n = N & \dfrac{a_{N+1}}{a_N} < r & \text{and} & a_{N+1} < ra_N \\
n = N + 1 & \dfrac{a_{N+2}}{a_{N+1}} < r & \text{and} & a_{N+2} < ra_{N+1} < r^2a_N \\
n = N + 2 & \dfrac{a_{N+3}}{a_{N+2}} < r & \text{and} & a_{N+3} < ra_{N+2} < r^3a_N \\
\cdots & \cdots & \cdots & \cdots
\end{array}
$$

We find, therefore, that beginning with a_N the terms of our given series are less than the terms of the series

$$a_N + a_Nr + a_Nr^2 + a_Nr^3 + \cdots$$

But this is a geometric series in which $|r| < 1$, and it therefore converges. Consequently, our original series converges by the comparison test.

2. Suppose $\rho > 1$. Then the ratio a_{n+1}/a_n eventually becomes and remains greater than 1, that is, $\dfrac{a_{n+1}}{a_n} > 1$ for all $n \geqq N$ where N is some definite positive integer. We have, therefore

$$
\begin{array}{lll}
\dfrac{a_{N+1}}{a_N} > 1 & \text{and} & a_{N+1} > a_N \\
\dfrac{a_{N+2}}{a_{N+1}} > 1 & \text{and} & a_{N+2} > a_{N+1} > a_N \\
\dfrac{a_{N+3}}{a_{N+2}} > 1 & \text{and} & a_{N+3} > a_{N+2} > a_N \\
\cdots & \cdots & \cdots
\end{array}
$$

Consequently, from $n = N + 1$ on, the terms of the series are all greater than the number a_N and so cannot approach zero. Hence, the series diverges.

3. Suppose $\rho = 1$. The test fails, as is clear from the following:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{1 + \dfrac{2}{n} + \dfrac{1}{n^2}} = 1$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Example 1. Test the series $1 + \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \dfrac{4}{2^4} + \cdots$ for convergence. By inspection, we observe that the general term is $a_n = n/2^n$. Notice that the first term,

1, will not be produced by assigning some value to n in this expression, but this is immaterial. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \frac{1}{2} = \frac{1}{2} = \rho$$

Since $\rho = \frac{1}{2} < 1$, the series converges.

Example 2. Test the series $1 + \frac{1}{1000} + \frac{1 \cdot 2}{1000^2} + \frac{1 \cdot 2 \cdot 3}{1000^3} + \cdots$ for convergence.

The general term is clearly $\frac{n!}{1000^n} = a_n$. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} = \lim_{n\to\infty} \frac{n+1}{1000} = +\infty$$

and the series diverges.

It might have been thought that this series would converge since the denominators of the terms are such very large numbers. But factorial n becomes so great as to destroy this effect of large denominators. The next example will further illuminate the nature of $n!$.

Example 3. Test for convergence the series

$$1 + \frac{k}{1} + \frac{k^2}{2!} + \frac{k^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{k^n}{n!}$$

where κ is any positive constant. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{k^{n+1}}{(n+1)!} \cdot \frac{n!}{k^n} = \lim_{n\to\infty} \frac{k}{n+1} = 0 = \rho$$

Since $\rho = 0 < 1$, the series converges.

From this result, we derive a useful piece of information. Since, if $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} \frac{k^n}{n!} = 0$ for any positive k. Since we are considering only series of positive terms, we are not yet ready to discuss the case of k negative. But in Art. 170 the ratio test will be extended to apply to series of both positive and negative terms, and the student will have no difficulty in seeing that the result holds for k negative.

Note: A good general rule to follow in testing series for convergence is first to apply the ratio test. If $\rho = 1$, then the comparison or integral tests may be tried. It is possible to detect divergence at once when a_n clearly fails to approach zero as n becomes infinite.

EXERCISES

Test the following series for convergence:

1. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ **2.** $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$

3. $\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \cdots$ 4. $1 + \frac{2}{2!} + \frac{3}{4!} + \frac{4}{6!} + \cdots$

5. $1 + \frac{1}{8} + \frac{2!}{8^2} + \frac{3!}{8^3} + \frac{4!}{8^4} + \cdots$

6. $\frac{3}{2} + \frac{9}{2 \cdot 4} + \frac{27}{3 \cdot 8} + \frac{81}{4 \cdot 16} + \frac{243}{5 \cdot 32} + \cdots$

7. $1 + \frac{4}{2 \cdot 5} + \frac{4^2}{3 \cdot 5^2} + \frac{4^3}{4 \cdot 5^3} + \frac{4^4}{5 \cdot 5^4} + \cdots$

8. $1 + \frac{6}{1} + \frac{6^2}{2!} + \frac{6^3}{3!} + \cdots$

9. $1 + \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{4}{4^4} + \cdots$

10. $\frac{1 + \sqrt{1}}{1} + \frac{1 + \sqrt{2}}{2} + \frac{1 + \sqrt{3}}{3} + \frac{1 + \sqrt{4}}{4} + \cdots$

11. $\frac{1 + \sqrt{1}}{1} + \frac{1 + \sqrt{2}}{2} + \frac{1 + \sqrt{3}}{4} + \frac{1 + \sqrt{4}}{8} + \cdots$

12. $\frac{1}{2} + \frac{3}{2^3} + \frac{5}{2^5} + \frac{7}{2^7} + \cdots$

13. $\frac{\sqrt{1}}{4} + \frac{\sqrt{2}}{9} + \frac{\sqrt{3}}{16} + \frac{\sqrt{4}}{25} + \cdots$

14. $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \cdots + \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} + \cdots$

15. $\frac{1}{2!} + \frac{1 \cdot 3}{4!} + \frac{1 \cdot 3 \cdot 5}{6!} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} + \cdots$

16. $1 + \frac{1 \cdot 4}{1 \cdot 3} + \frac{1 \cdot 4 \cdot 7}{1 \cdot 3 \cdot 5} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdots (3n+1)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} + \cdots$

17. $1 + \frac{1 \cdot 4}{1 \cdot 3} \cdot \frac{1}{1} + \frac{1 \cdot 4 \cdot 7}{1 \cdot 3 \cdot 5} \cdot \frac{1}{2^2} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdots (3n+1)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \cdot \frac{1}{n^2} + \cdots$

18. $\frac{2}{1} + \frac{2^2 \cdot 3}{2!} + \frac{2^3 \cdot 3^2}{3!} + \frac{2^4 \cdot 3^3}{4!} + \cdots$

19. $\frac{2^2 \cdot 3}{5^3} + \frac{2^3 \cdot 3^2}{5^4 \cdot 7} + \frac{2^4 \cdot 3^3}{5^5 \cdot 7^2} + \frac{2^5 \cdot 3^4}{5^6 \cdot 7^3} + \cdots$

20. $1 + \frac{5^2}{2^2} + \frac{5^4}{(2 \cdot 4)^2} + \frac{5^6}{(2 \cdot 4 \cdot 6)^2} + \frac{5^8}{(2 \cdot 4 \cdot 6 \cdot 8)^2} + \cdots$

B. SERIES WITH POSITIVE AND NEGATIVE TERMS

168. Alternating-series Test. A series in which the terms are alternately positive and negative is called an *alternating series.* Thus, if a_n is positive for every n,

$$a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n + \cdots$$

is an alternating series. There is a simple and easily applied test for the convergence of such a series

An alternating series is convergent if each term is numerically less than the preceding term and the limit of the nth term, as n increases indefinitely, is zero.

The proof is as follows:

For n odd, say $n = 2m - 1$,

$$s_n = s_{2m-1} = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2m-2} - a_{2m-1})$$

Since the a_n decrease numerically, each difference in parentheses is positive, so that s_{2m-1} *increases* when m increases. In particular,

$$s_{2m-1} \geqq s_1 = a_0 - a_1$$

For n even, say $n = 2m$,

$$s_n = s_{2m} = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{2m-1} - a_{2m})$$

Each of the differences in parentheses is positive, so that s_{2m} *decreases* when m increases. In particular,

$$s_{2m} \leqq s_0 = a_0$$

We next observe that $s_{2m-1} = s_{2m} - a_{2m} < s_{2m} \leqq s_0 = a_0$. Since s_{2m-1} always increases with increasing m but remains less than a_0, it must approach a limit according to the fundamental principle stated in Art. 165. Also

$$s_{2m} = s_{2m-1} + a_{2m} > s_{2m-1} \geqq s_1 = (a_0 - a_1)$$

Therefore, s_{2m} always decreases with increasing m but remains greater than $(a_0 - a_1)$. By an obvious modification of the fundamental principle, s_{2m} must approach a limit.

We now show that the limits of s_{2m-1} and s_{2m} are equal. We have $s_{2m} = s_{2m-1} + a_{2m}$. Since $\lim\limits_{n \to \infty} a_n = 0$ by hypothesis, we have

$$\lim_{m \to \infty} s_{2m} = \lim_{m \to \infty} s_{2m-1} + \lim_{m \to \infty} a_{2m} = \lim_{m \to \infty} s_{2m-1}$$

The partial sum s_n therefore has a limit as n increases indefinitely, and the series converges.

Example. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$ converges, for the terms decrease in numerical value, and $\lim\limits_{n \to \infty} \frac{1}{n} = 0$.

169. Remainder in the Alternating Series. An approximation to the sum of any convergent infinite series can be obtained by adding the first few terms of the series. For instance, $u_0 + u_1 + u_2 + \cdots + u_n$ is an approximation to the sum of the series $\sum\limits_{n=0}^{\infty} u_n$. The series

$$R_n = u_{n+1} + u_{n+2} + u_{n+3} + \cdots$$

is called the *remainder* and is the error committed in using the first $n + 1$ terms of the series as an approximation to the sum. In case of a convergent *alternating series* with numerically decreasing terms

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

we have a very convenient means of finding an upper limit to the numerical value of this remainder. For if we stop with the term a_n the numerical value of the remainder is less than a_{n+1}.

To prove this, we note that

$$|R_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots$$

is the numerical value of the remainder whether n is odd or even and that this convergent series necessarily has a positive sum. This is clear since the differences in parentheses

$$(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots$$

are all positive. Furthermore

$$\begin{aligned} |R_n| &= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots \\ &= a_{n+1} - |R_{n+1}| < a_{n+1} \end{aligned}$$

Example 1. Find the upper limit to the error if eight terms are used as an approximation to the sum of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots$$

The next term is $\frac{1}{9}$, and so the error is less than $\frac{1}{9}$. Using this rule, to be sure of an error of less than $\frac{1}{100}$, we should have to take 99 terms.

Example 2. Compute the sum of the series

$$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$$

correct to three places of decimals. We have

$$\begin{array}{ll} 1 = 1.0000 & \dfrac{1}{2!} = 0.5000 \\ \dfrac{1}{4!} = 0.0417 & \_ = 0.0014 \\ \dfrac{1}{8!} = 0.0000 & \\ \overline{\ 1.0417} & \overline{\ 0.5014} \end{array}$$

and $$S = 1.0417 - 0.5014 = 0.5403 \text{ approximately}$$

Since $1/8!$ is less than the required limit of error, we are sure that the remainder after that term cannot affect the third place of decimals. The sum of the series is therefore 0.540, correct to three places of decimals.

EXERCISES

Test the following series for convergence (Ex. 1 to 10):

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

3. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$

4. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$

5. $1 - \frac{1}{2} + \frac{2}{5} - \frac{3}{10} + \frac{4}{17} - \cdots$

6. $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \cdots$

7. $\frac{1}{2} - \frac{4}{9} + \frac{9}{28} - \frac{16}{65} + \cdots$

8. $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \cdots$

9. $\dfrac{1}{2} - \dfrac{4}{1 + 2\sqrt{2}} + \dfrac{9}{1 + 3\sqrt{3}} - \dfrac{16}{1 + 4\sqrt{4}} + \cdots$

10. $1 - \frac{3}{4} + \frac{5}{8} - \frac{7}{12} + \frac{9}{16} - \cdots$

11. Find an upper limit to the error if 10 terms are used in computing an approximate value for the sum of the series in Exercise 4.

12. Same as Exercise 11 for the series in Exercise 7.

13. Compute the sum of the series $1 - \dfrac{1}{2!} + \dfrac{1}{3!} - \dfrac{1}{4!} + \cdots$, correct to five places of decimals.

14. Compute approximately the sum of the series

$$1 - \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} - \cdots$$

using 15 terms, and state an upper limit to the error.

170. Absolute and Conditional Convergence. Consider a series $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ composed of infinitely many positive and negative terms. Let the positive terms be represented by $a_1, a_2, a_3, \ldots$ and the negative terms by $-b_1, -b_2, -b_3, \ldots$. Let the sum of the first p positive terms be

$$A_p = a_1 + a_2 + a_3 + \cdots + a_p$$

and the sum of the first q negative terms be

$$-B_q = -b_1 - b_2 - b_3 - \cdots - b_q$$

The sum s_n of the first $n + 1$ terms of the original series can be represented by

$$s_n = u_0 + u_1 + u_2 + \cdots + u_n = A_p - B_q$$

where p and q have suitable values (if there are no positive terms yet present in s_n, we take $p = 0$, and $A_0 = 0$; or similarly, $B_0 = 0$).

If we now let n increase indefinitely, both p and q will increase indefinitely, and we have the following possible results:

(1) $\lim_{p\to\infty} A_p = A$ and $\lim_{q\to\infty} B_q = B$. In this case,

$$\lim_{n\to\infty} s_n = \lim_{p\to\infty} A_p - \lim_{q\to\infty} B_q = A - B = S$$

and the series converges.

(2) $\lim_{p\to\infty} A_p = +\infty$ and $\lim_{q\to\infty} B_q = +\infty$. In this case, the difference $A_p - B_q$ may or may not have a limit; the series, therefore, may or may not be convergent.

(3) $\lim_{p\to\infty} A_p = A$ and $\lim_{q\to\infty} B_q = +\infty$, or $\lim_{p\to\infty} A = +\infty$ and

$$\lim_{q\to\infty} B_q = B$$

In these cases, it is clear that $A_p - B_q$ cannot approach a limit; the series, therefore, diverges.

The following simple examples illustrate these possibilities:

In the series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots + (-1)^n \cdot \frac{1}{2^n} + \cdots$, we have

$$A_p = 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2p-2}} = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{p-1}}$$

$$B_q = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2q-1}} = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{q-1}}\right)$$

Evidently, A_p and B_q, being geometric progressions with $|r| < 1$, both approach limits as p and q become infinite. Therefore, by case (1), the original series converges.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges by the alternating-series test, but

$$A_p = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2p-1}$$

$$B_q = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2q}$$

and both diverge to $+\infty$. On the other hand, the series $1 - 1 + 1 - 1 + \cdots$ diverges, and both $A_p = 1 + 1 + 1 + \cdots + 1$ and $B_q = 1 + 1 + 1 + \cdots + 1$ diverge to $+\infty$.

The series $1 - \frac{1}{2!} + 3 - \frac{1}{4!} + 5 - \frac{1}{6!} + \cdots$ has for its general term $-1/n!$ when n is even and n when n is odd. It diverges since its general term does not approach zero. Also

$$A_p = 1 + 3 + 5 + \cdots + (2p - 1)$$

while $B_q = \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2q)!}$. Evidently, $A_p \to +\infty$ while B_q converges for p and q increasing indefinitely.

We are now ready to prove an important **theorem**: *If the series* $|u_0| + |u_1| + |u_2| + \cdots + |u_n| + \cdots$ *converges, then* $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ *also converges.*

To prove this theorem, we use the same notation as before: If u_n is a positive term, write it as a_p; if u_n is a negative term, write it as $-b_q$. The nth partial sum of the series $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ is

$$s_n = A_p - B_q$$

We wish to show that this approaches a limit as $n \to \infty$, that is, as $p \to \infty$ and $q \to \infty$.

Now, the nth partial sum of the series

$$|u_0| + |u_1| + |u_2| + \cdots + |u_n| + \cdots$$

is, clearly,

$$s'_n = A_p + B_q \qquad \text{since } |-b_i| = b_i$$

This series is convergent by the hypothesis of the theorem, and therefore s'_n approaches a limit as n becomes infinite. Now, A_p and B_q are both series of positive terms and are therefore increasing as p and q increase. Since their sum is always less than S', the limit of s'_n, A_p and B_q both are increasing and are less than a constant S' and so must approach limits. That is, $\lim\limits_{p\to\infty} A_p = A$ and $\lim\limits_{q\to\infty} B_q = B$.

Consequently $\lim\limits_{n\to\infty} s_n = \lim\limits_{p\to\infty} A_p - \lim\limits_{q\to\infty} B_q = A - B$, and the series $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ converges. In other words, if in a given series we replace each term by its absolute value and this series of absolute values converges, the series itself converges. We give a special name to such a series:

A series is said to be **absolutely convergent** *if the series of absolute values of its terms is convergent. Other convergent series are said to be* **conditionally convergent.**

For example, the series $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$ is absolutely convergent since $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ is convergent. On the other hand, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent since $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

Evidently, any convergent series of positive terms is absolutely convergent.

We may now restate the *ratio test* for application to series of positive and negative terms:

If $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \rho(u_n \neq 0)$, *then the series*

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

is absolutely convergent if $\rho < 1$ and *divergent if* $\rho > 1$, *and the test fails if* $\rho = 1$.

For proof, it is sufficient to note that if $\rho < 1$ the series of absolute

values converges, by the test as proved for series of positive terms. If $\rho > 1$, then $|u_n|$ cannot approach zero, and hence u_n cannot approach zero, as already shown.

Example. For what values of x does the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

converge? In the first place, we note that if $x = 0$ the series converges with sum 1 since every term except the first is zero. For every $x \neq 0$, the ratio test gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x| = |x|$$

The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. The case $|x| = 1$ must be investigated by other means since the ratio test fails. For $x = 1$, the series becomes

$$1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

which diverges. For $x = -1$, the series becomes

$$1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + (-1)^n\frac{1}{n} + \cdots$$

which converges conditionally. We may summarize these results by stating that the series converges for $-1 \leq x < 1$ but diverges for all other values of x.

EXERCISES

Test the following series for absolute and conditional convergence:

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
2. $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \cdots$
3. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$
4. $1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \cdots$
5. $1 - \frac{1}{3} + \frac{3}{5} - \frac{5}{7} + \cdots$
6. $\frac{3}{1} - \frac{5}{4} + \frac{7}{9} - \frac{9}{16} + \cdots$
7. $2 + 2(\frac{2}{3})^2 + 3(\frac{2}{3})^3 + 4(\frac{2}{3})^4 + \cdots$
8. $\frac{3}{4} - \frac{5}{8} + \frac{7}{12} - \frac{9}{16} + \cdots$
9. $\dfrac{1}{1} - \dfrac{1\cdot 2}{1\cdot 3} + \dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} - \dfrac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots$
10. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$
11. $\dfrac{1}{10} - \dfrac{2!}{10^2} + \dfrac{3!}{10^3} - \dfrac{4!}{10^4} + \cdots$
12. $2 + \dfrac{3}{2\sqrt{2}} - \dfrac{4}{3\sqrt{3}} + \dfrac{5}{4\sqrt{4}} - \dfrac{6}{5\sqrt{5}} + \cdots$
13. $1 - \frac{4}{2} + \frac{9}{4} - \frac{16}{8} + \frac{25}{16} - \frac{36}{32} + \cdots$
14. $1 - \dfrac{2}{2!} + \dfrac{4}{3!} - \dfrac{8}{4!} + \dfrac{16}{5!} - \cdots$

171. Operations with Series. In working with finite sums, we may group the terms, rearrange the terms in any desired order, or combine two different sums by addition, subtraction, multiplication, or division.

Under certain circumstances, these operations may be extended to infinite series. A summary of some of the admissible operations will be given in this section, but proofs will not be supplied. The student is referred to more advanced texts* for a more extensive discussion.

The convergence or the sum of a convergent series will not be disturbed if the terms are grouped without changing their order.

Multiplication of every term of a series by a constant, k, does not affect the convergence but multiplies the sum of the series by k. We have already proved this in Art. 164.

In an absolutely convergent series, the terms may be rearranged in any way without affecting the convergence or the sum of the series. On the other hand, a change in the order of terms affects the sum of a conditionally convergent series and may destroy its convergence altogether.†

Addition and Subtraction. Given two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$. To form their *sum* means to construct a new series $\sum_{n=0}^{\infty} w_n$ where

$$w_n = u_n + v_n$$

Similarly, to form their *difference* means to construct a new series where $w_n = u_n - v_n$. If the two given series are convergent with sums U and V, respectively, then the series $\sum_{n=0}^{\infty} (u_n + v_n)$ is convergent with sum $U + V$. Similarly, the series $\sum_{n=0}^{\infty} (u_n - v_n)$ is convergent with sum $U - V$.

Multiplication. Given two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$. Their *product*‡ may be written as a series $\sum_{n=0}^{\infty} w_n$ whose terms are

$$\begin{aligned}
w_0 &= u_0v_0 \\
w_1 &= u_0v_1 + u_1v_0 \\
w_2 &= u_0v_2 + u_1v_1 + u_2v_0 \\
w_3 &= u_0v_3 + u_1v_2 + u_2v_1 + u_3v_0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
w_n &= u_0v_n + u_1v_{n-1} + u_2v_{n-2} + \cdots + u_{n-1}v_1 + u_nv_0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

* For example, to Smail, *op. cit.*, or to K. Knopp, *Theory and Application of Infinite Series*, Blackie & Son, Ltd., Glasgow, 1928.

† For an example, see Smail, *op. cit.*, pp. 117–118.

‡ Referred to as their *Cauchy product* after A. L. Cauchy (1789-1857).

If the two given series converge to sums U and V, respectively, the series $\sum_{n=0}^{\infty} w_n$ represents their product if it converges to a sum $W = U \cdot V$. This will occur under the following circumstances:

1. If $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are both absolutely convergent, then $\sum_{n=0}^{\infty} w_n$ is absolutely convergent with sum $W = UV$ (Cauchy's theorem).

2. If $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are convergent and at least one is absolutely convergent, then $\sum_{n=0}^{\infty} w_n$ is convergent with sum $W = UV$ (Mertens's theorem).*

3. If $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are convergent, with sums U and V, and if $\sum_{n=0}^{\infty} w_n$ is convergent, with sum W, then $W = UV$ (Abel's theorem).†

If neither $\sum_{n=0}^{\infty} u_n$ nor $\sum_{n=0}^{\infty} v_n$ is absolutely convergent, we cannot be sure from the (conditional) convergence of these series that $\sum_{n=0}^{\infty} w_n$ will converge, as can be shown by examples.‡ But if it does converge, its sum is $W = UV$.

Division. The question of division of series is more complicated and will not be discussed. It may be noted, however, that a new series

$$\sum_{n=0}^{\infty} t_n = \frac{\sum_{n=0}^{\infty} u_n}{\sum_{n=0}^{\infty} v_n}$$

can be formed in such a way that the product (as defined above)

$$\sum_{n=0}^{\infty} t_n \cdot \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} u_n$$

* Named for F. Mertens (1840–1927).

† Named for N. H. Abel (1802–1829).

‡ See Smail, *op. cit.*, pp. 123–124.

The sum of the series $\sum_{n=0}^{\infty} t_n$ will be U/V provided that $v_0 \neq 0$ and that $\sum_{n=0}^{\infty} t_n$ and $\sum_{n=0}^{\infty} v_n$ are absolutely convergent.

It is worth remarking that absolutely convergent series behave much like finite sums, whereas, in general, conditionally convergent series do not.

Example 1. Add the two series $\sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cdot$ Each series converges; and, adding them, we get

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{n+1}{n^3}$$

Example 2. Write the first four terms of the series giving the product of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by $\sum_{n=1}^{\infty} \frac{1}{n^3}\cdot$ These series can be written

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = v_0 + v_1 + v_2 + v_3 + \cdots$$

We have

$$u_0v_0 = 1$$
$$u_0v_1 + u_1v_0 = \frac{1}{2^3} + \frac{1}{2^2} = \frac{3}{8}$$
$$u_0v_2 + u_1v_1 + u_2v_0 = \frac{1}{3^3} + \frac{1}{2^2}\cdot\frac{1}{2^3} + \frac{1}{3^2} = \frac{155}{864}$$
$$u_0v_3 + u_1v_2 + u_2v_1 + u_3v_0 = \frac{1}{4^3} + \frac{1}{2^2}\cdot\frac{1}{3^3} + \frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1}{4^2} = \frac{175}{1728}$$

The product series is, therefore

$$1 + \tfrac{3}{8} + \tfrac{155}{864} + \tfrac{175}{1728} + \cdots$$

EXERCISES

Find the series formed by the sum or difference as indicated (Ex. 1 to 6).

1. $\sum_{n=2}^{\infty} \frac{1}{n^2+1} + \sum_{n=2}^{\infty} \frac{1}{n^2-1}$

2. $\sum_{n=2}^{\infty} \frac{1}{n^2-1} - \sum_{n=2}^{\infty} \frac{1}{n^2+1}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

4. $\sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{1}{n^3}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

7. Write the first four terms of the series resulting from the product

$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots)(1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \cdots)$$

8. Write the first four terms of the series resulting from the product

$$(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \cdots)(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots)$$

9. Write the first five terms of the series resulting from the product

$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots)(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots)$$

10. Find the first four terms of the series resulting from the product

$$(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \cdots)(1 - \tfrac{1}{9} + \tfrac{1}{25} - \tfrac{1}{49} + \tfrac{1}{81} - \tfrac{1}{121} + \cdots)$$

11. Find the first five terms of the series resulting from the division

$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots) \div (1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \cdots)$$

(*Hint:* Proceed as in ordinary long division.)

12. Find the first five terms of the series resulting from the division

$$(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \cdots) \div (1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots)$$

13. Find the first five terms of the series resulting from the division

$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots) \div (1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \cdots)$$

14. Find the first five terms of the series resulting from the division

$$(1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \cdots) \div (1 - \tfrac{1}{3} + \tfrac{1}{9} - \tfrac{1}{27} + \tfrac{1}{81} - \cdots)$$

C. POWER SERIES

172. Power Series. If x is a variable and $a_0, a_1, a_2, \ldots, a_n, \ldots$ are constants, a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

is called a *power series in x*. As heretofore understood, we restrict x and $a_0, a_1, a_2, \ldots$ to real values. Certain special cases in which complex numbers play a part will be taken up later.

A power series always converges for $x = 0$, since all its terms, except possibly a_0, are zero. It may converge for no other values of x, or it may converge for some values and diverge for others, or it may converge for all x. The set of values of x for which the power series is convergent is

called its *interval of convergence*, and we can often determine this interval by use of the ratio test. We have

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \cdot |x|$$

If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = l$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = l \cdot |x|$$

This will be less than 1 if $l \cdot |x| < 1$, that is, if $|x| < 1/l$, and the series will be absolutely convergent for such values of x. Note that, if $l = 0$, then $l \cdot |x| < 1$ for *all* x. If $|x| > 1/l$, the series diverges; and if $|x| = 1/l$, the test fails.

We may summarize these results as follows: *If*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$$

exists, then the power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

is absolutely convergent within the interval $-1/l < x < 1/l$ *and divergent outside this interval. If* $l = 0$, *the power series converges for all values of* x. *If* $\left|\dfrac{a_{n+1}}{a_n}\right| \to \infty$, *the power series converges only for* $x = 0$.

The series may be convergent or divergent at the end points of the interval, and these values of x must be investigated separately. The number $1/l = r$ is sometimes called the *radius of convergence* of the power series.

Example 1. The series $1 + x + 2!x^2 + 3!x^3 + \cdots + n!x^n + \cdots$ converges for $x = 0$ but diverges for all other values of x. For

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = \lim_{x\to\infty}\frac{(n+1)!}{n!}|x| = \lim_{n\to\infty}(n+1)\cdot|x| = +\infty$$

for every x different from zero.

Example 2. The interval of convergence for the series

$$1 + 2x + 4x^2 + 8x^3 + \cdots + 2^nx^n + \cdots$$

is found as follows: $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{2^{n+1}}{2^n} = 2 = l$. Consequently, the series is absolutely convergent in the interval $-\frac{1}{2} < x < \frac{1}{2}$. If $x = \frac{1}{2}$, the series reduces to $1 + 1 + 1 + 1 + \cdots$ which diverges; if $x = -\frac{1}{2}$, it reduces to $1 - 1 + 1 - 1 + \cdots$ which also diverges. The interval of convergence includes *neither* end point.

Example 3. To find the interval of convergence of the series

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} + \cdots$$

we have $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 = l$. The series converges absolutely for $-1 < x < 1$. Now, consider the end points of this interval. If $x = 1$, the series reduces to $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$ which converges, but only conditionally, by the alternating-series test. If $x = -1$, the series becomes the harmonic series which diverges. The interval of convergence is, therefore, $-1 < x \leqq 1$ which includes *one* end point.

Example 4. Consider the series

$$1 + \frac{x}{2\cdot 1^2} + \frac{x^2}{4\cdot 2^2} + \frac{x^3}{8\cdot 3^2} + \frac{x^4}{16\cdot 4^2} + \cdots + \frac{x^n}{2^n\cdot n^2} + \cdots$$

Here $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^n\cdot n^2}{2^{n+1}(n+1)^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n}{n+1}\right)^2 = \frac{1}{2}$$

The series is, therefore, absolutely convergent within the interval $-2 < x < 2$. Furthermore, if $x = 2$, the series becomes

$$1 + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

which is (absolutely) convergent. If $x = -2$, the series becomes

$$1 - 1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots + (-1)^n\cdot\frac{1}{n^2} + \cdots$$

which is (absolutely) convergent. The interval of convergence can be written as $-2 \leqq x \leqq 2$ which includes *both* end points.

Example 5. The series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

converges for all values of x, for

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0$$

Consequently $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = 0$ for *every* x. For this reason the interval of convergence is said to be $-\infty < x < \infty$.

If we choose to do so, we may set $x = x' - x_0$. Our power series can then be written

$$a_0 + a_1(x' - x_0) + a_2(x' - x_0)^2 + a_3(x' - x_0)^3 + \cdots + a_n(x' - x_0)^n + \cdots$$

This is called a *power series in* $x' - x_0$.

Multiplying these series together, we get

$$\frac{1}{1-x^2} = 1 + (-1+1)x + (1-1+1)x^2 + (-1+1-1+1)x^3 + (1-1+1-1+1)x^4 + \cdots = 1 + x^2 + x^4 + \cdots$$

which holds for $-1 < x < 1$. The student may verify this result by finding the series for $\frac{1}{1-x^2}$ by performing the indicated division.

5. One power series may be divided by another as if they were both polynomials (see Art. 171). If we use the notation of (3), the quotient

$$\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots}{b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots} = \frac{f(x)}{g(x)}$$

provided that $b_0 \neq 0$. However, there is no simple elementary method available for finding the interval of convergence of the resulting series.

6. A power series whose sum is $f(x)$ may be differentiated term by term, and the resulting series will represent the derivative of $f(x)$ *within* the interval of convergence of the originally given series. That is, if

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

with interval of convergence $-r < x < r$, then

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots$$

with interval of convergence $-r < x < r$.

For example, consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

Differentiating term by term, we have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

The student can verify directly by use of the ratio test that both series have the interval of convergence $-1 < x < 1$. It is likewise instructive to verify that the second series results from multiplying the first series by itself and also that it arises from carrying out the division of 1 by

$$(1-x)^2 = 1 - 2x + x^2$$

It should be noted that the series for $f'(x)$ converges and represents $f'(x)$ *within* the interval of convergence of the series for $f(x)$. A given power series may converge at an end point of its interval of convergence, whereas the series obtained by differentiation may diverge at this point.

7. A power series whose sum is $f(x)$ may be integrated term by term between any limits lying within its interval of convergence, and the resulting series will represent the integral of $f(x)$ between those limits.

For example, consider

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

which converges for $-1 < x < 1$. Consequently

$$\int_0^x \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \cdot \frac{x^{n+1}}{n+1} + \cdots$$

for all x in the interval $-1 < x < 1$. Since

$$\int_0^x \frac{dx}{1+x} = \ln(1+x)$$

we have a series that represents $\ln(1+x)$ in this interval.

By means of this series for $\ln(1+x)$, an interesting application of Abel's theorem (1) can be given. The series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

converges for $x = 1$; and since by (1) the power series represents a function that is continuous from the left at this end point, we have

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \lim_{x \to 1^-} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right) \\ &= \lim_{x \to 1^-} \ln(1+x) = \ln 2 \end{aligned}$$

It may be mentioned that, in the interior of its interval of convergence, a power series behaves in many ways like a finite polynomial.

EXERCISES

1. By division, $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Find a series for $\dfrac{1}{1-x^2}$ by adding these two series. Find the interval of convergence.

2. By division, find series for $\dfrac{x}{1+x^2}$ and $\dfrac{x}{1-x^2}$. Add the resulting series to get a series for $\dfrac{x}{1-x^4}$. Find the interval of convergence.

3. Add the power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ and

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

and find the interval of convergence of each series involved.

4. Subtract the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ from the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

and find the interval of convergence for each series involved.

5. Multiply the series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ by twice the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and find the interval of convergence of each series involved.

6. Multiply the series for $\frac{x}{1 + x^2}$ by the series for $\frac{x}{1 - x^2}$ to obtain a series for $\frac{x^2}{1 - x^4}$. Verify your result by dividing x^2 by $1 - x^4$. Compare with Exercise 2, and find the interval of convergence.

7. Find the series resulting from the division of

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

by $1 - x + x^2 - x^3 + x^4 - \cdots$. Find the interval of convergence of each series.

8. Find the series resulting from the division of

$$1 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

by $1 + x + x^2 + x^3 + x^4 + \cdots$, and find the interval of convergence of each series.

9. If $f(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$, find $f'(x)$ and its interval of convergence.

10. If $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots$, find $f'(x)$ and its interval of convergence.

D. EXPANSION OF FUNCTIONS

174. Maclaurin's Series. In Art. 83, we saw how a function that possesses n derivatives in an interval including $x = 0$ can be approximated by a polynomial, namely,

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} \cdot x^{n-1} + \frac{f^{(n)}(\xi)}{n!} \cdot x^n \qquad \text{for } \xi \text{ between 0 and } x \qquad (1)$$

The error, or *remainder* after n terms, is $R_n = \frac{f^{(n)}(\xi)}{n!} x^n$. Now, suppose that $f(x)$ possesses derivatives of all orders in an interval including $x = 0$. We need not stop, as in (1), with $n + 1$ terms but may allow n to increase indefinitely. The series (1) then becomes a power series, and we can state the following **theorem**: *The power series*

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \cdots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \cdots \quad (2)$$

represents the function $f(x)$ for those, and only those, values of x for which R_n approaches zero as n becomes infinite.

The function is said to be expanded in Maclaurin's series about $x = 0$.

It should be noted that the series (2) might converge for values of x for which R_n does not approach zero. Then the series, though convergent, does not represent the function for the values of x in question.

Example 1. In Example 1, Art. 83, $\sin x$ was expanded by Maclaurin's theorem to give

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^8}{8!} \sin \xi \qquad \text{for } \xi \text{ between 0 and } x$$

We now find the interval in which $\sin x$ can be represented by Maclaurin's series. We have at once

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!} \sin \xi \qquad \text{for } \xi \text{ between 0 and } x$$

The numerical value of the remainder is

$$\left| \frac{x^{2n}}{(2n)!} \right| \cdot |\sin \xi| = |R_n|$$

Now $|\sin \xi| \leqq 1$, and consequently $|R_n| \leqq |x^{2n}|/(2n)!$. But we saw in Example 3, Art. 167, that $\lim_{n \to \infty} \frac{k^n}{n!} = 0$ for any positive k. Taking $k = |x|$, we get $R_n \to 0$ when $n \to \infty$ for any fixed value of x, positive or negative. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

therefore represents $\sin x$, for all values of x, that is, for $-\infty < x < \infty$. Note that this interval coincides with the interval of convergence of the series.

Example 2. Expand e^x by Maclaurin's series, and find the interval in which the series represents the function. We have

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ &\cdots\cdots & &\cdots\cdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

Hence, Maclaurin's development gives

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and the remainder is $\frac{e^\xi}{n!}x^n = R_n$ where ξ is between 0 and x. The interval of convergence is easily found by the ratio test, for

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = \lim_{n\to\infty}\frac{n!}{(n+1)!}\cdot|x| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

so that the series converges for every x, that is, for $-\infty < x < \infty$.

We now consider the remainder. First, let x be any given positive number. Then

$$|R_n| = \frac{e^\xi}{n!}x^n \qquad \text{where } 0 < \xi < x$$

Hence $|R_n| < e^x(x^n/n!)$. Since x is a *fixed* number, e^x is some fixed constant. We have already noted that $x^n/n! \to 0$ when $n \to \infty$ for any fixed x. Consequently, $R_n \to 0$, and the series represents the function for all positive x.

Next, if $x = 0$, the series reduces to the first term, and $e^0 = 1$; therefore, the series represents the function.

Finally, let x be any negative number. Then

$$|R_n| = e^\xi \cdot \frac{|x|^n}{n!} \qquad \text{where } x < \xi < 0$$

Since $e^\xi < 1$,

$$|R_n| < \frac{|x|^n}{n!} \to 0 \qquad \text{as } n \to \infty$$

Hence $R_n \to 0$ as $n \to \infty$, and the series represents the function for any negative x.

Summarizing these results, we see that the series represents e^x for every value of x and that this interval coincides with the interval of convergence of the series.

175. Taylor's Series. If we wish to expand $f(x)$ in powers of $(x - a)$, we may use Taylor's series, provided that $f(x)$ possesses derivatives of all orders in an interval including $x = a$. We may then state the following **theorem:** *The power series*

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n + \cdots$$

represents the function $f(x)$ for those, and only those, values of x for which R_n approaches zero as n becomes infinite.

Here $R_n = \frac{f^{(n)}(\xi)}{n!}(x - a)^n$ (ξ between a and x), as found in Art. 82. In simple cases the interval of convergence of the series coincides with the interval in which the remainder term approaches zero.

As in Art. 82, we may write $x - a = h$ and obtain for Taylor's series

$$f(a + h) = f(a) + f'(a) \cdot h + \frac{f''(a)}{2!} \cdot h^2 + \cdots + \frac{f^{(n)}(a)}{n!} \cdot h^n + \cdots$$

a power series in h.

We may note again that Maclaurin's series is a special case of Taylor's series, namely, $a = 0$.

EXERCISES

Obtain the following expansions of functions by Maclaurin's series and find the interval of convergence (Ex. 1 to 10):

1. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all values of x

2. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$ for all values of x

3. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ for all values of x

4. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ for all values of x

5. $$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + \cdots$$

where m is a fixed constant. This is known as the *binomial series*. Show that it converges for $-1 < x < 1$.

6. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \leqq 1$

7. $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ for $-1 \leqq x < 1$

8. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \leqq x \leqq 1$

9. $\arcsin x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots$ for $-1 < x < 1$

10. $a^x = 1 + x \ln a + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \cdots$ for all values of x

11. Using the results of Exercises 6 and 7, show that

$$\ln \frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \quad \text{for } -1 < x < 1$$

12. Expand $1/\sqrt{1-x^2}$ by the binomial series (Exercise 5); then verify the result of Exercise 9 by finding $\int_0^x \frac{dx}{\sqrt{1-x^2}}$.

13. Expand $\frac{1}{1+x^2}$ by division; then verify the result of Exercise 8 by finding $\int_0^x \frac{dx}{1+x^2}$.

14. Expand $\frac{1}{1-x^2}$ by division, and use the fact that

$$\int_0^x \frac{dx}{1-x^2} = \operatorname{argtanh} x$$

to show that

$$\operatorname{argtanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad \text{for } -1 < x < 1$$

15. Expand $1/\sqrt{1+x^2}$ by the binomial series (Exercise 5). Use the fact that $\int_0^x \frac{dx}{\sqrt{1+x^2}} = \operatorname{argsinh} x$ to show that

$$\operatorname{argsinh} x = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9} - \cdots$$

for $-1 < x < 1$.

16. Use Maclaurin's series to show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

17. Use the series for $\sin x$ and $\cos x$ to derive the result of Exercise 16 by computing $\frac{\sin x}{\cos x}$.

18. Use Maclaurin's series to show that

$$\ln \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \cdots$$

19. Derive the result of Exercise 16 by differentiating the series of Exercise 18.

20. Show that $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$.

21. By multiplication of series, show that

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots$$

22. Show that $\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \cdots$.

23. Show that

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}\left(1 + \sqrt{3}\,x - \frac{x^2}{2!} - \sqrt{3}\,\frac{x^3}{3!} + \frac{x^4}{4!} + \sqrt{3}\,\frac{x^5}{5!} - \cdots\right)$$

24. Use the series for $\sin x$ to calculate $\sin 2°$ to five places of decimals. Check by reference to a table of sines. (*Hint*: $2° = 0.0349$ radian.)

25. Calculate $\cosh \frac{1}{10}$ to five places of decimals. Check by reference to a table of hyperbolic cosines.

26. Find a series whose sum is π by using the result of Exercise 8 and setting $x = 1$.

27. Find another series whose sum is π by setting $x = \frac{1}{2}$ in Exercise 9.

28. Show that $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \pi/4$. Use the result of Exercise 8 to show that

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7}\right) + \cdots$$

29. Using the series for e^x, calculate $\sqrt{e}$ to five places of decimals.

30. The binomial series (Exercise 5) can be used conveniently to make certain approximate computations. For example, find $\sqrt{408}$. Here,

$$\sqrt{408} = (400 + 8)^{1/2} = 20(1 + \tfrac{8}{400})^{1/2} = 20(1 + \tfrac{1}{50})^{1/2}$$

Expand $(1 + \frac{1}{50})^{1/2}$ by the binomial series, using $x = \frac{1}{50} = 0.02$ and $m = \frac{1}{2}$. Complete the calculation.

31. Use the binomial series to find $\sqrt{391}$ (see Exercise 30).

32. Use the binomial series to find $\sqrt{615}$ (see Exercise 30).

33. Use the binomial series to find $\sqrt{112}$ (see Exercise 30).

34. Use the binomial series to find $\sqrt[3]{1030}$.

35. Use the binomial series to find $\sqrt[3]{517}$.

36. Calculate $\displaystyle\int_0^{0.2} \frac{dx}{\sqrt{1 - x^2}}$ to five places of decimals by expanding the integrand into a power series.

37. Calculate $\displaystyle\int_0^{0.2} \frac{dx}{\sqrt{1 + x}}$ to five places of decimals by expanding the integrand into a power series.

38. Verify that

$$\int_0^x \sin x^2\, dx = \frac{1}{3}x^3 - \frac{1}{7}\cdot\frac{x^7}{3!} + \frac{1}{11}\cdot\frac{x^{11}}{5!} - \frac{1}{15}\cdot\frac{x^{15}}{7!} + \cdots$$

39. In Exercises 1 to 4, verify that R_n approaches 0 as n becomes infinite for all x in the interval of convergence of the series, and that, consequently, Maclaurin's series represents the function throughout this interval.

40. (*a*) Suppose $f(x)$ to be a function having the power series expansion

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

By calculating $f(0)$, $f'(0)$, $f''(0)$, . . . $f^{(n)}(0)$, . . . show that $c_0 = f(0)$, $c_1 = f'(0)$, $c_2 = \dfrac{f''(0)}{2!}$, . . . , $c_n = \dfrac{f^{(n)}(0)}{n!}$, . . . , and so discover the coefficients in Maclaurin's series.

(*b*) The same as (*a*), but suppose

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots$$

and so discover the coefficients in Taylor's series.

176. Trigonometric and Exponential Functions. The reader will recall (Art. 53) the definition of the hyperbolic functions in terms of exponential functions. We shall now discuss relations between the trigonometric or circular functions and the exponential functions. In Example 2, Art. 174, we saw that the series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \tag{3}$$

converges and represents e^z for all real values of z. We now make use of an important general principle of analysis. Although the exponential function e^z has as yet no definition for complex values of z, the infinite series on the right-hand side of (3) can be used to *define* e^z for arbitrary complex values of z.

As a consequence of this definition, we put $z = iy$, where y is real, and obtain

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots + \frac{(iy)^n}{n!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \cdots \end{aligned}$$

Separating real and imaginary parts, we recognize that the real terms furnish $\cos y$ and the imaginary terms $\sin y$; thus,

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

and $$e^{iy} = \cos y + i \sin y \qquad \text{for all real values of } y^* \tag{4}$$

Replacing y by $-y$ in (4), we have at once

$$e^{-iy} = \cos y - i \sin y \qquad \text{for all real values of } y \tag{5}$$

Now, subtract equation (5) from equation (4):

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \tag{6}$$

Next, add equations (4) and (5):

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \tag{7}$$

From equations (6) and (7), we can obtain expressions for the other trigonometric functions in terms of the exponential functions.

Until now, we have had no definitions for trigonometric functions of a complex variable. We use the above relations and define, for z complex,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{8}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{9}$$

and similarly for the other trigonometric functions.

* This is usually known as *Euler's theorem*, named for Leonhard Euler (1707–1783).

Note particularly that these definitions are *analytic;* that is, z is an arbitrary real or complex number and need not be the measure of an angle at all.

EXERCISES

Using the definitions of $\sin z$ and $\cos z$ in terms of exponential functions, prove the following identities (Ex. 1 to 7):

1. $\sin^2 x + \cos^2 x = 1$ **2.** $\sin 2x = 2 \sin x \cos x$
3. $\cos 2x = \cos^2 x - \sin^2 x$
4. $\sin (x + y) = \sin x \cos y + \cos x \sin y$
5. $\cos (x + y) = \cos x \cos y - \sin x \sin y$
6. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ **7.** $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

8. Show that $e^{x+iy} = e^x(\cos y + i \sin y)$.

9. Show that $e^{2k\pi i} = 1$ where k is any integer. Hence, show that $e^{z+2k\pi i} = e^z$ and that therefore e^z is *periodic* with period $2\pi i$.

10. By use of Euler's theorem, show that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta^*$$

177. Comparison of Trigonometric and Hyperbolic Functions. Since we now have definitions for the trigonometric functions of a complex variable, let us set $z = ix$ (x real) in the formulas (8) and (9) of the preceding section. We obtain

$$\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^x}{2i} \qquad \text{since } i^2 = -1$$

Multiplying numerator and denominator by i, we get

$$\sin ix = i\left(\frac{e^x - e^{-x}}{2}\right) = i \sinh x$$

Similarly

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x$$

Relations between the other trigonometric functions of ix and hyperbolic functions are easily formulated.

In Art. 53, the hyperbolic functions were defined in terms of the exponential functions of a *real* variable. Now that we have defined the exponential function of a *complex* variable, we can extend the definitions of the hyperbolic functions. If z is complex, we define

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

The other hyperbolic functions are defined as before in terms of $\sinh z$ and $\cosh z$.

* This formula is called *de Moivre's theorem* and is named for Abraham de Moivre (1667–1754).

Now, set $z = ix$ in the foregoing formulas:

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = i \sin x$$

$$\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

Relations of a similar character can readily be found between the other hyperbolic and trigonometric functions.

In Art. 53, it was stated that the hyperbolic functions are connected with a rectangular hyperbola in a way comparable with that in which the trigonometric or circular functions are connected with a circle. This comparison will now be made.

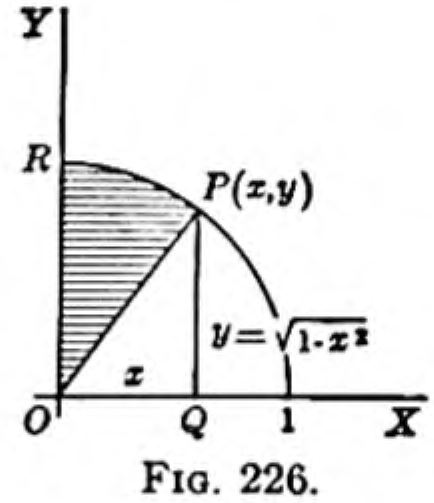


FIG. 226.

1. *Circle.* Consider the circle with radius 1 and center at the origin. Its equation is

$$x^2 + y^2 = 1$$

We calculate the area $OQPR$ under the circle (Fig. 226). It is

$$\int_0^x \sqrt{1 - x^2}\, dx = \tfrac{1}{2}x\sqrt{1 - x^2} + \tfrac{1}{2}\arcsin x$$

But since $OQ = x$ and $QP = \sqrt{1 - x^2}$, we see that the area of the triangle OQP is $\frac{1}{2}x\sqrt{1 - x^2}$. Hence, the area A of the (shaded) *circular sector* OPR is $A = \frac{1}{2}\arcsin x$, and we may write

$$\arcsin x = 2A = u$$

Therefore, $\quad x = \sin u \quad$ and $\quad y = \sqrt{1 - x^2} = \cos u$

These are the parametric equations of the circle where the parameter u represents twice the area of the shaded circular sector.

2. *Hyperbola.* Consider the rectangular hyperbola with center at the origin and semiaxes equal to 1. Its equation is

$$y^2 - x^2 = 1$$

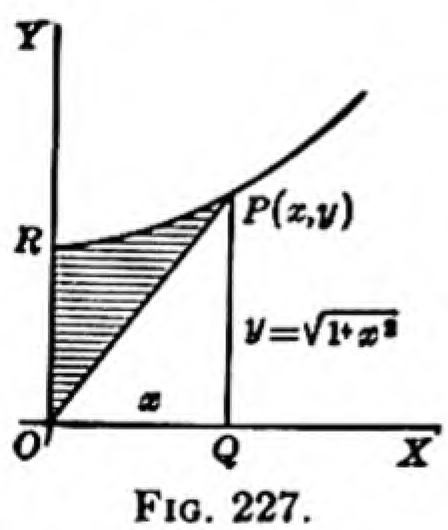


FIG. 227.

We calculate the area $OQPR$ under the hyperbola (Fig. 227). The student may verify that it is

$$\int_0^x \sqrt{1 + x^2}\, dx = \tfrac{1}{2}x\sqrt{1 + x^2} + \tfrac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right) = \tfrac{1}{2}x\sqrt{1 + x^2} + \tfrac{1}{2}\operatorname{argsinh} x$$

(see Arts. 92 and 93). But since $OQ = x$ and $QP = \sqrt{1 + x^2}$, we see that the area of the triangle OQP is $\frac{1}{2}x\sqrt{1 + x^2}$. Hence, the area B of the (shaded) *hyperbolic sector* OPR is $B = \frac{1}{2}\operatorname{argsinh} x$, and we may write

$$\operatorname{argsinh} x = 2B = v$$

Therefore $\quad x = \sinh v \quad$ and $\quad y = \sqrt{1 + x^2} = \cosh v$

These are the parametric equations of the hyperbola where the parameter v represents twice the area of the shaded hyperbolic sector.

3. Let φ be a function of x defined by the equation

$$\varphi = \arctan \sinh x$$

where the arc tangent has its principal value, $-\pi/2 < \varphi < \pi/2$. This function is called the *Gudermannian of* x* and is denoted by the symbol $\varphi = \text{gd } x$. It is sometimes called the *hyperbolic amplitude* and written $\varphi = \text{amh } x$. We have at once

$$\sinh x = \tan \varphi = \tan \text{gd } x$$

EXERCISES

Verify the following relations between circular (trigonometric) and hyperbolic functions (Ex. 1 to 9):

1. $\sin x = -i \sinh ix$
2. $\tan x = -i \tanh ix$
3. $\cot x = i \coth ix$
4. $\sec x = \text{sech } ix$
5. $\csc x = i \text{ csch } ix$
6. $\tanh x = -i \tan ix$
7. $\coth x = i \cot ix$
8. $\sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$
9. $\cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y$

10. Show that $\sinh (x + 2k\pi i) = \sinh x$, so that $\sinh x$ has period $2\pi i$ (k an integer).

11. Show that $\cosh (x + 2k\pi i) = \cosh x$, so that $\cosh x$ has period $2\pi i$ (k an integer).

12. Find the period of $\tanh x$, $\coth x$, $\text{sech } x$, $\text{csch } x$.

13. Find $\sinh (x + \pi i)$, $\cosh (x + \pi i)$, $\tanh (x + \pi i)$.

14. Find $\sinh \left(x + \frac{\pi}{2} i\right)$, $\cosh \left(x + \frac{\pi}{2} i\right)$, $\tanh \left(x + \frac{\pi}{2} i\right)$.

MISCELLANEOUS EXERCISES

1. Find the general term of the series

$$\ln 2 + \ln 3 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots$$

and express in the form Σa_n.

2. Find the general term of the series

$$-3 - \tfrac{1}{2} + 1 + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{9} + \tfrac{1}{12} + \cdots$$

and express in the form Σa_n.

3. Show that any "arithmetic series," namely,

$$a + (a + d) + (a + 2d) + \cdots$$

is divergent.

* Named for C. Gudermann (1798–1851).

Test for convergence by any available method (Ex. 4 to 20):

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

7. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{2}{4\cdot 5\cdot 6} + \dfrac{3}{7\cdot 8\cdot 9} + \dfrac{4}{10\cdot 11\cdot 12} + \cdots$

8. $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{9}} + \dfrac{1}{\sqrt{28}} + \dfrac{1}{\sqrt{65}} + \dfrac{1}{\sqrt{126}} + \cdots$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^k}{a^n}$ (k a positive integer and $a > 1$)

11. $\dfrac{1}{1+\sqrt{1}} + \dfrac{1}{4+\sqrt{2}} + \dfrac{1}{9+\sqrt{3}} + \dfrac{1}{16+\sqrt{4}} + \cdots$

12. $\dfrac{1}{1} + \dfrac{1\cdot 3}{1\cdot 4} + \dfrac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7} + \dfrac{1\cdot 3\cdot 5\cdot 7}{1\cdot 4\cdot 7\cdot 10} + \cdots$

13. $\dfrac{2}{1} + \dfrac{2\cdot 4}{1\cdot 5} + \dfrac{2\cdot 4\cdot 6}{1\cdot 5\cdot 9} + \dfrac{2\cdot 4\cdot 6\cdot 8}{1\cdot 5\cdot 9\cdot 13} + \cdots$

14. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \cdots$

15. $1 - \frac{3}{4} + \frac{5}{6} - \frac{7}{8} + \cdots$

16. $1 + \dfrac{3}{1\cdot 2} - \dfrac{5}{2\cdot 3} + \dfrac{7}{3\cdot 4} - \dfrac{9}{4\cdot 5} + \cdots$

17. $\dfrac{1}{2} - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt[3]{2}} - \dfrac{1}{\sqrt[4]{2}} + \cdots$

18. $\frac{1}{3} - \frac{3}{6} + \frac{5}{9} - \frac{7}{12} + \cdots$

19. $1 - \dfrac{3}{2!} + \dfrac{5}{3!} - \dfrac{7}{4!} + \cdots$

20. $4 - \dfrac{4}{\sqrt{2}} + \dfrac{4}{\sqrt{3}} - \dfrac{4}{\sqrt{4}} + \cdots$

21. Find an upper limit to the error if 10 terms are used in computing an approximate value for the sum of the series in Exercise 19. Find the sum to 10 terms.

22. Compute the sum $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \dfrac{1}{9!} - \cdots$ correct to five places of decimals.

Test for absolute or conditional convergence (Ex. 23 to 29).

23. $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$

24. $1 - \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} - \dfrac{1}{\sqrt{7}} + \cdots$

25. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots$

26. $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \cdots$

27. $1 - \dfrac{1}{1 \cdot 2} + \dfrac{1}{3 \cdot 4} - \dfrac{1}{5 \cdot 6} + \cdots$

28. $1 + \dfrac{1}{3!} - \dfrac{1}{5!} + \dfrac{1}{7!} - \cdots$

29. $1 - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \frac{1}{26} - \cdots$

Find the values of x for which the following power series are convergent (Ex. 30 to 36):

30. $1 - x + x^2 - x^3 + x^4 - \cdots$

31. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

32. $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots$

33. $1 + \dfrac{x}{3} + \dfrac{2!x^2}{9} + \dfrac{3!x^3}{27} + \dfrac{4!x^4}{81} + \cdots$

34. $\dfrac{3x}{2} + \dfrac{9x^2}{5} + \dfrac{27x^3}{10} + \dfrac{81x^4}{17} + \cdots$

35. $1 + \dfrac{2}{3}(x - 3) + \dfrac{2^2}{3^2}(x - 3)^2 + \dfrac{2^3}{3^3}(x - 3)^3 + \cdots$

36. $1 + \dfrac{x + 1}{1 \cdot 2} + \dfrac{(x + 1)^2}{2 \cdot 2^2} + \dfrac{(x + 1)^3}{3 \cdot 2^3} + \dfrac{(x + 1)^4}{4 \cdot 2^4} + \cdots$

37. Show that $(a + b)^m$ can be expressed as a power series by taking $x = b/a$.

38. Show that $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!} - \cdots$ for all values of x.

39. Use the result of Exercise 38 to find a series for $\int_0^x e^{-x^2}\,dx$ for all values of x.

40. Use Maclaurin's series to show that $\sec x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$.

41. Find a series for $\ln (x + \sqrt{1 + x^2})$ by use of integration.

42. Show that $x \sin x = x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!} - \dfrac{x^8}{7!} + \cdots$.

43. Calculate $\sin 10°$ to five places of decimals by use of the series for $\sin x$. Check by a table of sines.

44. Calculate $\cos 9°$ to five places of decimals.

45. Calculate $\int_0^{0.2} e^{-x^2}\,dx$ to five places of decimals, using the result of Exercise 39.

46. Assuming that $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ is convergent, use the series for $\sin x$ to obtain an approximate value for this integral.

47. Show that $e^{\sin x} = 1 + x + \dfrac{x^2}{2!} - \dfrac{3x^4}{4!} - \dfrac{8x^5}{5!} - \cdots$.

48. Show that $e^{\cos x} = e\left(1 - \dfrac{x^2}{2!} + \dfrac{4x^4}{4!} - \dfrac{31x^6}{6!} + \cdots\right)$.

CHAPTER 20

INTRODUCTION TO DIFFERENTIAL EQUATIONS

178. Differential Equations. As the name implies, a *differential equation* is an equation involving differentials or derivatives. We have already encountered simple types of such equations. For instance, in Example 1, Art. 87, we had an equation $\frac{dy}{dx} = 2x$ from which y was to be found as a function of x. In Example 2 of the same section, we used the differential equation $\frac{d^2y}{dt^2} = -32$ to express the fact that the acceleration of a vertically thrown projectile is -32 ft./sec.^{2} In Example 6, Art. 89, we encountered the equation $\frac{dx}{dt} = kx$ in finding the law of natural growth.

The fact that the derivative represents the slope of a curve indicates that the solution of certain geometrical problems, in which information is given about slopes, will involve differential equations. Since velocity and acceleration are first and second derivatives of distance with regard to time, differential equations will play an essential role in the study of dynamics. Many other important applications might be mentioned.

The following are additional examples of differential equations:

$$(2xy^3 + 16x^3y)\,dx + (3x^2y^2 + 4x^4 + 5y^4)\,dy = 0 \qquad \text{(Art. 145)} \qquad (1)$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad (2)$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \qquad (3)$$

$$(y^2 + x)\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 = 7 \qquad (4)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad (5)$$

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} - z^2\frac{\partial u}{\partial z} = 0 \qquad (6)$$

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad (7)$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x \qquad (8)$$

A differential equation involving a single independent variable, and hence only ordinary derivatives, is called an *ordinary differential equation*. If there are two or more independent variables present, so that the equation contains partial derivatives, it is called a *partial differential equation*. Equations (1), (2), (3), (4), (7), and (8) are ordinary and equations (5) and (6) are partial differential equations.

We classify differential equations according to *order* and *degree*. The order of the highest derivative present is called the *order* of the differential equation. Thus, (1), (3), and (6) are of first order; (2), (4), (5), and (7) are of second and (8) of third order. If the equation can be rationalized and cleared of fractions in regard to all the derivatives, the *degree* of the differential equation is the degree of the highest ordered derivative present. This rationalization may not be possible, in which case the term "degree" has no meaning. Here (1), (2), (4), (5), (6), and (8) are of first degree, and (3) and (7) are of second degree. Note that (4) is of first degree because the highest ordered derivative, namely, the second, enters to the first power. The fact that $\frac{dy}{dx}$ is raised to the second power is irrelevant. In (7), we first square both sides to rationalize the equation. The highest ordered derivative is raised to the second power, and the equation is of second degree.

A *solution* of an ordinary differential equation means a function of the *independent* variable, say $f(x)$, such that, when this function is substituted for y and its successive derivatives for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ in the differential equation, this equation is *satisfied* in the sense that it becomes an identity in x.

Thus, the function $f(x) = A \sin x + B \cos x$ (A and B any constants) and its second derivative when substituted for y and $\frac{d^2y}{dx^2}$ in equation (2) above produce the following result:

$$y = A \sin x + B \cos x \qquad \frac{d^2y}{dx^2} = -A \sin x - B \cos x$$

Hence
$$\frac{d^2y}{dx^2} + y = A \sin x + B \cos x - A \sin x - B \cos x$$
$$= 0 \text{ identically}$$

We say, therefore, that this function $f(x)$ is a solution of the differential equation.

Again, the function $\sqrt{2Cx + C^2}$ is a solution of (3).

The solution of an ordinary differential equation may be a function of x that is given implicitly by an equation $F(x,y) = 0$. The relation $F(x,y) = 0$ determines a solution of the given differential equation under the following circumstances:

1. The relationship $F(x,y) = 0$ must define a function of x (known or unknown), say $y = f(x)$. The first derivative $\frac{dy}{dx}$, and derivatives of higher order, can then be found in terms of both x and y.

2. When the expressions for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are substituted into the differential equation, the result must be either an identity in x and y or an equation satisfied by all values of x and y that satisfy the relationship $F(x,y) = 0$.

For example, the relation

$$F(x,y) = x^2y^3 + 4x^4y + y^5 + C = 0$$

defines a function $y = f(x)$ that is a solution of the differential equation (1).

From this discussion, we see that a solution of an ordinary differential equation may involve arbitrary constants. In fact, it is shown in books on differential equations* that the most general solution of an ordinary differential equation of nth order involves exactly n arbitrary constants.† Such a solution is called the *general* or *complete* solution.

The existence of such a solution for every ordinary differential equation (under certain restrictions) is proved in more advanced texts. There may be solutions that are distinct from the general solution, but we shall not discuss them. A *particular solution*, or *particular integral*, is obtained by assigning definite values to the arbitrary constants. It is to be noted that finding the integral $\int f(x)\,dx$ is equivalent to solving the differential equation $\frac{dy}{dx} = f(x)$. To distinguish this case from the solution of differential equations in general, we call such ordinary integrations *quadratures*.

As in our study of the problem of integration, we shall see that there is no general straightforward process by which the solutions of all differential equations can be found. We shall consider only some of the methods for finding the general solutions of certain simple but useful types of *ordinary differential equations*. We shall not take up the study of partial differential equations at all.

EXERCISES

State the order and, where the term has meaning, the degree of the following differential equations (Ex. 1 to 10):

* For example, in A. R. Forsyth, *A Treatise on Differential Equations*, 5th ed., The Macmillan Company, New York, 1921.

† This means, of course, that the n constants cannot be replaced by a smaller number of constants. For instance, $y = ax + b + \ln c$ involves essentially only two constants, for $b + \ln c$ is simply one constant and could be written $b + \ln c = k$.

1. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = -4\sin x - 2\cos x$

2. $\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + x = 0$

3. $\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$

4. $\left(\dfrac{dy}{dx}\right)^2 + y^2 - 1 = 0$

5. $\dfrac{d^3y}{dx^3} + 3\left(\dfrac{d^2y}{dx^2}\right)^2 + \left(\dfrac{dy}{dx}\right)^3 + xy = 0$

6. $\dfrac{d^4y}{dx^4} = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}$

7. $\sqrt{\dfrac{d^2y}{dx^2}} + \dfrac{dy}{dx} + xy^2 = 0$

8. $e^{\frac{dy}{dx}} + \dfrac{dy}{dx} = x$

9. $\ln\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + y = x$

10. $(x - y)\,dx + x\,dy = 0$

Verify that the following are solutions of the indicated differential equations (Ex. 11 to 18). Arbitrary constants are denoted by $C_1, C_2, C_3, \ldots$.

11. The equation of Exercise 1: $y = C_1e^{-x} + C_2e^{3x} + \sin x$
12. The equation of Exercise 4: $y = \sin(x + C)$
13. The equation of Exercise 3: $4(x^2 + y)^3 - (2x^3 + 3xy + C)^2 = 0$
14. The equation of Exercise 10: $\dfrac{y}{x} + \ln|x| = C$
15. $(x^2 + y^2 + y)\,dx - x\,dy = 0$: $x - \arctan(y/x) = C$
16. $\dfrac{d^2y}{dx^2} + y = 2e^x$: $y = C_1\sin x + C_2\cos x + e^x$
17. $\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 1$: $y = C_1 + \ln|\sec(x + C_2)|$
18. $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = 2x - 3 + 4\sin x + 2\cos x$:

$$y = (C_1 + C_2x)e^x + C_3e^{-2x} + x + \cos x$$

179. Differential Equation Found from the Primitive. Although we shall be largely concerned with finding the solution of a given differential equation, it is frequently necessary to answer the question "Given a relation between x and y involving n arbitrary constants, what is the lowest ordered differential equation of which this is the general solution?" The given relation is called the *primitive* giving rise to the differential equation. In general, this differential equation can be found by differentiating the primitive n times and then eliminating the n constants from the $n + 1$ equations. The method will be indicated by examples.

Example 1. We shall find the differential equation arising from the primitive

$$x^2 + y^2 - 2cx = 0 \tag{9}$$

in which c is an arbitrary constant. Differentiating with respect to x, we have

$$2x + 2y\frac{dy}{dx} - 2c = 0 \tag{10}$$

From (10), we find that

$$c = x + y\frac{dy}{dx}$$

Replacing c in (9) by this, we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

or

$$y^2 - x^2 - 2xy\frac{dy}{dx} = 0 \tag{11}$$

The student may observe that we might have eliminated the constant c by another differentiation. From (10), we should have

$$1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

But this is not the *lowest ordered* differential equation that can be obtained.

Notice that (9) is the equation of a family of circles with centers on the x axis (at c, 0) and passing through the origin. We call (11) the *differential equation of the family.*

Example 2. We shall find the differential equation of the family of central conics

$$Ax^2 + By^2 = 1 \tag{12}$$

(A and B arbitrary constants). Differentiating twice with respect to x, we obtain

$$Ax + Byy' = 0 \tag{13}$$
$$A + B(y')^2 + Byy'' = 0 \tag{14}$$

From (14), we have $A = -B(y'^2 + yy'')$. Substituting this into (13), we get

$$xyy'' + xy'^2 - yy' = 0$$

a differential equation of second order. This is to be expected, since the primitive (12) contained two arbitrary constants.

EXERCISES

Find the differential equations corresponding to the following primitives (Ex. 1 to 10):

1. $y = A \sin x + B \cos x$
2. $y = Cx + e^x$
3. $y = x + Ce^x$
4. $y = C_1e^x + C_2e^{-x}$
5. $y = C_1x^2 + C_2$
6. $x^2y + x = Cy$
7. $y = Ae^x + Be^{2x} + Ce^{3x}$
8. $y = (C_1 + C_2x)e^{2x}$
9. $y = \cos(x + C)$
10. $y = \cos(x + C) + \sin(x + C)$

11. Find the differential equation of the family of lines passing through the origin.

12. Find the differential equation of the family of hyperbolas having the coordinate axes as asymptotes.

13. Find the differential equation of the family of parabolas with vertex at the origin and axis on the x axis.

14. Find the differential equation of the family of circles with centers on the x axis.

15. Find the differential equation of the family of circles with centers at the origin.

16. Find the differential equation of the family of parabolas with their foci at the origin and their axes lying along the x axis.

A. DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

We shall consider certain types of differential equations of first degree only. Methods for finding solutions of those of first order and higher degree are given in more advanced texts.

180. Exact Equation. The most general form of a differential equation of first order and first degree is

$$M\,dx + N\,dy = 0 \tag{15}$$

that is,

$$M + N\frac{dy}{dx} = 0$$

where M and N are functions of x and y. In Art. 145 it was stated that, if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then (15) is the differential of some function $z = f(x,y)$. That is, (15) is equivalent to the equation $dz = 0$ whose general solution is $z = c$, or $f(x,y) = c$, where c is an arbitrary constant. We shall now prove this statement by finding the function $z = f(x,y)$. Such a method is illustrated in Art. 145. The student should study this illustrative example again. The steps required in finding the solution are the following (observe that they are the steps taken in Art. 145). Keep in mind that *M and N are given*, and that they satisfy the equation $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

1. Integrate M with regard to x, holding y fixed.* The result will be designated by z:

$$z = \int_x M\,dx + \varphi(y) \tag{16}$$

where $\varphi(y)$ is an arbitrary function of y alone.

2. Denote by A the difference $N - \dfrac{\partial}{\partial y}\left(\displaystyle\int_x M\,dx\right) = A$. Observe that

$$\frac{\partial A}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\,\partial y}\left(\int_x M\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\,\partial x}\left(\int_x M\,dx\right)$$

since the order of differentiation is immaterial. But $\dfrac{\partial}{\partial x}\displaystyle\int_x M\,dx = M$, so that $\dfrac{\partial^2}{\partial y\,\partial x}\left(\displaystyle\int_x M\,dx\right) = \dfrac{\partial M}{\partial y}$. *Consequently,* $\dfrac{\partial A}{\partial x} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 0$, *and A is free of x.* We shall now determine $\varphi(y)$ so that

$$\varphi'(y) = A = N - \frac{\partial}{\partial y}\left(\int_x M\,dx\right)$$

If we do this, then $N = \dfrac{\partial}{\partial y}\left(\displaystyle\int_x M\,dx\right) + \varphi'(y)$. Also, from (16), we have

* $\int_x M\,dx$ is a convenient symbol for this process.

$\dfrac{\partial z}{\partial x} = M$, and $\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left(\int_x M\,dx\right) + \varphi'(y) = N$, so that

$$M\,dx + N\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = dz$$

the differential of a function z. We have only to find $\varphi(y)$.

3. To find $\varphi(y)$, integrate $\varphi'(y)$ with respect to y.

$$\varphi(y) = \int \left(N - \frac{\partial}{\partial y}\int_x M\,dx\right) dy$$

The resulting value of z is found by putting this value for $\varphi(y)$ into (16). Since $z = c$ is our solution, we have

$$z = \int_x M\,dx + \int \left(N - \frac{\partial}{\partial y}\int_x M\,dx\right) dy = c \tag{17}$$

Thus, if $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, we can find a function of which $M\,dx + N\,dy$ is the differential. This completes the proof mentioned in Art. 145.

Example. Show that

$$\cos y\,dx + (2y - x\sin y)\,dy = 0$$

is exact, and find the general solution by use of (17). We have

$$M = \cos y \qquad N = 2y - x\sin y$$

Hence
$$\frac{\partial M}{\partial y} = -\sin y = \frac{\partial N}{\partial x}$$

Also
$$\int_x M\,dx = \int_x \cos y\,dx = x\cos y$$

$$\frac{\partial}{\partial y}\int_x M\,dx = -x\sin y$$

$$z = x\cos y + \int (2y - x\sin y + x\sin y)\,dy$$

and
$$z = x\cos y + y^2 = c$$

is the general solution.

EXERCISES

Show that the following differential equations are exact, and find their solutions (compare Art. 145):

1. $y\,dx + (x + y)\,dy = 0$
2. $(2xy + y + 1)\,dx + (x^2 + x + 1)\,dy = 0$
3. $\left(y + \dfrac{1}{x}\right) dx + x\,dy = 0$
4. $\left(1 - \dfrac{y}{x^2} + \ln x\right) dx + \dfrac{1}{x}\,dy = 0$
5. $(2xy^3 - 2y^2 + 4x^3)\,dx + (3x^2y^2 - 4xy - 3y^2)\,dy = 0$

6. $(e^{y^2} + ye^x)\,dx + (2xye^{y^2} + e^x)\,dy = 0$

7. $\left(y^2 - \dfrac{y}{x^2}e^{\frac{y}{x}}\right)dx + \left(2xy + \dfrac{1}{x}e^{\frac{y}{x}}\right)dy = 0$

8. $(2x\cos x^2 + 5\cos y)\,dx - 5x\sin y\,dy = 0$

9. $[\cos(x+y) - y\sin xy]\,dx + [\cos(x+y) - x\sin xy]\,dy = 0$

10. $y\left(2x - \dfrac{1}{x^2+y^2}\right)dx + x\left(x + \dfrac{1}{x^2+y^2}\right)dy = 0$

181. Variables Separable. If M is a function of x only and N is a function of y only, we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 0$. This is the simplest form of exact differential equation, and the *variables are said to be separated.* The general solution can be written down at once, thus:

$$\int M\,dx + \int N\,dy = c$$

For example, suppose we have

$$e^{-x^2}\,dx + \tan y\,dy = 0$$

The general solution is

$$\int e^{-x^2}\,dx - \ln|\cos y| = c$$

Although we cannot express $\int e^{-x^2}\,dx$ in terms of elementary functions, we say that the solution of the differential equation has been found. In general, we regard the solution of any differential equation as achieved if a relation between the variables can be written down in which functions of the variables expressed as *quadratures* appear. It may not be possible to carry out these quadratures in terms of elementary functions. Nevertheless, the differential equation is solved, for such integrals define functions of the indicated variables.

If the variables are not already separated, it may be possible to separate them by inspection. The method is best made clear by examples.

Example 1. In the equation

$$e^{x^2}\sec y\,dx + \frac{1}{x}\sin y\,dy = 0$$

the variables can be separated by multiplying through by $x\cos y$. The result is

$$xe^{x^2}\,dx + \sin y\cos y\,dy = 0$$

in which the variables are separated. The solution is found by integrating,

$$\tfrac{1}{2}e^{x^2} + \tfrac{1}{2}\sin^2 y = c$$

which may be written

$$e^{x^2} + \sin^2 y = k \qquad \text{where } k = 2c$$

Example 2. In the equation

$$x(y^2 - 9)\,dx + y\,(1 + x)\,dy = 0$$

the variables are separated by dividing through by $(1 + x)(y^2 - 9)$, thus

$$\frac{x\,dx}{1+x} + \frac{y\,dy}{y^2-9} = 0 \qquad \text{or} \qquad \left(1 - \frac{1}{1+x}\right)dx + \frac{y\,dy}{y^2-9} = 0$$

Integrating, we get

$$x - \ln|1+x| + \tfrac{1}{2}\ln|y^2-9| = c$$

for the general solution. There are various equivalent forms in which this may be written. For instance

$$x + \frac{1}{2}\ln\frac{|y^2-9|}{(1+x)^2} = c$$

or

$$\begin{aligned} 2x &= 2\ln|1+x| - \ln|y^2-9| + 2c \\ &= \ln\frac{(1+x)^2}{|y^2-9|} + \ln k^2 = \ln\frac{k^2(1+x)^2}{|y^2-9|} \qquad \text{where } 2c = \ln k^2 \end{aligned}$$

from which we get

$$\frac{k^2(1+x)^2}{|y^2-9|} = e^{2x}$$

If we prefer, we may set $k^2 = 1/c'$ and write

$$\frac{(1+x)^2}{y^2-9} = c'e^{2x}$$

The student will observe that, by using various forms for the arbitrary constant of integration, he may express the general solution in various ways. It is sometimes very convenient in practice to change the form of the solution in this manner.

EXERCISES

Solve the following differential equations:

1. $x + y\dfrac{dy}{dx} = 0$

2. $x\,dy + y\,dx = 0$

3. $(y+1)\,dx + (x-3)\,dy = 0$

4. $x(y^2+4)\,dx + y(x+2)\,dy = 0$

5. $\sqrt{y^2-1}\,dx + y\sqrt{1-x^2}\,dy = 0$

6. $u\tan v\,du + dv = 0$

7. $du + u\tan v\,dv = 0$

8. $\dfrac{dr}{d\theta} = \dfrac{\sin\theta\cos\theta}{r^2}$

9. $\dfrac{dr}{d\theta} = \dfrac{r}{\cos\theta}$

10. $\dfrac{dz}{dy} = \sec z\tan^2 y$

11. $\dfrac{dz}{dy} = \dfrac{e^{2z}}{4+y^2}$

12. $y\,dx + (1+x^2)\arctan x\,dy = 0$

13. $e^{x+y}\,dx + e^{2y-x}\,dy = 0$

14. $(1-y^2)x\,dx + y(1+x^2)(1+y^2)\,dy = 0$

15. $(xy^2+y^2)\,dx + (xy^2-4x)\,dy = 0$

16. $xe^{x+2y} + \dfrac{dy}{dx} = 0$

17. $\dfrac{dy}{dx} = \dfrac{3x^2\sqrt{1 - y^4}}{2y}$

18. $2x(1 + y^2)\dfrac{dx}{dy} + (x^2 + 1)^2 = 0$

19. $(1 + x^2)\dfrac{dy}{dx} + xy = x$

20. $x\,dx + y^3 \operatorname{sech} x\,dy = 0$

182. Homogeneous Equations. The reader will recall from his study of algebra that a function such as

$$x^3y^2 + 3xy^4 + y^5 = f(x,y)$$

was called *homogeneous* of degree 5. We may give a general definition of a homogeneous function of x and y of degree n* as follows: *Replace x by tx and y by ty, and simplify the expression. If the result is the original function multiplied by t^n, then that function is homogeneous of degree n.*

In symbolic form, $f(x,y)$ is homogeneous of degree n if $f(tx,ty) = t^nf(x,y)$ where t is any number other than zero.

Now, if M and N are homogeneous functions of the *same degree*, then the variables can be separated in the equation $M\,dx + N\,dy = 0$ by use of the transformation $y = vx$, as we see in the following example.

Example. In the equation

$$(y^2 + xy)\,dx + 3x^2\,dy = 0$$

M and N are homogeneous of second degree. Setting $y = vx$, we have

$$dy = v\,dx + x\,dv$$

and the equation becomes $(v^2x^2 + vx^2)\,dx + 3x^2(v\,dx + x\,dv) = 0$. Dividing out the x^2, this reduces to

$$(v^2 + 4v)\,dx + 3x\,dv = 0 \qquad \text{or} \qquad \frac{dx}{x} + \frac{3dv}{v^2 + 4v} = 0$$

This may be written

$$\frac{dx}{x} + 3\,\frac{dv}{(v + 2)^2 - 4} = 0$$

Integrating, we get

$$\ln|x| + \frac{3}{4}\ln\left|\frac{v}{v + 4}\right| = C$$

Now, let $v = y/x$, and simplify. We obtain

$$\ln|x| + \frac{3}{4}\ln\left|\frac{y}{y + 4x}\right| = C \qquad \text{or} \qquad 4\ln|x| + 3\ln\left|\frac{y}{y + 4x}\right| = C'$$

This result can be put in a different form if desired. For instance, let $C' = \ln|C''|$, and combine terms in the left-hand member by the usual laws of logarithms:

$$\ln\left|\frac{x^4y^3}{(y + 4x)^3}\right| = \ln|C''| \qquad \text{or} \qquad \frac{x^4y^3}{(y + 4x)^3} = C''$$

Let the reader verify that this solution is correct.

* The values $n \leqq 0$ are included.

EXERCISES

Solve the following differential equations:

1. $(x + y)\,dx + (x - y)\,dy = 0$
2. $(x^2 - y^2)\,dx - xy\,dy = 0$
3. $(x^2 + y^2)\,dx + xy\,dy = 0$
4. $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$
5. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$
6. $xy\,dx + (y\sqrt{x^2 + y^2} - x^2)\,dy = 0$
7. $(y + \sqrt{x^2 - y^2})\,dx - x\,dy = 0$
8. $(x + 2y)\,dx + (y - 2x)\,dy = 0$
9. $\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{xy}}$
10. $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2 + xy}$
11. $\dfrac{dy}{dx} = -\dfrac{2x^2 + y^2}{2xy - 3y^2}$
12. $(t + 2s)\,dt + 2(t + 4s)\,ds = 0$
13. $2(t + 2s)\,dt + (6t + 25s)\,ds = 0$
14. $[x + (x - y)e^{\frac{y}{x}}]\,dx + xe^{\frac{y}{x}}\,dy = 0$
15. $\left(x - y \arcsin \dfrac{y}{x}\right) dx + x \arcsin \dfrac{y}{x}\,dy = 0$
16. $\left(x - y \sin \dfrac{y}{x}\right) dx + x \sin \dfrac{y}{x}\,dy = 0$
17. $y(y + xe^{\frac{x}{y}})\,dx - x^2e^{\frac{x}{y}}\,dy = 0$
18. $y \cos \dfrac{x}{y}\,dx + \left(y \sin \dfrac{x}{y} - x \cos \dfrac{x}{y}\right) dy = 0$
19. $y\,dx + (y - x)\,dy = 0$. Find the particular solution for which $x = 4$, $y = 1$.
20. $(y + \sqrt{x^2 + y^2})\,dx - x\,dy = 0$. Find the particular solution for which $x = 3$, $y = 4$. Draw this curve, also two or three others of the family.

183. Linear Equations. A differential equation of the form

$$\frac{dy}{dx} + Py = Q \tag{18}$$

where P and Q are functions of x alone, is called a *linear* differential equation of first order. Observe that the derivative and the dependent variable y enter only to the first degree. It is easy to verify that, if we multiply both sides by the factor $e^{\int P\,dx}$, the left-hand member becomes the derivative of $ye^{\int P\,dx}$ while the right-hand member remains a function of x only. For

$$\frac{d}{dx}(ye^{\int P\,dx}) = ye^{\int P\,dx} \cdot P + e^{\int P\,dx} \cdot \frac{dy}{dx} = e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right)$$

Consequently, if we multiply equation (18) by $e^{\int P\,dx}$, we get

$$\frac{d}{dx}(ye^{\int P\,dx}) = Qe^{\int P\,dx}$$

Integrating, we obtain

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx \tag{19}$$

The constant of integration is included in the indefinite integral on the right-hand side.

Example 1. To solve the equation

$$\frac{dy}{dx} + y \tan x = \sec x$$

we note that $P = \tan x$ and $Q = \sec x$. Therefore

$$\int P\,dx = \int \tan x\,dx = \ln|\sec x|$$

Hence
$$e^{\int P\,dx} = e^{\ln|\sec x|} = |\sec x|$$

Using this in (19), we obtain

$$y|\sec x| = \int \sec^2 x\,dx$$

and the solution is

$$y \sec x = \tan x + C$$

If desired, this can be written

$$y = \sin x + C \cos x$$

Example 2. Solve the equation

$$(1 + x)\frac{dy}{dx} + xy = (1 + x)^2$$

This can be put into the form (18) by dividing through by $(1 + x)$. This gives

$$\frac{dy}{dx} + \frac{x}{1 + x} y = (1 + x)^2$$

Here $P = \dfrac{x}{1 + x}$, so that

$$\int P\,dx = \int \frac{x}{1 + x}\,dx = \int \left(1 - \frac{1}{1 + x}\right) dx = x - \ln|1 + x|$$

Hence
$$e^{\int P\,dx} = e^{x - \ln|1+x|} = \frac{e^x}{|1 + x|}$$

The solution is therefore given by

$$\frac{ye^x}{1 + x} = \int (1 + x)^2 \cdot \frac{e^x}{1 + x}\,dx = \int (1 + x)e^x\,dx$$

The right-hand side is readily integrated by parts, and we get

$$\frac{ye^x}{1 + x} = xe^x + C$$

which may be written

$$ye^x = x(1 + x)e^x + C(1 + x) \qquad \text{or} \qquad y = (1 + x)(x + Ce^{-x})$$

184. Bernoulli's Equation. A differential equation that is not linear may sometimes be reduced to the linear type by a suitable transformation of variables. An equation of some importance is *Bernoulli's equation:**

$$y^{n-1}\frac{dy}{dx} + Py^n = Q \tag{20}$$

where P and Q are functions of x alone and n is any number different from zero. If we set $y^n = v$, the equation will be transformed to the linear type. For

$$ny^{n-1}\frac{dy}{dx} = \frac{dv}{dx} \qquad \text{and} \qquad y^{n-1}\frac{dy}{dx} = \frac{1}{n}\frac{dv}{dx}$$

This gives

$$\frac{1}{n}\frac{dv}{dx} + Pv = Q$$

$$\frac{dv}{dx} + nPv = nQ \tag{21}$$

which is linear.

Example. Solve $y^2\frac{dy}{dx} + \frac{1}{x}y^3 = x^2 + 4$. We set $y^3 = v$, $3y^2\frac{dy}{dx} = \frac{dv}{dx}$, and the equation becomes

$$\frac{1}{3}\frac{dv}{dx} + \frac{1}{x}v = x^2 + 4 \qquad \text{or} \qquad \frac{dv}{dx} + \frac{3}{x}v = 3(x^2 + 4)$$

This is linear of the form $\frac{dv}{dx} + Pv = Q$, and we have

$$\int P\,dx = \int \frac{3}{x}\,dx = 3\ln|x| = \ln|x^3|$$

Hence $e^{\int P\,dx} = |x^3|$ and

$$vx^3 = 3\int (x^2 + 4)x^3\,dx = \tfrac{1}{2}x^6 + 3x^4 + C$$

Recalling that $v = y^3$, we obtain from this

$$x^3y^3 = \tfrac{1}{2}x^6 + 3x^4 + C$$

as the solution.

EXERCISES

Solve the following differential equations:

1. $\frac{dy}{dx} - \frac{y}{x} = x$
2. $\frac{dy}{dx} - y = x + 1$
3. $\frac{dy}{dx} + y = e^x$
4. $x\frac{dy}{dx} + y = e^x + \cos x$
5. $(x^2 + 1)\frac{dy}{dx} + xy = x^2$
6. $(x^2 - 1)\frac{dy}{dx} + 2xy = e^x$

* Named for Jakob Bernoulli (1654–1705).

7. $\dfrac{dy}{dx} = y + \sin x$

8. $(x^3 \cos x + y)\, dx - x\, dy = 0$

9. $dy + (y - 2 \sin x) \cos x\, dx = 0$

10. $x\, dy + (y - x \ln x)\, dx = 0$

11. $\dfrac{ds}{dt} - s \tan t = 4(\cos t + \sin t)$

12. $\dfrac{ds}{dt} + s \cot t = t + \sin t$

13. $(1 - t^2) \dfrac{ds}{dt} + st = 4t$

14. $(x + 1) \dfrac{dy}{dx} - 2y = (x + 1)^4$

15. $x^2 \ln x \dfrac{dy}{dx} + xy = 1$

16. $(a^2 - x^2) \dfrac{dy}{dx} + y = (a^2 - x^2)^2$

17. $x \dfrac{dy}{dx} + y = xy^3$

18. $\dfrac{ds}{dt} + s = s^2 t$

19. $\dfrac{ds}{dt} = \dfrac{t^2 + s^2}{st}$

20. $x \dfrac{dy}{dx} + y = y^2 \ln x$

185. Integrating Factor. We have already seen that we can sometimes separate the variables in the equation $M\, dx + N\, dy = 0$ by multiplying through by a suitable factor. In other words, we made the differential equation *exact* by this process. For instance, in solving a linear equation of first order, we multiplied through by a factor $e^{\int P\,dx}$, and this had the effect of making the equation exact. Such a factor is called an *integrating factor*. It can be shown that the differential equation

$$M\, dx + N\, dy = 0$$

has an indefinite number of integrating factors. Hence, if

$$M\, dx + N\, dy = 0$$

is not exact $\left(\text{that is, if } \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}\right)$, it may be possible to discover a factor μ such that

$$\mu M\, dx + \mu N\, dy = 0$$

is exact.

Unfortunately, there is no simple method available for finding integrating factors. In certain cases, factors are known, but a discussion of these cases is beyond the scope of this book. Sometimes, however, such a factor can be found by inspection. It is not possible to give any general rule for doing this, but familiarity with some of the more commonly occurring differentials will expedite the search. For instance,

(1) $$x\, dy + y\, dx = d(xy)$$

(2) $$\frac{x\, dy - y\, dx}{x^2} = d\left(\frac{y}{x}\right)$$

(3) $$\frac{x\, dy - y\, dx}{x^2 + y^2} = d\left(\arctan \frac{y}{x}\right)$$

$$(4) \qquad \frac{x\,dy - y\,dx}{x^2 - y^2} = d\left(\frac{1}{2}\ln\left|\frac{x+y}{x-y}\right|\right)$$

$$(5) \qquad 2x\,dx + 2y\,dy = d(x^2 + y^2)$$

The following examples will illustrate ways in which integrating factors can be found.

Example 1. To find an integrating factor for

$$\frac{1}{x}\,dy + \frac{y}{x^2}\,dx - 2x\,dx = 0$$

we observe that multiplying through by x^2 will give $x\,dy + y\,dx$ plus a function of x times dx. Thus

$$x\,dy + y\,dx - 2x^3\,dx = 0$$

that is

$$d(xy) - 2x^3\,dx = 0$$

Integrating, we get

$$xy - \tfrac{1}{2}x^4 = C$$

which is the solution.

Example 2. Solve

$$x\,dy - y\,dx + x(x+1)\,dx = 0$$

Here, $1/x^2$ is an integrating factor, for

$$\frac{x\,dy - y\,dx}{x^2} + \left(1 + \frac{1}{x}\right)dx = 0$$

gives

$$d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x}\right)dx = 0$$

The solution is

$$\frac{y}{x} + x + \ln|x| = C$$

Example 3. To solve

$$x\,dy - y\,dx + f(x^2 + y^2)\cdot(x\,dx + y\,dy) = 0$$

we may use $\dfrac{1}{x^2 + y^2}$ as an integrating factor, for this reduces the equation to

$$d\left(\arctan\frac{y}{x}\right) + F(x^2 + y^2)\cdot d(x^2 + y^2) = 0$$

which is exact. Similar treatment will reduce

$$x\,dy - y\,dx + f(x^2 - y^2)\cdot(x\,dx - y\,dy) = 0$$

to an exact equation.

EXERCISES

Find an integrating factor by inspection and solve the equation in each of the following cases:

1. $x^2\,dy + xy\,dx + (x^3 + 1)\,dx = 0$ **2.** $x\,dy - y\,dx + x^2 \ln y\,dy = 0$
3. $x\,dy - y\,dx + (x^2 + y^2)\tan y\,dy = 0$

4. $x\,dy - y\,dx + (x^2 - y^2)\sec x\,dx = 0$
5. $x\,dy - y\,dx + (x^2 + y^2)^{3/2}(x\,dx + y\,dy) = 0$
6. $\dfrac{y}{x^2}\,dx + \dfrac{1}{x}\,dy + \dfrac{dx}{x^2 + 1} = 0$
7. $ye^{2y}\,dx + (xe^{2y} - 1)\,dy = 0$
8. $x\,dy - y\,dx + x^2y\,dy + y^2x\,dx = 0$
9. $(x^2 + y)\,dx + x(xy - 1)\,dy = 0$
10. $(x\,dy + y\,dx)\sqrt{y} + \sqrt{1 - x^2y^2}\,dy = 0$

186. Transformation of Variable; Summary of Methods. If none of the methods of Arts. 180 to 184 applies and no integrating factor is evident, it may be possible to reduce a given equation $M\,dx + N\,dy = 0$ to a manageable form by a transformation of variables. Such substitutions as $y/x = v$ (Art. 182), $xy = v$, $x + y = v$ are frequently useful.

If M and N are linear in x and y, but not homogeneous, a "translation of axes" will reduce the equation to homogeneous type. For example, suppose

$$(x + 2y + 1)\,dx + (x - y - 2)\,dy = 0$$

Let

$$x = x' + h \qquad \text{and} \qquad y = y' + k$$

where h and k are suitable constants. Then,

$$dx = dx' \qquad \text{and} \qquad dy = dy'$$

Also

$$x + 2y + 1 = x' + 2y' + h + 2k + 1$$
$$x - y - 2 = x' - y' + h - k - 2$$

We can choose h and k so that these expressions are homogeneous in x' and y'. This requires that

$$h + 2k + 1 = 0 \qquad h - k - 2 = 0$$

Solving these equations, we get $h = 1$ and $k = -1$. The differential equation becomes

$$(x' + 2y')\,dx' + (x' - y')\,dy' = 0$$

This can be solved by the method of Art. 182, and the relations between x and x' and y and y' can be used to express the solution in terms of x and y.

It is very important to be able to recognize the type of a given differential equation so that an effective method can be adopted for its solution. If given an equation of first order and first degree, the following questions might well be asked: Are the variables separated or easily separable? Are M and N homogeneous of the same degree? Is the equation linear, or is it Bernoulli's equation? Is it an exact equation, that is, do we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$? If it belongs to none of these types, is an integrating factor apparent, or can it be brought into conformity with one of these types by a transformation of variable?

MISCELLANEOUS EXERCISES

Solve the following differential equations (Ex. 1 to 28):

1. $\frac{dy}{dx} + 1 = e^y$

2. $(x - 1)y\,dx + (y + 1)x\,dy = 0$

3. $x\frac{dy}{dx} + y = x \sin x$

4. $(x^2 - y^2)\,dx + 2xy\,dy = 0$

5. $\left(\frac{1}{x} - \frac{y}{x^2 + y^2}\right) dx + \left(\frac{1}{y} + \frac{x}{x^2 + y^2}\right) dy = 0$

6. $x\,dx + (y - 2x)\,dy = 0$

7. $\cot x \frac{dy}{dx} + y = 1$

8. $y\frac{dy}{dx} = \sqrt{16 - y^4}$

9. $y\,dx + (x^3 + x^2y^2)\,dy = 0$

10. $\left(x - y \cot \frac{y}{x}\right) dx + x \cot \frac{y}{x}\,dy = 0$

11. $x\frac{dy}{dx} + y = \cosh x$

12. $\left(4x^3y^3 + \frac{2x}{y^2} + \frac{y}{x^2}\right) dx + \left(3x^4y^2 - \frac{x^2}{y^3} - \frac{1}{x}\right) dy = 0$

13. $\left(\frac{x}{y} + y\right) dy + dx = 0$

14. $(5x + 3y - 4)\,dx + (x + y - 2)\,dy = 0$

15. $nx\frac{dy}{dx} + 2y = xy^{n+1}$

16. $(x^5 + x^3y^2 - y)\,dx + x\,dy = 0$

17. $(x + 2y - 2)\,dx + (2x - 3y + 10)\,dy = 0$

18. $(x - ye^{\frac{y}{x}})\,dx + \left(xe^{\frac{y}{x}} - \frac{x^2}{y}\right) dy = 0$

19. $(x - ye^{\frac{y}{x}})\,dx + xe^{\frac{y}{x}}\,dy = 0$

20. $y\,dx - x\,dy + y^2 \cot y\,dy = 0$

21. $(x^3 + y^3)\,dx + xy^2\,dy = 0$

22. $(1 - x^2)\frac{dy}{dx} - xy + xy^2 = 0$

23. $\frac{dy}{dx} + y \tan x + 1 = 0$

24. $x\,dy - y\,dx + e^x(x^2 - y^2)\,dx = 0$

25. $y^2\,dx - (x^2 + xy)\,dy = 0$

26. $(2x + y - 4)\,dx + (x + 3y - 7)\,dy = 0$

27. $\cos x \frac{dy}{dx} + y = \sin x$

28. $(x^2y + 1)\,dx + x^4\,dy = 0$

29. Solve the linear equation $\frac{dy}{dx} + y = x$, and add a constant when finding $\int P\,dx$. Observe what happens to this constant in the final solution. Investigate the general case.

30. The differential equation of a family of curves is $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$. Find the equation of the family, and sketch several of the curves.

31. Find the differential equation of the family of ellipses with centers at the origin and with a common axis of length 8 lying on the x axis.

32. Find the differential equation of the family of lines tangent to a circle of radius 5 and center at the origin.

33. Find the differential equation of the family of all circles of radius 5.

34. Find the differential equation of the family of circles tangent to the coordinate axes.

187. Geometric Applications; Orthogonal Trajectories. The fact that $\frac{dy}{dx}$ is the slope of the curve $y = f(x)$ at the point x, y leads to numerous interesting applications of differential equations. The method of attack is best made clear by examples.

Example 1. Find the equation of the family of curves the length of whose subnormals is constant. From Art. 30, the length of the subnormal to a curve $y = f(x)$ at point x_1, y_1 is $Q_1N_1 = m_1y_1$. At the point x, y, this is $y\,\frac{dy}{dx}\cdot$ Our condition states that this is a constant, say a.

$$y\,\frac{dy}{dx} = a \qquad y\,dy = a\,dx \qquad \frac{1}{2}\,y^2 = ax + k'$$

or

$$y^2 = 2ax + k \qquad \text{where } k = 2k'$$

a family of parabolas with vertices at the points $(-k/2a,0)$.

An interesting application to geometry is the finding of a family of curves each one of which cuts every curve of a given family at right angles. Each of these families is said to be *orthogonal* to the other, or to form a set of *orthogonal trajectories* of the other. We proceed to show how to determine a family orthogonal to a given family.

Suppose we have a family of curves

$$F(x,y,c) = 0 \tag{22}$$

where c is the parameter (an arbitrary constant). From this we can find the differential equation of the family by the process described in Art. 179. Let this differential equation be

$$f\left(x,y,\frac{dy}{dx}\right) = 0 \tag{23}$$

Now this equation enables us to find the slope of any curve of the family at a point x, y on that curve, for we can calculate $\frac{dy}{dx}$ at that point. But the curve that cuts this given curve at *right angles* at the point x, y has for its slope $-\frac{dx}{dy}\cdot$ Consequently, every curve satisfying the differential equation

$$f\left(x,y,-\frac{dx}{dy}\right) = 0 \tag{24}$$

will have its slope at x, y the negative reciprocal of the slope of the curve of the given family that passes through x, y. Hence, (24) is the differential equation of the orthogonal family. It remains to solve (24) to determine the required orthogonal trajectories.

Notice that the process is comparatively simple. First, find the

differential equation of the given family. Next, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, obtaining a new differential equation. Solve this equation.

Example 2. Find the orthogonal trajectories of the family of hyperbolas $xy = c$. For $c > 0$ the curves are in the first and third quadrants (Fig. 228), for $c < 0$ in the second and fourth quadrants, and for $c = 0$ they are the x and y axes.

The differential equation of this family is $x\frac{dy}{dx} + y = 0$. Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$-x\frac{dx}{dy} + y = 0 \qquad \text{or} \qquad -x\,dx + y\,dy = 0$$

The solution of this is $-x^2 + y^2 = -k$, a family of rectangular hyperbolas. For $k > 0$ the curves are shown by solid lines in Fig. 228, for $k < 0$ by dotted lines, for $k = 0$ by the lines $y = \pm x$.

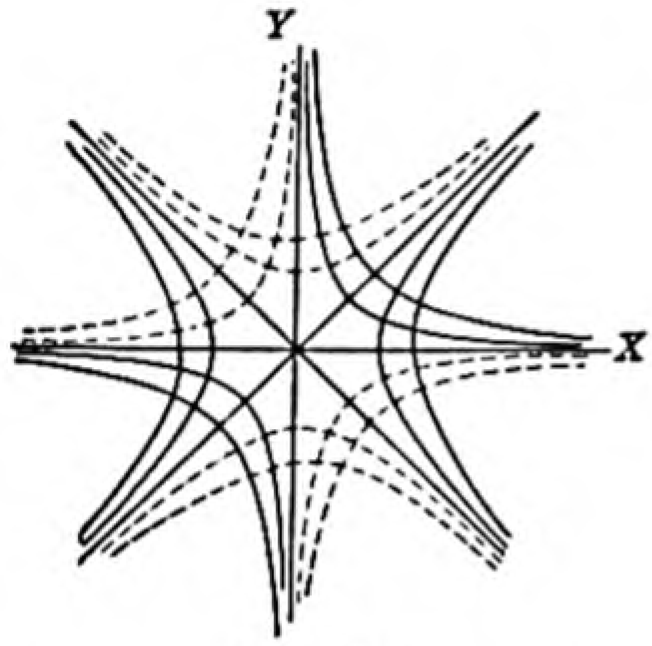


FIG. 228.

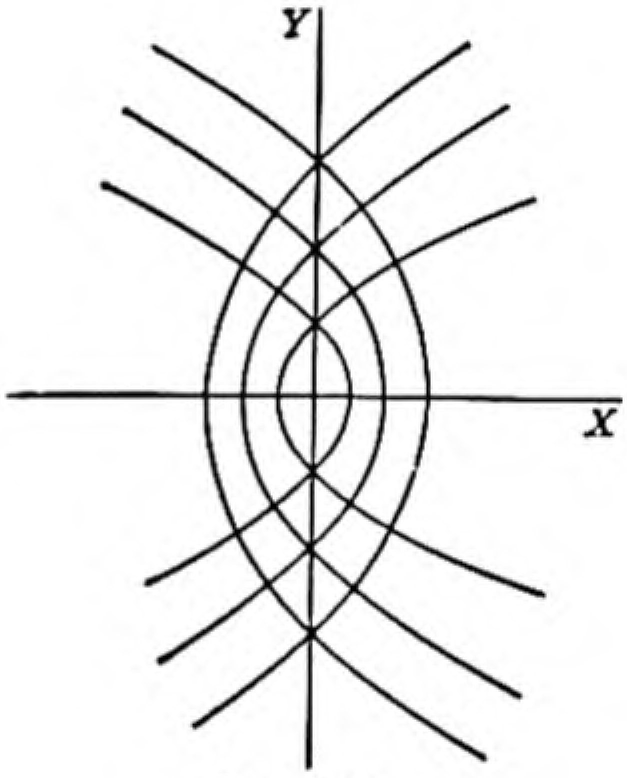


FIG. 229.

Example 3. A family of parabolas has a common focus and common axis. Find the orthogonal family. For convenience, let the origin be at the common focus and let the x axis be along the common axis. If the distance from vertex to focus is p, then the vertex of any one of the parabolas of the family is at the point $-p$, 0, and the equation of the family is

$$y^2 = 4p(x + p) = 4px + 4p^2$$

For $p > 0$, the curves open to the right (Fig. 229); for $p < 0$, they open to the left. Differentiating,

$$2y\frac{dy}{dx} = 4p$$

The differential equation of the family is therefore

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

which may be written

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0 \tag{25}$$

We now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, obtaining

$$y\left(\frac{dx}{dy}\right)^2 - 2x\left(\frac{dx}{dy}\right) - y = 0 \tag{26}$$

Since we have studied no methods for solving differential equations of higher than first degree, we are not prepared to solve this equation. We can, however, solve the problem of finding the orthogonal trajectories. For if we multiply equation (26) through by $-\left(\frac{dy}{dx}\right)^2$, we get

$$-y + 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 = 0$$

which is just equation (25). Consequently, the orthogonal family has the same differential equation as the originally given family. From this we conclude that this family of parabolas is *self-orthogonal*. This means that any two curves of the family that intersect one another intersect at right angles.

We omit a discussion of finding orthogonal trajectories when the equations are given in polar coordinates.

EXERCISES

1. At any point P of a given curve, the line OP is perpendicular to the tangent. Find the equation of the curve.

2. At any point $P(x,y)$ of a curve the subtangent equals x^3. Find the equation of the curve.

3. At any point $P(x,y)$ of a curve the subtangent equals $-1/x$. Find the equation of the curve.

4. At any point $P(x,y)$ of a curve the tangent line and the line OP form the sides of an isosceles triangle the base of which lies on the x axis. Find the equation of the curve.

5. The slope of a curve at point $P(x,y)$ is equal to xy. Find the equation of the curve.

6. At any point $P(x,y)$ of a curve the length of the normal (Art. 30) is proportional to y^2. Find the equation of the curve of this family which intersects the y axis at right angles.

7. At any point $P(r,\theta)$ of a curve the polar subtangent (Art. 72) is proportional to r. Find the equation of the curve.

8. At any point $P(r,\theta)$ of a curve the angle between the tangent and the radius vector is $\frac{1}{2}\theta$. Find the equation of the curve.

9. At any point $P(r,\theta)$ of a curve the angle between the tangent and the radius vector is 2θ. Find the equation of the curve.

10. At any point $P(r,\theta)$ of a curve the polar subtangent is k^2 times the polar subnormal. Find the equation of the curve.

11. Find the orthogonal trajectories of a family of concentric circles (take centers at O).

12. Find the orthogonal trajectories of the family of conics $ax^2 + by^2 = C$ where a and b are fixed numbers. Sketch.

13. Find the orthogonal trajectories of the family of parabolas $y^2 = Cx$, and sketch.

14. The equation of a family of circles is $x^2 + y^2 = Cx$. Find the orthogonal trajectories, and sketch.

15. Find the orthogonal trajectories of the family of curves $x - y + Ce^y = 0$.

16. The equation of a family of confocal conics is $\frac{x^2}{C} + \frac{y^2}{C - k} = 1$ (C is the arbitrary constant). Show this family to be self-orthogonal, and sketch.

17. A ship moving through still water is subject to a retardation proportional to its velocity at time t. If v_0 is the velocity at the instant the power is shut off, find the velocity t sec. later.

18. In the theory of electricity the following differential equation is important: $L\frac{di}{dt} + Ri = E$. Here i is the current, L the (constant) coefficient of self-induction, R the (constant) resistance, and E the electromotive force which may be a constant or a function of the time t. Find i if E = constant, and if $i = 0$ when $t = 0$.

19. A particle falls from rest through the air. If air resistance is proportional to the square of the velocity, the differential equation of motion is $\frac{dv}{dt} = g - kv^2$. Find v in terms of t, using the fact that $v_0 = 0$.

20. Using the result of Exercise 19, find the relation between x and t if x is the distance from the starting point.

B. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

188. Linear Equations. An equation such as

$$x^2 \frac{d^3y}{dx^3} + (2x - 1)\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + x^3y = x^2 \sin x$$

is called a *linear differential equation*. Note that it is linear in the dependent variable and all its derivatives and that the coefficients and the right-hand member are functions of x only. Such an equation is of the general type

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = Q \quad (27)$$

where $P_0, P_1, P_2, \ldots, P_n, Q$ are all functions of x only (including, of course, constants). The equation of Art. 183 is a special case of (27).

For convenience, we shall introduce the symbol D to represent the *operator* $\frac{d}{dx}$. Thus, $\frac{d^ny}{dx^n} = D^ny$. For example

$$\begin{aligned} 2\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} + y &= 2D^3y + xD^2y - 3x^2Dy + y \\ &= (2D^3 + xD^2 - 3x^2D + 1)y \end{aligned}$$

We call the expression in parentheses a *differential operator*. We may,

therefore, write equation (27) in the form

$$(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots + P_{n-1}D + P_n)y = Q$$

or

$$F(D)y = Q$$

We shall prove two important properties of this differential operator.

(1) $F(D)(cy) = cF(D)(y)$ where c is any constant. That is, operating with $F(D)$ upon cy gives the same result as operating with $F(D)$ upon y and then multiplying by c. This is almost evident, for

$$D^k(cy) = cD^k(y)$$

for every $k = 0, 1, 2, \ldots, n$. Note that $D^0y = y$ is a useful convention.

(2) $F(D)(c_1y_1 + c_2y_2) = c_1F(D)(y_1) + c_2F(D)(y_2)$. That is, operating with $F(D)$ upon a sum of two functions gives the same result as operating with $F(D)$ separately upon each of the functions and adding the results. Again, this is almost evident since the derivative of a sum is the sum of the derivatives. Thus

$$\begin{aligned} D^k(c_1y_1 + c_2y_2) &= D^k(c_1y_1) + D^k(c_2y_2) \\ &= c_1D^k(y_1) + c_2D^k(y_2) \end{aligned}$$

for all $k = 0, 1, 2, \ldots, n$. The extension to any finite number of functions $y_1, y_2, y_3, \ldots, y_m$ is obvious.

Homogeneous linear equation. If $Q = 0$ in (27), the equation is called *homogeneous*, that is, homogeneous in y and its derivatives. The following properties of this homogeneous equation $F(D)y = 0$ are of prime importance:

(3) If $y_1, y_2, y_3, \ldots, y_k$ are any k particular solutions of $F(D)y = 0$, then $y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ky_k$ is also a solution. This follows at once from (1) and (2). For since

$$\begin{aligned} F(D)y_1 &= 0 \\ F(D)y_2 &= 0 \\ &\cdots\cdots \\ F(D)y_k &= 0 \end{aligned}$$

we have

$$\begin{aligned} F(D)(c_1y_1 + c_2y_2 + \cdots + c_ky_k) = c_1F(D)y_1 + c_2F(D)y_2 + \cdots \\ + c_kF(D)y_k = 0 \end{aligned}$$

(4) Suppose $y_1, y_2, y_3, \ldots, y_n$ are n linearly independent particular integrals of the homogeneous linear equation $F(D)y = 0$ of nth order. By this we mean that none of these integrals is a linear combination of the others (for instance, none of them is a multiple of another). Then

$$Y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n$$

is the general solution of $F(D)y = 0$ containing n arbitrary constants. It is important that these particular integrals be independent, for otherwise there would be present fewer than n arbitrary constants.

Now, suppose we are given a linear differential equation

$$F(D)y = Q$$

where Q is not identically zero. One step toward solving this is to replace Q by zero and find the general solution of the so-called *reduced equation* $F(D)y = 0$. This solution is

$$Y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n$$

and is called the *complementary function*. Now, if $y = u$ is any particular integral of the *complete equation* $F(D)y = Q$, then

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n + u = Y + u$$

is its general solution, or *complete integral*. This is easily seen to be true, for

$$F(D)(Y + u) = 0 + Q = Q$$

since $F(D)y = 0$, and $F(D)u = Q$.

We may summarize the results of this section as follows: *The general solution of any linear differential equation is the complementary function plus any particular integral of the complete equation.*

189. Linear Equations with Constant Coefficients; Complementary Function. There is no general method available for finding the complementary function or complete integral of a linear differential equation in which the coefficients $P_0, P_1, P_2, \ldots, P_n$ of y and its derivatives are functions of x. But if these coefficients are *constants*, there are simple and quite general methods for finding the solution. This class of differential equations is of considerable importance in applications, and the type form is

$$k_0\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_{n-1}\frac{dy}{dx} + k_ny = Q \quad (28)$$

which can be written

$$(k_0D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_{n-1}D + k_n)y = Q$$

or, simply

$$F(D)y = Q$$

where $k_0, k_1, k_2, \ldots, k_n$ are any constants.

We may find the complementary function; for if

$$F(D)y = 0 \quad (29)$$

and we set $y = e^{mx}$, we have

$$Dy = me^{mx}, \qquad D^2y = m^2e^{mx}, \qquad \ldots, \qquad D^ny = m^ne^{mx}$$

Consequently

$$F(D)y = k_0m^ne^{mx} + k_1m^{n-1}e^{mx} + \cdots + k_{n-1}me^{mx} + k_ne^{mx} = 0$$

or

$$e^{mx}(k_0m^n + k_1m^{n-1} + \cdots + k_{n-1}m + k_n) = 0$$

Hence, if we demand that e^{mx} be an integral of (29), we must find m to satisfy the equation

$$k_0m^n + k_1m^{n-1} + \cdots + k_{n-1}m + k_n = 0 \tag{30}$$

This is, of course, $F(m) = 0$, and it is called the *auxiliary equation* (or, occasionally, the *characteristic equation*).

If $m_1, m_2, \ldots, m_n$ are the roots of the auxiliary equation (30), then

$$y_1 = e^{m_1x}, \qquad y_2 = e^{m_2x}, \qquad \ldots, \qquad y_n = e^{m_nx}$$

will all be particular integrals of (29). We have two cases to consider.

1. *Roots of the auxiliary equation distinct.* Here, $m_1, m_2, \ldots, m_n$ are all different; therefore, $e^{m_1x}, e^{m_2x}, \ldots, e^{m_nx}$ are linearly independent. The complementary function of (28) is

$$Y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

Example 1. To solve the equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$$

(and so to find the complementary function for any linear equation with this for left-hand member), we have the auxiliary equation

$$m^2 + 5m - 6 = 0 \qquad (m + 6)(m - 1) = 0$$

whose roots are $m = 1, -6$. Hence

$$y = c_1e^x + c_2e^{-6x}$$

is the required solution.

Example 2. Solve the equation

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

The auxiliary equation is $m^3 - 4m^2 + 3m + 2 = 0$. One root of this equation is $m = 2$. Hence we have

$$(m - 2)(m^2 - 2m - 1) = 0$$

and the roots are

$$m_1 = 2 \qquad m_2 = 1 + \sqrt{2} \qquad m_3 = 1 - \sqrt{2}$$

and the solution of the differential equation is

$$y = c_1e^{2x} + c_2e^{(1 + \sqrt{2})x} + c_3e^{(1 - \sqrt{2})x} = c_1e^{2x} + e^x(c_2e^{\sqrt{2}x} + c_3e^{-\sqrt{2}x})$$

2. *Roots of the auxiliary equation repeated.* In this case, not all of $m_1, m_2, \ldots, m_n$ are distinct. Suppose $m_i = m_j$. Then $c_i e^{m_i x}$ and $c_j e^{m_j x}$ are essentially the same, and the method of (1) does not give a solution with n arbitrary constants. It can be shown that if $(m - m_r)^s$ appears in the factorization of the auxiliary equation, so that m_r is a repeated root of multiplicity s, then not only is $e^{m_r x}$ an integral of (29), but so also are $xe^{m_r x}, x^2 e^{m_r x}, \ldots, x^{s-1}e^{m_r x}$. The proof will not be given here. Corresponding to the repeated root m_r, therefore, there will appear in the complementary function terms

$$(c_r + c_{r+1}x + c_{r+2}x^2 + \cdots + c_{r+s-1}x^{s-1})e^{m_r x}$$

Example 3. To solve

$$y^{(4)} - 8y'' + 16y = 0$$

we have the auxiliary equation

$$m^4 - 8m^2 + 16 = 0$$

that is

$$(m^2 - 4)^2 = 0 \qquad \text{or} \qquad (m - 2)^2(m + 2)^2 = 0$$

Hence, $m = 2, 2, -2, -2$ are the roots. Corresponding to $m = 2$, we have the terms $(c_1 + c_2 x)e^{2x}$, and corresponding to $m = -2$ we have $(c_3 + c_4 x)e^{-2x}$. The required solution is

$$y = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-2x}$$

Example 4. To solve

$$\frac{d^4y}{dx^4} - 6\frac{d^2y}{dx^2} - 8\frac{dy}{dx} - 3 = 0$$

we have the auxiliary equation

$$m^4 - 6m^2 - 8m - 3 = 0$$

which can be written $(m - 3)(m + 1)^3 = 0$. Here 3 is a simple root, but -1 is a repeated root of multiplicity three. Hence, the solution is

$$y = c_1 e^{3x} + (c_2 + c_3 x + c_4 x^2)e^{-x}$$

EXERCISES

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - a^2y = 0$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$

3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$

4. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - 13\dfrac{d^2y}{dx^2} - 38\dfrac{dy}{dx} - 24y = 0$

5. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - 4y = 0$

6. $\dfrac{d^4y}{dx^4} - 8\dfrac{d^3y}{dx^3} + 18\dfrac{d^2y}{dx^2} - 27y = 0$

7. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - 20\dfrac{dy}{dx} = 0$

8. $\dfrac{d^3y}{dx^3} - 25\dfrac{dy}{dx} = 0$

9. $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$

10. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = 0$

11. $(D^2 + 6D - 6)y = 0$

12. $(D^3 - 6D^2 - D + 30)y = 0$ (Check your result by substituting it into the original equation.)

13. $D^3(D - 2)^4y = 0$

14. $(D^2 + 3D + 1)^2y = 0$

15. $(D^4 - 2D^2)y = 0$ (Check your result by substituting it into the original equation.)

190. Auxiliary Equation with Complex Roots. If the auxiliary equation has real coefficients, any complex roots will appear in pairs of conjugate complex numbers. For example, if $\alpha + i\beta$ is a root, then its *complex conjugate* $\alpha - i\beta$ is also a root. This fact, together with the results of Art. 176, will enable us to have only real terms appearing in the complementary function. In the case under discussion, two of the terms of the complementary function are

$$\begin{aligned} c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} &= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}) \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] \\ &= e^{\alpha x}(A\cos\beta x + B\sin\beta x) \end{aligned}$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ are simply arbitrary constants.

To see what happens if such a pair of roots is repeated, let us suppose that $\alpha + i\beta$ and $\alpha - i\beta$ are repeated once. Then the corresponding part of the complementary function would be

$$\begin{aligned} &(c_1 + c_2x)e^{(\alpha+i\beta)x} + (c_3 + c_4x)e^{(\alpha-i\beta)x} \\ &\quad = e^{\alpha x}[(c_1 + c_2x)e^{i\beta x} + (c_3 + c_4x)e^{-i\beta x}] \\ &\quad = e^{\alpha x}[(c_1 + c_2x)(\cos\beta x + i\sin\beta x) + (c_3 + c_4x)(\cos\beta x - i\sin\beta x)] \\ &\quad = e^{\alpha x}[(A_1 + A_2x)\cos\beta x + (B_1 + B_2x)\sin\beta x] \end{aligned}$$

where A_1, A_2, B_1, B_2 are arbitrary constants.

In general, if $\alpha + i\beta$ and $\alpha - i\beta$ are roots of multiplicity s, the corresponding terms in the complementary function are

$$e^{\alpha x}[(A_1 + A_2x + \cdots + A_sx^{s-1})\cos\beta x + (B_1 + B_2x + \cdots + B_sx^{s-1})\sin\beta x]$$

Example 1. To solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

we have

$$m^2 + m + 1 = 0$$

from which

$$m = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Hence

$$y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

Example 2. To solve

$$(D^3 + 4D)y = 0$$

we have $$m^3 + 4m = 0 \qquad m(m^2 + 4) = 0$$

from which $m = 0, 2i, -2i$. Note that $m_1 = 0, \alpha = 0, \beta = 2$. Hence,

$$y = c_1e^0 + e^0(A \cos 2x + B \sin 2x) = c_1 + A \cos 2x + B \sin 2x$$

It may be desirable to express $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$ in a somewhat different form. For instance, let the reader show that either

$$ae^{\alpha x} \sin (\beta x + b)$$

or $ae^{\alpha x} \cos (\beta x + b)$, where a and b are arbitrary constants, may be used instead of $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$.

The results of Example 1 can therefore be written

$$y = ae^{-\frac{1}{2}x} \cos \left(\frac{\sqrt{3}}{2} x + b\right)$$

or in one of the other equivalent forms. The solution of Example 2 can be written

$$y = c_1 + a \cos (2x + b)$$

In every case, observe that the solution contains a number of arbitrary constants equal to the order of the differential equation.

EXERCISES

Solve the following differential equations (Ex. 1 to 10):

1. $\dfrac{d^2y}{dx^2} + 25y = 0$

2. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 4y = 0$

3. $\dfrac{d^4y}{dx^4} - 16y = 0$

4. $\dfrac{d^3x}{dt^3} + 25\dfrac{dx}{dt} = 0$

5. $\dfrac{d^4x}{dt^4} + 8\dfrac{d^2x}{dt^2} + 16x = 0$

6. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

7. $(D^3 - 5D^2 + 10D - 8)y = 0$

8. $(D^4 + 4D^3 + 8D^2)y = 0$

9. $(D^2 + 6D + 11)^2y = 0$

10. $D^2(D^2 + 2D + 10)^3y = 0$

11. The differential equation $\dfrac{d^2x}{dt^2} = -k^2x$ expresses the conditions of simple harmonic motion, namely, that the acceleration is proportional to the displacement x from the center of the motion and directed toward this center. Solve the equation, and identify the constants in the solution with the amplitude, period, and phase (see Exercises 35 to 39, p. 162).

12. Show that if m and $-m$ are a pair of real roots of the auxiliary equation, then the corresponding terms of the complementary function can be expressed in the following equivalent forms:

$$c_1e^{mx} + c_2e^{-mx}$$
$$A \cosh mx + B \sinh mx$$
$$a \cosh (mx + b)$$
$$c \sinh (mx + d)$$

191. Particular Integral. We have seen how to find the complementary function for a linear differential equation with constant coefficients,

$$F(D)y = Q \tag{31}$$

This can always be done if we can find the roots of the auxiliary equation $F(m) = 0$. If we can now find any *particular integral* of the complete equation (31), it can be added to the complementary function, and the result will be the general solution of (31). There is a quite general method for expressing such a particular integral, involving the use of the differential operator $F(D)$. This and certain other methods are developed in detail in books on differential equations. We shall discuss only one method, comparatively easy to apply and sufficient for many of the cases that arise in practice. It is the method of *undetermined coefficients* and can be used when the function Q contains only terms that have a finite number of distinct derivatives. Such terms are x^n where n is a positive integer; e^{ax}, $\sin bx$, $\cos cx$, where a, b, c are any numbers; or any terms that are products of these.

The first step in this method is to write down a *trial* particular integral whose terms are multiplied by undetermined coefficients. This trial integral is then substituted for y in $F(D)y = Q$, and the coefficients are determined so that the integral is actually a solution of the differential equation. Examples will be given to illustrate the method, and then a general rule will be stated.

Example 1. Solve the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + \sin x$$

that is $\qquad (D^2 - 3D + 2)y = x^2 + \sin x$

The auxiliary equation is $\qquad m^2 - 3m + 2 = 0$

with roots $m = 2, 1$. The complementary function is

$$y = c_1e^{2x} + c_2e^x$$

Here $Q = x^2 + \sin x$. We note that the only functions of x that arise from the differentiation of Q are (except for constant factors) x, a constant, and $\cos x$. We shall take the terms of Q plus all those which can arise from them by differentiation, multiply each by an undetermined coefficient, add all together, and call this our *trial integral*. Thus,

$$u = ax^2 + bx + c + f\sin x + g\cos x$$

Differentiating this, we obtain

$$Du = 2ax + b - g\sin x + f\cos x$$
$$D^2u = 2a - f\sin x - g\cos x$$

Consequently

$$(D^2 - 3D + 2)u = 2ax^2 + (-6a + 2b)x + (2a - 3b + 2c) + (-f + 3g + 2f)\sin x + (-g - 3f + 2g)\cos x$$

We wish this expression to reduce to $x^2 + \sin x$ for all values of x. Hence, coefficients of x^2 must add to 1, of $\sin x$ must add to 1, of $\cos x$ and of x must add to 0, and the constant term must be 0. This gives

$$\begin{aligned}
2a &= 1 & a &= \tfrac{1}{2} \\
-6a + 2b &= 0 & b &= 3a = \tfrac{3}{2} \\
2a - 3b + 2c &= 0 & c &= -a + \tfrac{3}{2}b = -\tfrac{1}{2} + \tfrac{9}{4} = \tfrac{7}{4} \\
f + 3g &= 1 & & \\
-3f + g &= 0 & g &= 3f \qquad \text{so } f + 9f = 1, \text{ and } f = \tfrac{1}{10} \\
& & g &= \tfrac{3}{10}
\end{aligned}$$

Therefore

$$u = \tfrac{1}{2}x^2 + \tfrac{3}{2}x + \tfrac{7}{4} + \tfrac{1}{10}\sin x + \tfrac{3}{10}\cos x$$

and the general solution of the differential equation is

$$y = c_1e^{2x} + c_2e^x + \tfrac{1}{2}x^2 + \tfrac{3}{2}x + \tfrac{7}{4} + \tfrac{1}{10}\sin x + \tfrac{3}{10}\cos x$$

Example 2. Solve the equation

$$y''' - 3y'' + 2y' = x^2 + e^x$$

that is

$$(D^3 - 3D^2 + 2D)y = x^2 + e^x$$

The auxiliary equation is

$$m^3 - 3m^2 + 2m = 0$$

whose roots are $m = 0, 1, 2$. Hence, the complementary function is

$$Y = c_1e^0 + c_2e^x + c_3e^{2x} = c_1 + c_2e^x + c_3e^{2x}$$

The scheme for setting up the trial integral used in Example 1 fails, as the student can readily verify. In fact, the method will fail in the following two cases.

1. If a term in Q is also a term in the complementary function, the method fails. For if we introduce such a term into the trial integral, we shall have zero as the result of operating upon it by $F(D)$. In Example 2, e^x is a term of the complementary function corresponding to a simple root $m = 1$ of the auxiliary equation. We shall introduce, therefore, the term xe^x (plus terms arising from differentiating this) into the trial integral. If $m = 1$ had been a root of multiplicity s, we should have introduced x^se^x into the trial integral.

2. If a term in Q is of the type x^tv where v is a term of the complementary function, the method fails. In our case, x^2 is such a term of Q. For here $v = 1$ since $c_1 \cdot 1$ is a term of the complementary function corresponding to a simple root $m = 0$ of the auxiliary equation. We shall introduce $x^{2+1} \cdot 1$ (plus terms arising from differentiating this) into our trial integral. If $m = 0$ had been a root of multiplicity s, we should have introduced x^{2+s} into the trial integral. In general, if v corresponds to a root of multiplicity s and $x^t \cdot v$ is a term of Q, we introduce $x^{t+s} \cdot v$ into the trial integral. The proper trial integral in Example 2 is therefore

$$u = ax^3 + bx^2 + cx + g + fxe^x + he^x$$

But operating upon g and he^x by $D^3 - 3D^2 + 2D$ will produce zero since $c_1 \cdot 1$ and c_2e^x are terms of the complementary function. We can therefore use for our trial integral

$$u = ax^3 + bx^2 + cx + fxe^x$$

From this we get

$$\begin{aligned} Du &= 3ax^2 + 2bx + c + fe^x + fxe^x \\ D^2u &= 6ax + 2b + 2fe^x + fxe^x \\ D^3u &= 6a + 3fe^x + fxe^x \\ (D^3 - 3D^2 + 2D)u &= 6ax^2 + (4b - 18a)x + (2c - 6b + 6a) - fe^x \end{aligned}$$

Hence

$$\begin{aligned} 6a &= 1 \quad &\text{and}\quad a &= \tfrac{1}{6} \\ 4b - 18a &= 0 \quad &\text{and}\quad b &= \tfrac{3}{4} \\ 2c - 6b + 6a &= 0 \quad &\text{and}\quad c &= \tfrac{7}{4} \\ -f &= 1 \quad &\text{and}\quad f &= -1 \end{aligned}$$

The general solution of the complete equation is therefore

$$y = c_1 + c_2e^x + c_3e^{2x} + \tfrac{1}{6}x^3 + \tfrac{3}{4}x^2 + \tfrac{7}{4}x - xe^x$$

Example 3. Solve the equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x + \cos x$$

that is

$$(D^3 - 2D^2 + D)y = e^x + \cos x$$

We have

$$m^3 - 2m^2 + m = 0$$

for the auxiliary equation whose roots are $m = 0, 1, 1$. Since $m = 1$ is a root of multiplicity 2, the complementary function is

$$Y = c_1 + (c_2 + c_3x)e^x$$

In choosing our trial integral, we note that one of the terms of Q is $\cos x$. This does not appear in the complementary function; we shall therefore include it, and any terms obtained from it by differentiation, namely, $\sin x$. But the other term of Q, e^x, appears in the complementary function. Furthermore, it corresponds to a root of the auxiliary equation of multiplicity two. For the trial particular integral, we shall therefore multiply e^x by the *second* power of x and then include terms arising from differentiating x^2e^x. These would be xe^x and e^x; but since these already appear in the complementary function, operating upon them with $D^3 - 2D^2 + D$ would produce zero. Consequently, we may omit them from the trial integral. We take

$$u = ax^2e^x + b\sin x + c\cos x$$

The student can easily verify that $a = \frac{1}{2}$, $b = 0$, $c = \frac{1}{2}$ and that the general solution of the differential equation is

$$y = c_1 + (c_2 + c_3x)e^x + \tfrac{1}{2}x^2e^x + \tfrac{1}{2}\cos x$$

Example 4. Solve the equation

$$(D^4 + 8D^2 + 16)y = x\sin 2x$$

The auxiliary equation is

$$m^4 + 8m^2 + 16 = 0 \qquad (m^2 + 4)^2 = 0$$

and the roots are $m = 2i, 2i, -2i, -2i$. The complementary function is

$$Y = (A_1 + A_2x)\cos 2x + (B_1 + B_2x)\sin 2x$$

Observe that Q is $\sin 2x$ multiplied by an integral power of x, namely, the first power. Also, note that $\sin 2x$ is a term of the complementary function which corresponds to a root of multiplicity 2. We shall therefore multiply $x \sin 2x$ by the *second* power of x in making up the trial particular integral. We must also add terms obtained by differentiation. This gives

$$u = ax^3 \sin 2x + bx^2 \sin 2x + cx^3 \cos 2x + fx^2 \cos 2x$$

We also get $x \sin 2x$, $\sin 2x$, $x \cos 2x$, $\cos 2x$ by differentiation; but we need not include them in u, for since they are all terms of the complementary function they would contribute nothing to the result.

If we operate upon u by $D^4 + 8D^2 + 16$ and determine coefficients so that the result is $x \sin 2x$, we get

$$a = -\tfrac{1}{96} \qquad b = c = 0 \qquad f = -\tfrac{1}{64}$$

The general solution is therefore

$$y = (A_1 + A_2x) \cos 2x + (B_1 + B_2x) \sin 2x - \tfrac{1}{96}x^3 \sin 2x - \tfrac{1}{64}x^2 \cos 2x$$

We may summarize the procedure used in these examples in the following general rule for finding the particular integral of a linear differential equation $F(D)y = Q$ in which Q consists of terms having a finite number of distinct derivatives:

In general, take for the trial particular integral the sum of all the terms of Q plus all terms obtained from these by differentiation, each multiplied by an undetermined multiplier. Substitute the trial integral into the differential equation, and determine the coefficients so that the result will be equal to Q.

Modification. Suppose Q contains an exponential or trigonometric term of the complementary function (or an integral power of x times such a term) that corresponds to a root of multiplicity s in the auxiliary equation. The trial integral should contain this term (or the integral power of x times this term) multiplied by x^s, plus terms obtained by differentiation. Each term is to be multiplied by an undetermined coefficient.

Hint: It is useless to include in the trial integral any term that appears in the complementary function. No harm is done by such inclusion; but since operating upon any term of the complementary function by $F(D)$ produces zero, the amount of calculation is needlessly increased.

EXERCISES

Solve the following differential equations:

1. $\dfrac{d^2y}{d^2x} - 4y = x + e^x$

2. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = \sin x + \cos x$

3. $\dfrac{d^2y}{dx^2} + y = x^2 + 2x + e^x$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = e^x - e^{-2x} + x^2$

5. $(D^3 - 4D^2 - 5D)y = e^{-x} + \sin x$

6. $(D^3 + D^2 + D + 1)y = 2 \sin x + 3 \cos x$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2e^x + x - 1$

8. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = x^2 + 2 \sin 2x$

9. $(D + 1)^2y = e^{-x} \cos x$

10. $(D^4 + D^2)y = x^3 + 15e^{3x}$

11. $\dfrac{d^2s}{dt^2} - 5\dfrac{ds}{dt} + 6s = 3e^t$

12. $\dfrac{d^3s}{dt^3} + 4\dfrac{d^2s}{dt^2} + \dfrac{ds}{dt} = t^2 + \sin 2t$

13. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = e^{-x} + \sinh x + 2 \cosh x$

14. $\dfrac{d^4s}{dt^4} + 2\dfrac{d^2s}{dt^2} + s = \cos t$

15. $\dfrac{d^2s}{dt^2} - 3\dfrac{ds}{dt} + 2s = e^{3t} \cos 2t$

16. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2 + xe^{-x} + e^{2x}$

17. $\dfrac{d^2y}{dx^3} - 2\dfrac{dy}{dx} + y = x^2e^{3x}$

18. $2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 2y = xe^{2x}$

C. SECOND-ORDER EQUATIONS OF SPECIAL TYPES

Certain special types of differential equations of second order and first degree frequently occur in applications. We consider two such general types whose solutions can be found. The methods explained can be used in attacking equations of higher order and of similar type, and in certain cases solutions can be obtained. We shall, however, restrict ourselves to equations of the second order.

192. Dependent Variable Absent. It may happen that the dependent variable y does not appear in the differential equation. There are two cases to consider.

1. The first derivative $\dfrac{dy}{dx}$ may also be absent so that we have $\dfrac{d^2y}{dx^2} = f(x)$. The solution is at once evident, for

$$\frac{dy}{dx} = \int f(x)\,dx + c_1 \qquad y = \int [\textstyle\int f(x)\,dx]\,dx + c_1x + c_2$$

if we write the constants of integration explicitly.

It is obvious that the differential equation of nth order $\dfrac{d^ny}{dx^n} = f(x)$ can be solved by a similar method.

Example 1. Solve $\dfrac{d^2y}{dx^2} = \sin x$. We have

$$\frac{dy}{dx} = -\cos x + c_1 \qquad y = -\sin x + c_1x + c_2$$

2. Equations of the form $\dfrac{d^2y}{dx^2} = f\left(x, \dfrac{dy}{dx}\right)$. Here y is absent, but the first derivative is present. To solve the equation, set $\dfrac{dy}{dx} = p$. Then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$, and we have

$$\frac{dp}{dx} = f(x,p)$$

a differential equation of first order in the variables x and p. If this first-order equation can be solved, we get for its solution $p = g(x,c_1)$ where c_1 is an arbitrary constant. From this we have

$$p = \frac{dy}{dx} = g(x,c_1) \qquad y = \int g(x,c_1)\,dx + c_2$$

if we write the constant of integration explicitly.

Example 2. Solve $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = x$. Since y is absent, we set $\dfrac{dy}{dx} = p$, $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$ and the equation becomes

$$\frac{dp}{dx} + xp = x$$

This is a linear differential equation of first order in x and p, and

$$e^{\int x\,dx} = e^{\frac{1}{2}x^2}$$

is an integrating factor (Art. 183). The solution is

$$pe^{\frac{1}{2}x^2} = \int xe^{\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}x^2} + c_1 \qquad \text{and} \qquad p = 1 + c_1e^{-\frac{1}{2}x^2}$$

Therefore $\qquad \dfrac{dy}{dx} = 1 + c_1e^{-\frac{1}{2}x^2} \qquad$ and $\qquad y = x + c_1\int e^{-\frac{1}{2}x^2}\,dx + c_2$

Since $\int e^{-\frac{1}{2}x^2}\,dx$ cannot be expressed in terms of elementary functions, we leave the solution in this form.

Note that this method depends upon *reducing the order* of the given equation by substituting $\dfrac{dy}{dx} = p$. If this new equation of lower order can be solved, the solution of the original equation can be expressed in terms of quadratures. If, in a differential equation of nth order, the dependent variable is absent, the transformation $\dfrac{dy}{dx} = p$ will produce an equation of order $n - 1$. If this can be solved, the solution of the original equation can be found.

193. Independent Variable Absent. Again, two cases may be considered.

1. Equations of the form $\dfrac{d^2y}{dx^2} = f(y)$. Here the first derivative $\dfrac{dy}{dx}$ is absent, as well as x. If we multiply both sides by the integrating factor $2\,\dfrac{dy}{dx}\,dx$, we get

$$2\,\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}\,dx = 2f(y)\,\frac{dy}{dx}\,dx$$

But the left-hand member is the differential of $\left(\dfrac{dy}{dx}\right)^2$. Thus

$$d\left[\left(\frac{dy}{dx}\right)^2\right] = 2f(y)\,dy$$

Integrating, we have

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c_1 = g(y) + c_1$$

If we write the constant of integration explicitly. Hence,

$$\frac{dy}{dx} = \pm\sqrt{g(y) + c_1}$$

Separating the variables, we obtain

$$\pm\frac{dy}{\sqrt{g(y) + c_1}} = dx \qquad \text{and} \qquad x = \pm\int\frac{dy}{\sqrt{g(y) + c_1}} + c_2$$

Example 1. Solve the equation $\dfrac{d^2y}{dx^2} = y - a$. Multiplying through by $2\,\dfrac{dy}{dx}\,dx$,

we get
$$2\,\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}\,dx = 2(y - a)\,dy$$

Integrating,
$$\left(\frac{dy}{dx}\right)^2 = (y - a)^2 + c_1$$

Hence
$$\frac{dy}{dx} = \pm\sqrt{(y - a)^2 + c_1}$$

Separating the variables, we find

$$\frac{dy}{\sqrt{(y - a)^2 + c_1}} = \pm dx$$
$$\ln|y - a + \sqrt{(y - a)^2 + c_1}| = \pm x + c_2$$

2. Equations of the form $\dfrac{d^2y}{dx^2} = f\left(y, \dfrac{dy}{dx}\right)$ in which y, $\dfrac{dy}{dx}$ appear but x does not appear. Again we set $\dfrac{dy}{dx} = p$. However, if we write

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}\cdot p$$

our equation becomes
$$p\,\frac{dp}{dy} = f(y,p)$$

which is an equation of first order in the variables y and p. If the solution of this equation can be found,

$$p = g(y,c_1)$$

we shall have
$$\frac{dy}{dx} = g(y,c_1) \qquad \text{or} \qquad \frac{dy}{g(y,c_1)} = dx$$

from which we obtain
$$\int \frac{dy}{g(y,c_1)} = x + c_2$$

Example 2. Solve the equation $2y\,\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 4$. We set $\dfrac{dy}{dx} = p$ and

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = p \cdot \frac{dp}{dy}$$

This gives
$$2yp\,\frac{dp}{dy} = p^2 + 4$$

Separating the variables, we obtain

$$\frac{dy}{y} = \frac{2p}{p^2+4}\,dp \qquad \ln |y| = \ln (p^2 + 4) - \ln |c_1|$$

This form of the constant of integration is convenient, and we rewrite this result,

$$c_1y = p^2 + 4$$

From this we get
$$\frac{dy}{dx} = p = \pm \sqrt{c_1y - 4} \qquad \frac{dy}{\sqrt{c_1y-4}} = \pm dx$$

Integrating, we find

$$\frac{2\sqrt{c_1y-4}}{c_1} = \pm x + c$$
$$2\sqrt{c_1y-4} = \pm c_1x + c_2 \qquad \text{where } c_2 = cc_1$$

for the general solution.

Again the student will note that the method consists of a device for reducing the order of the given differential equation by a transformation of variable. If the given equation is of the nth order and the independent variable x is absent, the substitution $\dfrac{dy}{dx} = p$ can be used to obtain an equation of order $n - 1$ in p and y. If this equation can be solved, we can express the solution of the original equation in terms of quadratures.

The student is referred to books on differential equations for a more extensive discussion of the methods here given and for additional methods

for solving ordinary differential equations which we have omitted altogether.

It should be mentioned that, in case a differential equation does not yield to solution by the usual methods, numerical and graphical methods can often be used to very good advantage in special types of problems to obtain results that possess a high degree of accuracy.*

It may be pointed out that the solution $y = f(x)$ of a differential equation can often be advantageously represented in the form of an infinite series. This is done by assuming

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

and then determining the coefficients a_0, a_1, a_2, a_3, . . . of the power series so that the differential equation is satisfied.†

EXERCISES

Solve the following differential equations:

1. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$
2. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$
3. $\frac{d^4y}{dx^4} = e^x + \sinh x$
4. $\frac{d^3y}{dx^3} = \frac{1}{x^2}$
5. $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1$
6. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$
7. $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$
8. $(x^2 + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = x^4 - 1$
9. $\frac{d^2y}{dx^2} = e^y$
10. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 1$
11. $y\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$
12. $(x + 1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$

MISCELLANEOUS EXERCISES

Solve the following differential equations (Ex. 1 to 17):

1. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$
2. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$
3. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x\sin x + \cos x$
4. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^x$
5. $\frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + aby = 0$
6. $(D^2 - 4D + 6)y = \sin x$
7. $(D^2 - 2D)y = 1 + e^{2x}$
8. $(D^3 - 1)y = e^x \sin x$
9. $\frac{d^3y}{dx^3} - 8y = 0$
10. $\frac{d^4y}{dx^4} = \sin x$

* See, for example, Lester R. Ford, *Differential Equations*, 2d ed., McGraw-Hill Book Company, Inc., New York, 1955.

† See, for example, *ibid.*, pp. 13–14, and later chapters.

11. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} = 1$

12. $x \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = x^2e^x$

13. $\dfrac{d^2y}{dx^2} - 5 \dfrac{dy}{dx} + 6y = \cos x - e^{2x}$

14. $\dfrac{d^4y}{dx^4} + 18 \dfrac{d^2y}{dx^2} + 81y = 0$

15. $y \dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 1$

16. $\dfrac{d^2y}{dx^2} = y^2$

17. $\dfrac{d^4y}{dx^4} + 16 \dfrac{d^2y}{dx^2} = 0$

18. In finding the curve assumed by a perfectly flexible cable supported at two points on the same horizontal line and subject to a force due to its own weight, it is necessary to solve the differential equation $\dfrac{d^2y}{dx^2} = \dfrac{w}{h}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Here, w is the weight per unit length and h is the horizontal tension at the lowest point. If we take the (vertical) y axis through the lowest point, we have $\dfrac{dy}{dx} = 0$ when $x = 0$. Further, let $y = h/w$ for $x = 0$. Show that the solution, under these conditions, of this differential equation is $y = \dfrac{h}{w} \cosh \dfrac{w}{h} x$ (catenary).

19. To find the velocity v_1 acquired by a body in falling to the earth from an infinite distance, assuming that the attraction is inversely proportional to the square of the distance x from the center of the earth, we have the differential equation of motion $\dfrac{d^2x}{dt^2} = \dfrac{k}{x^2}$. Set $v = \dfrac{dx}{dt}$; note that, when $x = \infty$, $v = 0$. If the radius of the earth is $r = 4000$ miles and $-g = -32$ ft./sec.^{2} is the value of $\dfrac{d^2x}{dt^2}$ when $x = r$, show that

$$v_1 = \sqrt{2gr} = 7 \text{ miles/sec. approximately}$$

BRIEF TABLE OF INTEGRALS

The constant of integration has been omitted in each case.

CERTAIN ELEMENTARY FORMS

(1) $\int k\,dx = k \int dx$

(2) $\int (du + dv + \cdots + dw) = \int du + \int dv + \cdots + \int dw$

(3) $\int u\,dv = uv - \int v\,du$

(4) $\int u^n\,du = \frac{u^{n+1}}{n+1} \qquad n \neq -1$

(5) $\int \frac{du}{u} = \ln |u|$

FORMS CONTAINING $(a + bx)$

(6) $\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2} \ln |a + bx|$

(7) $\int \frac{x\,dx}{(a+bx)^2} = \frac{a}{b^2(a+bx)} + \frac{1}{b^2} \ln |a + bx|$

(8) $\int x(a+bx)^n\,dx = \frac{x(a+bx)^{n+1}}{b(n+1)} - \frac{(a+bx)^{n+2}}{b^2(n+1)(n+2)} \qquad n \neq -1, -2$

(9) $\int \frac{dx}{x(a+bx)} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|$

(10) $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2} \ln \left| \frac{x}{a+bx} \right|$

(11) $\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2}$

(12) $\int x^n \sqrt{a+bx}\,dx = \frac{2x^n(a+bx)^{3/2}}{(2n+3)b} - \frac{2an}{(2n+3)b} \int x^{n-1}\sqrt{a+bx}\,dx$

$2n + 3 \neq 0$

(13) $\int \frac{x\,dx}{\sqrt{a+bx}} = \frac{2}{3b^2}(bx - 2a)\sqrt{a+bx}$

$$(14)\quad \int \frac{x^n\,dx}{\sqrt{a+bx}} = \frac{2x^n\sqrt{a+bx}}{(2n+1)b} - \frac{2an}{(2n+1)b}\int \frac{x^{n-1}\,dx}{\sqrt{a+bx}} \qquad 2n+1 \neq 0$$

$$(15)\quad \int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln \left| \dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} \right| & a > 0 \\ \text{or} \\ \dfrac{2}{\sqrt{-a}} \arctan \sqrt{\dfrac{a+bx}{-a}} & a < 0 \end{cases}$$

$$(16)\quad \int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{2(n-1)a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}} \qquad n \neq 1$$

FORMS CONTAINING $(a^2 - x^2)$

$$(17)\quad \int \frac{dx}{a^2-x^2} = \begin{cases} \dfrac{1}{2a}\ln\dfrac{a+x}{a-x} = \dfrac{1}{a}\operatorname{argtanh}\dfrac{x}{a} & x^2 < a^2 \\ \text{or} \\ \dfrac{1}{2a}\ln\dfrac{x+a}{x-a} = \dfrac{1}{a}\operatorname{argcoth}\dfrac{x}{a} & x^2 > a^2 \end{cases}$$

$$(18)\quad \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

$$(19)\quad \int (a^2-x^2)^{3/2}\,dx = \frac{x}{4}(a^2-x^2)^{3/2} + \frac{3}{8}a^2x\sqrt{a^2-x^2} + \frac{3}{8}a^4\arcsin\frac{x}{a}$$

$$(20)\quad \int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}(a^2-x^2)^{3/2} + \frac{a^2}{8}x\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}$$

$$(21)\quad \int x^n\sqrt{a^2-x^2}\,dx = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{(n-1)a^2}{n+2}\int x^{n-2}\sqrt{a^2-x^2}\,dx \qquad n \neq -2$$

$$(22)\quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$(23)\quad \int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

$$(24)\quad \int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

$$(25)\quad \int \frac{x^n\,dx}{\sqrt{a^2-x^2}} = -\frac{x^{n-1}\sqrt{a^2-x^2}}{n} + \frac{(n-1)a^2}{n}\int \frac{x^{n-2}\,dx}{\sqrt{a^2-x^2}} \qquad n \neq 0$$

$$(26)\quad \int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$(27)\quad \int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

$$(28)\quad \int \frac{dx}{x^n\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(n-1)a^2x^{n-1}} + \frac{n-2}{(n-1)a^2}\int \frac{dx}{x^{n-2}\sqrt{a^2-x^2}} \qquad n \neq 1$$

FORMS CONTAINING $(x^2 \pm a^2)$

(29) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a}$$

(30) $$\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln |x + \sqrt{x^2 \pm a^2}|$$

(31) $$\int (x^2 \pm a^2)^{3/2}\, dx = \frac{x}{4} (x^2 \pm a^2)^{3/2} \pm \frac{3}{8} a^2 x \sqrt{x^2 \pm a^2} + \frac{3}{8} a^4 \ln |x + \sqrt{x^2 \pm a^2}|$$

(32) $$\int x^2 \sqrt{x^2 \pm a^2}\, dx = \frac{x}{4} (x^2 \pm a^2)^{3/2} \mp \frac{a^2}{8} x \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln |x + \sqrt{x^2 \pm a^2}|$$

(33) $$\int x^n \sqrt{x^2 \pm a^2}\, dx = \frac{x^{n-1}(x^2 \pm a^2)^{3/2}}{n + 2} \mp \frac{(n-1)a^2}{n+2} \int x^{n-2} \sqrt{x^2 \pm a^2}\, dx \qquad n \neq -2$$

(34) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln (x + \sqrt{x^2 + a^2}) = \operatorname{argsinh} \frac{x}{a}$$

(35) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| = \operatorname{argcosh} \frac{x}{a}$$

(36) $$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}$$

(37) $$\int \frac{x^2\, dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \ln |x + \sqrt{x^2 \pm a^2}|$$

(38) $$\int \frac{x^n\, dx}{\sqrt{x^2 \pm a^2}} = \frac{x^{n-1} \sqrt{x^2 \pm a^2}}{n} \mp \frac{(n-1)a^2}{n} \int \frac{x^{n-2}\, dx}{\sqrt{x^2 \pm a^2}} \qquad n \neq 0$$

(39) $$\int \frac{dx}{x \sqrt{x^2 + a^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{x^2 + a^2}}{|x|} = -\frac{1}{a} \operatorname{argcsch} \frac{x}{a}$$

(40) $$\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a}$$

(41) $$\int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$$

(42) $$\int \frac{dx}{x^n \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(n-1)a^2 x^{n-1}} \mp \frac{(n-2)}{(n-1)a^2} \int \frac{dx}{x^{n-2} \sqrt{x^2 \pm a^2}} \qquad n \neq 1$$

BINOMIAL DIFFERENTIALS

(43) $$\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a\sqrt{ab}} \arctan \sqrt{\frac{b}{a}}\, x$$

(44) $$\int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \ln \frac{x^2}{|a + bx^2|}$$

(45) $$\int x^m(a+bx^n)^r\,dx = \frac{x^{m-n+1}(a+bx^n)^{r+1}}{(nr+m+1)b} - \frac{(m-n+1)a}{(nr+m+1)b}\int x^{m-n}(a+bx^n)^r\,dx \qquad nr+m+1 \neq 0$$

(46) $$\int x^m(a+bx^n)^r\,dx = \frac{x^{m+1}(a+bx^n)^r}{nr+m+1} + \frac{anr}{nr+m+1}\int x^m(a+bx^n)^{r-1}\,dx \qquad nr+m+1 \neq 0$$

(47) $$\int x^m(a+bx^n)^r\,dx = \frac{x^{m+1}(a+bx^n)^{r+1}}{a(m+1)} - \frac{b(nr+n+m+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^r\,dx \qquad m \neq -1$$

(48) $$\int x^m(a+bx^n)^r\,dx = -\frac{x^{m+1}(a+bx^n)^{r+1}}{n(r+1)a} + \frac{nr+n+m+1}{n(r+1)a}\int x^m(a+bx^n)^{r+1}\,dx \qquad r \neq -1$$

EXPONENTIAL AND LOGARITHMIC FORMS

(49) $$\int e^x\,dx = e^x$$

(50) $$\int a^x\,dx = \frac{a^x}{\ln a} = a^x \log_a e \qquad a > 0,\ a \neq 1$$

(51) $$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

(52) $$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

(53) $$\int \ln x\,dx = x\ln x - x$$

(54) $$\int x\ln x\,dx = \frac{x^2}{4}(2\ln x - 1)$$

(55) $$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} \qquad n \neq -1$$

(56) $$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \qquad n > 0$$

FORMS CONTAINING TRIGONOMETRIC FUNCTIONS

(57) $$\int \sin x\,dx = -\cos x$$

(58) $$\int \cos x\,dx = \sin x$$

(59) $$\int \tan x\,dx = \ln|\sec x|$$

(60) $\int \cot x\,dx = \ln|\sin x|$

(61) $\int \sec x\,dx = \ln|\sec x + \tan x|$

(62) $\int \csc x\,dx = \ln|\csc x - \cot x| = -\ln|\csc x + \cot x|$

(63) $\int \sec^2 x\,dx = \tan x$

(64) $\int \csc^2 x\,dx = -\cot x$

(65) $\int \sec x \tan x\,dx = \sec x$

(66) $\int \csc x \cot x\,dx = -\csc x$

(67) $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$

(68) $\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x$

(69) $\int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

(70) $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad n \text{ integer} > 0$

(71) $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$

(72) $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x$

(73) $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

(74) $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \qquad n \text{ integer} > 0$

(75) $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx$

$m + n \neq 0$

(76) $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\,dx$

$m + n \neq 0$

(77) $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\,dx$

$n \neq -1$

(78) $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\,dx$

$m \neq -1$

(79) $$\int \sin mx \sin nx\, dx = \frac{\sin (m-n)x}{2(m-n)} - \frac{\sin (m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$

(80) $$\int \sin mx \cos nx\, dx = -\frac{\cos (m-n)x}{2(m-n)} - \frac{\cos (m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$

(81) $$\int \cos mx \cos nx\, dx = \frac{\sin (m-n)x}{2(m-n)} + \frac{\sin (m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$

(82) $$\int x \sin x\, dx = \sin x - x \cos x$$

(83) $$\int x^n \sin x\, dx = -x^n \cos x + n \int x^{n-1} \cos x\, dx$$

(84) $$\int x \sin^n x\, dx = \frac{\sin^{n-1} x(\sin x - nx \cos x)}{n^2} + \frac{n-1}{n} \int x \sin^{n-2} x\, dx \qquad n \neq 0$$

(85) $$\int x \cos x\, dx = \cos x + x \sin x$$

(86) $$\int x^n \cos x\, dx = x^n \sin x - n \int x^{n-1} \sin x\, dx$$

(87) $$\int x \cos^n x\, dx = \frac{\cos^{n-1} x(\cos x + nx \sin x)}{n^2} + \frac{n-1}{n} \int x \cos^{n-2} x\, dx \qquad n \neq 0$$

(88) $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

(89) $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

(90) $$\int \tan^2 x\, dx = \tan x - x$$

(91) $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx \qquad n \neq 1$$

(92) $$\int \cot^2 x\, dx = -\cot x - x$$

(93) $$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx \qquad n \neq 1$$

(94) $$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx \qquad n \neq 1$$

(95) $$\int \csc^n x\, dx = -\frac{\cot x \csc^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x\, dx \qquad n \neq 1$$

(96) $$\int \frac{dx}{1 + \sin x} = \tan \left(\frac{1}{2}x - \frac{\pi}{4}\right)$$

(97) $$\int \frac{dx}{1 - \sin x} = \tan \left(\frac{1}{2}x + \frac{\pi}{4}\right)$$

$$(98)\quad \int \frac{dx}{a + b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2 - b^2}} \arctan\left(\dfrac{a\tan\frac{1}{2}x + b}{\sqrt{a^2 - b^2}}\right) & a^2 > b^2 \\ \dfrac{1}{\sqrt{b^2 - a^2}} \ln\left|\dfrac{a\tan\frac{1}{2}x + b - \sqrt{b^2 - a^2}}{a\tan\frac{1}{2}x + b + \sqrt{b^2 - a^2}}\right| & a^2 < b^2 \end{cases}$$

$$(99)\quad \int \frac{dx}{1 + \cos x} = \tan \tfrac{1}{2}x$$

$$(100)\quad \int \frac{dx}{1 - \cos x} = -\cot \tfrac{1}{2}x$$

$$(101)\quad \int \frac{dx}{a + b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2 - b^2}} \arctan\left(\dfrac{\sqrt{a^2 - b^2}\tan\frac{1}{2}x}{a + b}\right) & a^2 > b^2 \\ \dfrac{1}{\sqrt{b^2 - a^2}} \ln\left|\dfrac{\sqrt{b^2 - a^2}\tan\frac{1}{2}x + a + b}{\sqrt{b^2 - a^2}\tan\frac{1}{2}x - a - b}\right| & a^2 < b^2 \end{cases}$$

$$(102)\quad \int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \ln\left|\tan\left(\frac{1}{2}x + \frac{\pi}{8}\right)\right|$$

FORMS CONTAINING INVERSE TRIGONOMETRIC FUNCTIONS

$$(103)\quad \int \arcsin x\,dx = x\arcsin x + \sqrt{1 - x^2}$$

$$(104)\quad \int x^n \arcsin x\,dx = \frac{x^{n+1}}{n+1}\arcsin x - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1 - x^2}} \qquad n \neq -1$$

$$(105)\quad \int \arctan x\,dx = x\arctan x - \ln\sqrt{1 + x^2}$$

$$(106)\quad \int \operatorname{arcsec} x\,dx = x\operatorname{arcsec} x - \ln|x + \sqrt{x^2 - 1}|$$

FORMS CONTAINING HYPERBOLIC FUNCTIONS

$$(107)\quad \int \sinh x\,dx = \cosh x$$

$$(108)\quad \int \cosh x\,dx = \sinh x$$

$$(109)\quad \int \tanh x\,dx = \ln\cosh x$$

$$(110)\quad \int \coth x\,dx = \ln|\sinh x|$$

$$(111)\quad \int \operatorname{sech} x\,dx = 2\arctan e^x$$

$$(112)\quad \int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|$$

$$(113)\quad \int \operatorname{sech}^2 x\,dx = \tanh x$$

(114) $\displaystyle\int \operatorname{csch}^2 x\,dx = -\coth x$

(115) $\displaystyle\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$

(116) $\displaystyle\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x$

(117) $\displaystyle\int \sinh^2 x\,dx = \tfrac{1}{2}\sinh x \cosh x - \tfrac{1}{2}x$

(118) $\displaystyle\int \cosh^2 x\,dx = \tfrac{1}{2}\sinh x \cosh x + \tfrac{1}{2}x$

(119) $\displaystyle\int x \sinh x\,dx = x \cosh x - \sinh x$

(120) $\displaystyle\int x^2 \sinh x\,dx = (x^2 + 2)\cosh x - 2x \sinh x$

(121) $\displaystyle\int x \cosh x\,dx = x \sinh x - \cosh x$

(122) $\displaystyle\int \tanh^2 x\,dx = x - \tanh x$

(123) $\displaystyle\int \coth^2 x\,dx = x - \coth x$

(124) $\displaystyle\int \operatorname{argsinh} x\,dx = x \operatorname{argsinh} x - \sqrt{1 + x^2}$

(125) $\displaystyle\int \operatorname{argcosh} x\,dx = x \operatorname{argcosh} x - \sqrt{x^2 - 1} \qquad x^2 > 1$

(126) $\displaystyle\int \operatorname{argtanh} x\,dx = x \operatorname{argtanh} x + \tfrac{1}{2}\ln(1 - x^2) \qquad x^2 < 1$

ANSWERS TO ODD-NUMBERED EXERCISES

Page 8

13. $f(x + 1) = x^3 - x^2 - 6x$ **15.** $g(3 - x) = g(x)$ **17.** $f(\pi - x) = -f(x)$

Page 11

1. (a) $y = \pm \sqrt{x^3 - 8}$ (b) $y = \dfrac{3(1 - x)}{x^2 + x - 1}$

(c) $y = \pm\frac{1}{3}\sqrt{16 - x^2}$ (d) $y = \dfrac{-(3x + 1) \pm \sqrt{x^2 + 14x + 9}}{4}$

3. (a) $y^4 - 4xy^2 + (4x^2 - x) = 0$
(b) $y^8 - 12xy^6 + (54x^2 - 4x)y^4 - (108x^3 - 24x^2)y^2$
$+ (81x^4 - 36x^3 + 4x^2 - x) = 0$

Pages 20–22

7. -1 **9.** 13 **11.** $\dfrac{3\pi}{4\sqrt{2}}$ **13.** $\frac{21}{4}$ **15.** No limit **17.** 0 **21.** No limit
23. $\sqrt[3]{3}$ **25.** $\frac{1}{3}$ **27.** -1 **29.** -1

Page 25

1. 0 **3.** 7 **5.** ∞ **7.** (a) 0; (b) a_0/b_0; (c) ∞ **9.** 0, ∞ **11.** ∞
13. 1 **15.** 0 **17.** No limit **19.** No limit **21.** (a) $y = 1, x = -1, x = 4$;
(b) $y = 5, x = -2, x = -1, x = 1, x = 3$

Page 27

5. $x = -3$ **7.** None

Pages 28–29

1. $s = kw(9 - w^2)$ **3.** $f(\pi - x) = 1/f(x)$ **5.** $F(\sin x) = x$ **9.** $\frac{5}{3}$ **11.** ∞
13. -2 **15.** 2 **17.** No limit **19.** No limit **21.** 0 **23.** $-\frac{5}{4}$ **25.** $\frac{4}{5}$
27. $-\frac{1}{6}$ **29.** (a) $y = 0$; $x = 1, 2, 3$; (b) $y = 2, x = 3, -3$; (c) $x = -3$; (d) $y = 0$
31. $x = 2$, no; $x = -\frac{1}{4}$, no

Page 32

1. $y = 2x - 3$ **3.** $3x - y + 6 = 0$

Page 34

1. 7; 4 **3.** -7; -8 **5.** 8π sq. in./sec.

Pages 40–41

1. $5x^4$ **3.** $24x^2$ **5.** $-6t$ **7.** $-\dfrac{1}{x^2}$ **9.** $-\dfrac{4}{x^5}$

11. $\dfrac{-16x}{(x^2 - 4)^2}$ **13.** $\dfrac{1}{2\sqrt{x}}$ **15.** $\dfrac{x}{\sqrt{x^2 - 25}}$ **17.** $\dfrac{1}{3t^{2/3}}$ **19.** $-\dfrac{1}{2x^{3/2}}$

21. $-\dfrac{1}{3x^{4/3}}$ **23.** $-\dfrac{15x^2}{2(x^3 - 27)^{3/2}}$ **25.** $8\pi r$, $4\pi r^2$

27. (*a*) Increasing except at (0,0); (*b*) increasing at any point; (*c*) decreasing except at $x = 0$; (*d*) increasing for $x < 0$, decreasing for $x > 0$

29. -1 **31.** 2 **33.** $-\frac{1}{4}$ **35.** $-\frac{3}{4}$ ft. per sec. **37.** -1 ft. per sec.

39. $\frac{18}{16}$ ft./sec.

Pages 47–49

1. 7 **3.** $20x^4 - 9x^2 + 5$ **5.** $15x^4 - 16x^3 + 6x^2 + 14x - 5$

7. $x^3 - 7x^2 + 12x - 10$ **9.** $6 + \dfrac{2}{3x^3}$ **11.** $27x^2 - 8x + 1 + \dfrac{1}{x^2} - \dfrac{8}{x^3} + \dfrac{27}{x^4}$

13. $-3x^{-7/4} + 3x^{-5/2}$ **15.** $-\dfrac{1}{2t^2} - \dfrac{6}{5t^3}$ **17.** $-\dfrac{10}{3x^3} + \dfrac{21}{4x^4}$

19. $-\dfrac{1}{(2x)^{3/2}} + 20x^{1/4}$ **29.** $-\dfrac{53}{(5x - 9)^2}$ **31.** $\dfrac{6x(2x + 1)}{(4x + 1)^2}$

33. $\dfrac{-x(x^3 - 48x - 54)}{(x^3 + 27)^2}$ **35.** $\dfrac{2(3t^2 - 9t - 1)}{(2t - 3)^2}$ **39.** $\dfrac{315x^2}{4(x^3 + 8)^2}$

41. $\dfrac{x^2 + 6x + 7}{(x + 3)^2}$ **47.** (*a*) 34°; (*b*) 63°; (*c*) 72°; (*d*) 83°

49. -40 ft./sec. **51.** Increasing **53.** Decreasing

55. Decreasing for $x < 3$; increasing for $x > 3$

57. Increasing for $x < -1$ and for $x > 2$, decreasing for $-1 < x < 2$

Pages 51–52

1. $\dfrac{1}{\sqrt{2x + 3}}$ **3.** $3x\sqrt{x^2 + 9}$ **5.** $-5x(a^2 - x^2)^{3/2}$ **7.** $\frac{15}{2}(3x + 4)^{3/2}$

9. $\frac{8}{3}x(x^2 + 1)^{1/3}$ **11.** $\dfrac{21}{5(7x - 9)^{2/5}}$ **13.** $-\frac{6}{5}(x + 2)(x^2 + 4x + 13)^{-8/5}$

15. $-\dfrac{a}{2(ax + b)^{3/2}}$ **17.** $-\dfrac{2ax + b}{3(ax^2 + bx + c)^{5/3}}$ **19.** $5(a - x)(2ax - x^2)^{3/2}$

21. $3t^2(2t^2 + 1)\sqrt{t^2 + 1}$ **23.** $6x(x^3 + 1)(x^2 - 4)^2(2x^3 - 4x + 1)$

25. $\dfrac{x(4x^2 - 13)\sqrt{x^2 - 1}}{\sqrt{x^2 - 4}}$ **27.** $\dfrac{-5(6x^2 + 8x + 5)}{2(2x^3 + 4x^2 + 5x + 7)^{3/2}}$

29. $\dfrac{(z + 7)(z^2 + 2z + 7)}{z^2(2z + 7)^{3/2}}$ **31.** $\dfrac{2a^2 - x^2}{x^3\sqrt{x^2 - a^2}}$ **33.** $-\dfrac{18}{(x^2 - 9)^{3/2}}$

35. $\dfrac{(4t^2 + 9t - 2)\sqrt{t^2 + 1}}{(2t + 3)^2}$ **37.** $-\dfrac{4z(z^2 - 2z + 5)}{(z + 2)^3(4z - 5)^2}$

39. $\dfrac{3(z - 1)^2}{2\sqrt{(z - 1)^3 - 1}}$ **41.** $\dfrac{1}{4\sqrt{x - a}\sqrt{a + \sqrt{x - a}}}$

43. $-\frac{3}{4}, \frac{4}{3}$ **45.** $-\frac{27}{64}$ **47.** 22.6 cu. in./sec.

Pages 55–56

1. $20x^3 - 24x^2 + 24x$; 0 **3.** $9/x^4$ **5.** $\dfrac{12}{\sqrt{4x + 7}}$

7. $2x(490x^2 + 84x + 3)$ **9.** $-\dfrac{18t}{(t^2 + 9)^{5/2}}$ **11.** $-\dfrac{48x}{(x^2 + 16)^{5/2}}$

13. $\dfrac{8(3x^6 + 40x^3 + 64)}{(x^3 + 8)^{5/3}}$ **15.** $n!$ **17.** $\dfrac{(-1)^n n!}{(x - 1)^{n+1}}\ (n > 1)$

19. Leibnitz's formula:

$$\frac{d^n}{dx^n}(uv) = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v''' + \cdots + nu'v^{(n-1)} + uv^{(n)}$$

Pages 58–59

1. $\frac{2}{y}$ **3.** $-\frac{4x}{9y}$ **5.** $-\frac{y}{x}$ **7.** $-\frac{b^2x}{a^2y}$ **9.** $-\frac{x}{a+y}$ **11.** $-\sqrt{y/x}$

13. $-\frac{x^2}{y^2}$ **15.** $-\frac{3x^2+4xy+y^2}{2x^2+2xy+3y^2}$ **17.** $\frac{-2y}{3x}$

19. (1) $-\frac{16}{y^3}$; (2) $-\frac{16}{y^3}$; (3) $-\frac{16}{9y^3}$; (4) $-\frac{81}{16y^3}$; (5) $2y/x^2$; (6) $-\frac{4a^2}{y^3}$;

(7) $-\frac{b^4}{a^2y^3}$; (8) $-\frac{b^4}{a^2y^3}$; (9) $-\frac{a^2}{(a+y)^3}$; (10) $2y/x^2$; (11) $a^{1/2}/2x^{3/2}$;

(12) $a^{2/3}/3y^{1/3}x^{4/3}$; (13) $-\frac{2a^3x}{y^5}$

21. 0 **23.** $\frac{3}{4}$ **25.** $-\frac{1}{15}$ **27.** $\frac{20}{3}$ **29.** $\frac{5}{3}$

Pages 62–63

1. $2t, 2$ **3.** $\frac{t}{2}, -\frac{t^3}{16a}$ **5.** $\frac{3}{2}t, \frac{3}{4t}$ **7.** $-\frac{2}{3}t^5, \frac{10}{9}t^8$ **9.** $\frac{432}{625t^{14}}$

11. $$\frac{\frac{dx}{dt}\left[\frac{d^3y}{dt^3}\frac{dx}{dt} - \frac{d^3x}{dt^3}\frac{dy}{dt}\right] - 3\frac{d^2x}{dt^2}\left[\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right]}{\left(\frac{dx}{dt}\right)^5}$$

Pages 64–65

1. $6x^2 - 8x + 5 + \frac{8}{x^3}$ **3.** $\frac{3}{2\sqrt{t}} - \frac{14}{3(t+1)^{5/3}}$ **5.** $15x^4 + 144x^3 - 14x$

7. $-\frac{177}{(14x-3)^2}$ **9.** $-\frac{x}{\sqrt{a^2-x^2}}$ **11.** $-\frac{1}{3}(2x+3)(x^2+3x+11)^{-4/3}$

13. $-\frac{5}{7}h(hx+k)^{-12/7}$ **15.** $-\frac{3x^3+a^2x}{\sqrt{a^2+x^2}}$ **17.** $t(7t^2+8)(t^2+4)^{3/2}$

19. $2(2r-1)^2(r^2+1)(7r^2-2r+3)$ **21.** $\frac{3t+10}{2(t+1)^{3/2}}$

23. $\frac{4t^2-3t-3}{(4t-1)^2\sqrt{2t+3}}$ **25.** $-\frac{12}{x^2\sqrt{x^2+9}}$ **27.** $y'' = \frac{2(4-3y)}{(3y-2)^3}$

29. $y'' = \frac{15y}{4x^2}$ **31.** $y'' = \frac{5}{9t^2}$ **33.** $y'' = \frac{1}{2\alpha^3}$ **35.** $y'' = -2\left(\frac{\beta}{\beta+1}\right)^3$

37. $y'' = \frac{6t(t^2-1)}{(t^2+1)^{5/2}}$ **39.** $-\frac{6y}{x^3}$ **41.** $-\frac{45}{64t^{11}}$

43. $\frac{1}{2}$ **45.** $\frac{4}{11}$ **47.** -1 **49.** $\frac{3}{4}$

Pages 69–71

1. $2x - y + 3 = 0$, $x + 2y - 11 = 0$, $\frac{5}{2}\sqrt{5}$, $\sqrt{5}$, $\frac{5}{2}$, 10
3. $3x + 4y - 31 = 0$, $4x - 3y - 8 = 0$, $\frac{20}{3}$, 5, $-\frac{16}{3}$, -3
5. $2x - 9y + 20 = 0$, $9x + 2y + 5 = 0$, $\sqrt{85}$, $\frac{2}{9}\sqrt{85}$, 9, $\frac{4}{9}$
7. $12x - y + 16 = 0$, $x + 12y + 98 = 0$, $\frac{2}{3}\sqrt{145}$, $8\sqrt{145}$, $-\frac{2}{3}$, -96
9. $3x + 4y - 18 = 0$, $4x - 3y + 1 = 0$, 5, $\frac{15}{4}$, -4, $-\frac{9}{4}$
11. $x - 3y + 9 = 0$, $3x + y - 33 = 0$, $6\sqrt{10}$, $2\sqrt{10}$, 18, 2
13. At (2,2): $x + y - 4 = 0$, $y = x$, $2\sqrt{2}$, $2\sqrt{2}$, -2, -2.
15. $2x + y \pm 8 = 0$ 17. None 19. $2x + y - 6 = 0$.
21. $5x - 6y - 30 = 0$, $5x - 6y - 4 = 0$
23. $x - y + 5 = 0$, $27x - 27y + 103 = 0$ 25. arctan $\frac{5}{3}$ 27. arctan $\frac{3}{4}$, 90°
29. arctan $\frac{3}{5}$ 31. arctan 3 33. $x + 3y - 20 = 0$, $31x - 27y - 260 = 0$
35. $5x + 3y - 16 = 0$, $13x - 5y + 48 = 0$ 37. $y = 2x^2 - 3x + 5$
39. $y = x^4 - x^3 + 2x^2 - 3x + 4$

Pages 75–76

1. max. at (1,1) 3. min. at $x = -1$ 5. max. at $(-1,5)$, min. at (0,4)
7. min. at $x = \dfrac{-\sqrt{5}}{5}$, max. at $x = \dfrac{\sqrt{5}}{5}$ 9. min. at $(\pm a,0)$, max. at $(0,a^4)$
11. max. at $x = 1$, min. at $x = 3$ 13. min. at $x = -1$, max. at $x = 2$
15. min. at $x = -1$ and $x = 2$, max. at $x = 1$ and $x = -2$
17. min. at $x = -3$ and $x = 1$, max. at $x = -2$ 19. min. at (0,0)
21. No max. or min. 23. min. at $x = \frac{1}{8}$ 25. max. at $x = \frac{2}{5}$, min. at $x = 2$
27. max. at $x = 1$, min. at $x = 3$ 29. max. of 7 at $x = 1$, min. of 3 at $x = 3$
31. max. of 1 at $t = 1$ 33. No max. or min. 35. min. of $-\frac{27}{512}$ at $w = \frac{1}{8}$
37. min.: $f(1) = -3$, max.: $f(3) = 49$ 39. min.: $f(0) = 0$, no max.

Pages 80–81

1. max. at $x = -1$, infl. at $x = 1$, min. at $x = 3$
3. max. at $x = -1$, infl. at $x = \frac{1}{2}$, min. at $x = 2$
5. min. at $x = \pm 2$, max. at $x = -1$, infl. at $x = \dfrac{-1 \pm \sqrt{13}}{3}$
7. min. at $x = 1$, infl. at $x = 3$, max. at $x = 5$
9. min. at $x = -a$, max. at $x = a$, infl. at $x = 0$ and $x = \pm a\sqrt{3}$
11. infl. at $x = 0$ 13. max. at $x = -a$, min. at $x = a (a > 0)$
15. infl. at $x = a$ 17. min. at $x = a$
19. min. at $x = -\frac{1}{4}$, infl. at $x = -\frac{5}{2}$ and $x = -1$ 21. infl. at $x = a$
23. min. at $x = a$ 33. $y = 2x^3 - 3x^2 - 12x + 2$ 35. $y = -x^4 + 8x^2$
37. $y = 2x^3 - 3x^2 - 12x + 6$ 39. $y = x^4 - 2x^2 + 4$

Pages 89–92

1. A square 3. $\frac{1}{2}a$, $\frac{1}{2}a$ 5. $\frac{1}{3}a$, $\frac{2}{3}a$ 7. 1 9. A square
11. 60 × 120 ft. 13. 18 cu. in. 15. 8 × 8 × 4 in., $14.40
17. 5 × 10 in. 19. Height = diameter 21. Height of cylinder = radius
23. $2a$ 25. Depth = $\sqrt{3}$ times breadth 27. Width = height
29. $h = \frac{1}{3}H$ 31. $h = \frac{4}{3}a$ 35. $a + b$ 37. 3 miles from camp
39. 50 41. 900 43. 205 45. $r = \frac{1}{4}a$, $A = \frac{1}{16}a^2$

47. $h = \frac{1}{3}H$ **49.** Altitude of cylinder = 3 times radius

51. At 2:24 P.M., distance = $12\sqrt{5}$ miles **53.** $h = \dfrac{1}{\sqrt{2}}r$

Pages 96–97

1. $\dfrac{16}{\pi}$ in./min. **3.** $\frac{3}{5}$ ft./min. **5.** (a) 2 m.p.h.; (b) 5 m.p.h.

7. 16π cu. in./sec. **9.** $\frac{72}{13}$ ft./sec. **11.** $\dfrac{3}{16\pi}$ ft./min.

13. 4 ft./sec. **15.** $\frac{25}{54}$ per sec. **17.** 88.4 ft./sec.

19. After $\frac{40}{17}$ sec., 41.8 ft. **21.** (a) 8.66 knots; (b) $1\frac{1}{2}$ hr. after starting

23. $211\frac{1}{3}$ ft./min.

Pages 98–100

1. $x + 4y = 9,\ 4x - y = 2$ **3.** $10x - y = 5,\ x + 10y = 51$

5. $3x + 2y = 13,\ 2x - 3y = 13$ **7.** $2x - y + 2 = 0$ **9.** 0, $\arctan \frac{2}{11}$

11. 90° **13.** $2y^2 - 6y - 3x + 1 = 0$ **15.** $y = x^3 + x^2 + x + 1$

25. Min. at $x = -2$, max. at $x = 2$, infl. at $x = 0$ **27.** Infl. at $x = 4$

29. Max. at $x = 0$, min. at $x = \pm 1$, infl. at $x = \pm 1/\sqrt{3}$

31. Min. at $x = \dfrac{-1 - \sqrt{5}}{2}$, max. at $x = \dfrac{-1 + \sqrt{5}}{2}$, infl. at $x = -\dfrac{1}{2}$

33. Min. at $x = 1$ **35.** $y = (x - 1)^3$ **37.** $y = 3x^5 - 10x^3 + 15x - 7$

41. $\frac{1}{2}$ **43.** $5 \times 5 \times 4$ ft. **45.** Height = $\frac{5}{3}$ times radius

47. 3.62 in.

49. $b - \dfrac{al}{\sqrt{w^2 - l^2}}$ miles on land, $\dfrac{aw}{\sqrt{w^2 - l^2}}$ miles under water **51.** (a) $\frac{3}{4}a$; (b) $\frac{2}{3}a$

53. (a) Radius of semicircle = $\dfrac{a}{\pi}$; (b) height of rectangle = radius of semicircle

55. 3 ft./sec. **57.** $\frac{5}{8}$ ft./sec. **59.** $\frac{4}{5}$ ft./sec.; 5 ft./sec.

Pages 106–107

1. $4 \cos 4x$ **3.** $2ax \sec^2 (ax^2)$ **5.** $-\dfrac{1}{2} \sin \left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

7. $-\csc (x + k) \cot (x + k)$ **9.** $\cos x - x \sin x$

11. $-\dfrac{v}{2} \csc^2 \dfrac{v}{2} + \cot \dfrac{v}{2}$ **13.** $-12 \cos^2 4y \sin 4y$

15. $2 \tan x \sec^2 x - 2 \cot x \csc^2 x$ **17.** $\dfrac{12 \sin^3 3\theta}{\cos^5 3\theta}$ **19.** $-\cot^2 \theta$

21. $\tan \theta \sec \theta + \sin \theta$ **23.** $-\sec \left(\dfrac{\pi}{4} - t\right) \tan \left(\dfrac{\pi}{4} - t\right)$

25. $-(2x \csc^2 x + \cot x) \dfrac{\cot x}{x^3}$ **27.** $\dfrac{2 \sin x(x \cos x - \sin x)}{x^3}$

29. $(1 + \cot x)^2(1 + \cot x - 3x \csc^2 x)$ **31.** $6 \sin 4x \sqrt{1 - \cos 4x}$

33. $\dfrac{-4 \sin x \cos x}{(1 + \sin^2 x)^2}$ **35.** $\sec x(\sec x + \tan x)$

37. $-\frac{b}{a}\cot\theta,\ -\frac{b}{a^2}\csc^3\theta$ **39.** $-\tan\theta,\ \frac{1}{3a}\sec^4\theta\csc\theta$

41. $\frac{\sin\theta}{1-\cos\theta},\ -\frac{1}{a(1-\cos\theta)^2}$ **43.** $\frac{b}{a}\csc\varphi,\ -\frac{b}{a^2}\cot^3\varphi$

45. $-\frac{b}{a}\cot(kt-\alpha),\ -\frac{b}{a^2}\csc^3(kt-\alpha)$

47. Min. value $=0$, max. value $=1$, infl. at $x=\pm\frac{\pi}{4},\ \pm\frac{3\pi}{4},\ \ldots$

49. Min. value $=-\sqrt{a^2+b^2}$, max. value $=\sqrt{a^2+b^2}$

51. No max. or min. **53.** $\arctan\frac{1}{2}$, $\arctan 4\sqrt{2}$ **55.** $\arctan\frac{4}{3}$

Pages 109–110

1. $90°$ **7.** $h=\frac{4}{3}a$ **9.** $h=a\sqrt{2}$ **11.** $(a^{2/3}+b^{2/3})^{3/2}$

13. $5(2^{2/3}+3^{2/3})^{3/2}=35.2$ ft. **17.** $\arccos\frac{2}{3}$, 25 in.

19. a/π, that is, semicircular **21.** Decreasing at 15.2 ft.^{2}/min.

23. 8.7 in.^{2}/sec.

Pages 116–117

1. $\frac{4}{\sqrt{1-16x^2}}$ **3.** $-\frac{3}{\sqrt{16-9x^2}}$ **5.** $\frac{2x}{1+x^4}$ **7.** $-\frac{\sqrt{3}}{2\sqrt{x}\,(x+3)}$

9. $-\frac{1}{\sqrt{1-x^2}}$ **11.** $-\frac{3}{x\sqrt{x^6-1}}$ **13.** $\frac{x}{\sqrt{3x^2-x^4-2}}$

15. $\frac{3}{(3x-5)\sqrt{(3x-4)(3x-6)}}$ **17.** $-\frac{2a^2x}{a^4+x^4}$ **19.** $\frac{1}{\sqrt{t-t^2}}$

21. $\frac{1}{\sqrt{2ax-x^2}}$ **23.** $\frac{2}{2+7x^2}$ **25.** $\frac{1}{1+y^2}$ **27.** $\frac{x^2+1}{x\sqrt{3x^2-x^4-1}}$

29. $\frac{2a}{x^2+a^2}$ **31.** $\frac{3u^3}{9+u^2}+3u^2\arctan\frac{u}{3}$ **33.** $2\arcsin 2x$

35. $2x\arctan(x/a)$ **37.** $2\sqrt{a^2-x^2}$ **39.** $\frac{\sqrt{x}}{1+x}$

41. $\frac{6\arctan 3x}{1+9x^2}$ **45.** $\frac{3}{50}$ radians/min. **47.** Decreasing at $\frac{11}{375}$ radians/sec.

Pages 123–125

1. $\frac{1}{x-2}$ **3.** $1/w$ **5.** $\frac{3x^2}{x^3-9}$ **7.** $\frac{6u-1}{u(3u-1)}$ **9.** $\frac{15}{5x+1}$

11. $\frac{5x^4+3x^2-16x}{(x^2+1)(x^3-8)}$ **13.** $\frac{7-6x}{(3-x)(2+3x)}$ **15.** $\frac{20}{4t^2-25}$ **17.** $\frac{x}{x^2+16}$

19. $\frac{3(6t-5)M}{2(3t^2-5t+1)}$ **21.** $\frac{(15x^2+7)M}{5x^3+7x+1}$ **23.** $\cot\theta$ **25.** $-3\tan 3\varphi$

27. $\frac{M}{2}\tan\frac{1}{2}\varphi$ **29.** $-2M\cot\theta$ **31.** $6\csc 2x\sec 2x\ln^2\tan 2x$

33. $\frac{\sec\theta\csc\theta}{\ln\tan\theta}$ **35.** $\frac{1}{2t\sqrt{\ln t}}$ **37.** $-\frac{\sec\theta\csc\theta}{\ln 3\cot\theta}$

39. $\sec x \tan x(1 + \ln \sec x)$ **41.** $\dfrac{1}{u^2} \log \dfrac{e}{u}$ **43.** $\dfrac{\sec^2 \theta}{1 - \tan^2 \theta}$

45. $\sec \frac{1}{2}x \csc \frac{1}{2}x \log e$ **47.** $-\csc^2 x,\ 2 \csc^2 x \cot x$ **49.** $-\dfrac{2}{x^2}, \dfrac{4}{x^3}$

51. $(7x + 5)(x + 3)^3(x - 1)^2$ **53.** $\dfrac{7x + 9}{(2x + 5)^{1/2}(3x - 1)^{1/3}}$

55. $\dfrac{3x(9x^3 - 135x + 10)(3x^2 - 5)^4}{(x^3 - 9)^4}$ **61.** $2x - y - 1 - \ln 2 = 0$

63. $2x - y = 4 + \ln 2$ **65.** $(-1)^{n-1} \cdot \dfrac{(n - 1)!}{x^n}$

Pages 126–128

1. $3e^{3x}$ **3.** $2e^{2x+1}$ **5.** $2te^{t^2+4}$ **7.** $e^{\tan \theta} \sec^2 \theta$
9. $xe^{-x}(2 - x)$ **11.** $3(2^{3x}) \ln 2$ **13.** $-e^{\cos^2 t} \sin 2t$ **15.** 1
17. $e^{2u}(2 \cos 3u - 3 \sin 3u)$ **19.** $e^{\tan \theta}(1 + \theta \sec^2 \theta)$

21. $-6e^{3x} \sin e^{3x} \cos e^{3x}$ **23.** $\dfrac{2e^{2x}}{\sqrt{1 - e^{4x}}}$ **25.** $\dfrac{1}{\sqrt{e^{2t} - 1}}$

27. $\dfrac{-e^{-x^2}(2x^2 + 1)}{x^2}$ **29.** $3x^{3x}(1 + \ln x)$ **31.** $[1 + (\ln x) \ln \ln x](\ln x)^{x-1}$

33. $x^{(x^x+x)}\left[\dfrac{1}{x} + \ln x + \ln^2 x\right]$ **35.** 0 **37.** $e^x(x + n)$

39. $x - y + 1 = 0$ **41.** (b) $\left(\pm \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{e}}\right)$

Pages 129–130

17. $\dfrac{1}{2} \cosh \dfrac{x}{2}$ **19.** $-3 \sinh (4 - 3t)$ **21.** $\dfrac{16}{5} \operatorname{sech}^2 \left(\dfrac{16y - 1}{5}\right)$

23. $\dfrac{1}{x^2} \operatorname{csch}^2 \dfrac{1}{x}$ **25.** $-\dfrac{1}{2\sqrt{t}} \operatorname{sech} \sqrt{t} \tanh \sqrt{t}$

27. $2t \operatorname{csch} (16 - t^2) \coth (16 - t^2)$ **29.** $2 \tanh x \operatorname{sech}^2 x$

31. $x \cosh^3 \dfrac{x^2}{8} \sinh \dfrac{x^2}{8}$ **33.** $y \cosh y + \sinh y$ **35.** $e^u(\operatorname{sech}^2 u + \tanh u)$

37. $e^x \sinh e^x$ **39.** $\tanh x$ **41.** $\operatorname{csch} x$

Pages 133–134

17. $\dfrac{2x}{\sqrt{x^4 + 1}}$ **19.** $\dfrac{\pm 1}{\sqrt{x^2 - 2x}}$ **21.** $\dfrac{3x^2}{1 - x^6}$ **23.** $\dfrac{2}{4 - x^2}$

25. $-\dfrac{4}{x\sqrt{4 + x^4}}$ for $x < 0$ **27.** $\dfrac{2x^3}{1 - x^4} + 2x \operatorname{argtanh} x^2$ **29.** $\dfrac{6\left(\operatorname{argtanh} \dfrac{x}{2}\right)^2}{4 - x^2}$

Pages 136–137

1. $\frac{2}{3}x^{-1/3} + 6x - x^{-3/2}$ **3.** $2(2x - 1)(x^2 + 1)^2(8x^2 - 3x + 2)$
5. $\dfrac{6 - 4y - 6y^2}{(y^2 + 1)^2}$ **7.** $\dfrac{M}{1 - u^2}$ **9.** $-\dfrac{\ln u}{u^2}$ **11.** $\dfrac{1}{\ln y} + \ln \ln y$

13. $x(3x+2)e^{3x+1}$ **15.** $\dfrac{2}{t^2+2t+5}$ **17.** $-3\csc^2 \frac{3}{2}\theta \cot \frac{3}{2}\theta$

19. $x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$ **21.** $x^{e^x}e^x\left(\dfrac{1}{x} + \ln x\right)$

23. $\dfrac{\sqrt{1-t^2}-t}{1-t^2}$ **25.** $\dfrac{1}{2}x \sinh \dfrac{x^2}{4}$ **27.** $\sinh^2 \dfrac{x}{3} \cosh \dfrac{x}{3}$

29. $e^{\cosh x} \sinh x$ **31.** $\dfrac{\tanh y}{\ln \cosh y}$ **33.** $\dfrac{M}{u\sqrt{1+\log^2 u}}$

35. $\sec \varphi$ **37.** $\dfrac{a^x \ln a}{\sqrt{1-a^{2x}}}$ **39.** $\dfrac{3\sec^2 3x}{2\sqrt{\tan 3x}}$ **41.** $e^{\frac{1}{x}}\left(2x - 1 + \dfrac{1}{x^2}\right)$

43. $\csc \theta$ **45.** $e^{2x} \sin x$ **47.** $\sqrt{y^2+a^2}$ **49.** $y \sin y$

51. $-\dfrac{x}{4y}, -\dfrac{1}{y^3}$ **53.** $-\dfrac{y}{2x}, \dfrac{3y}{4x^2}$ **55.** $-\cos t + \sin t,\ -e^t(\sin t + \cos t)$

57. $te^t,\ e^t(t^2+t)$ **59.** $\tanh u,\ \dfrac{1}{3a}\operatorname{sech}^4 u \operatorname{csch} u$

Pages 141–142

1. $(4x^3 - 21x^2)\,dx$ **3.** $\dfrac{x\,dx}{\sqrt{x^2+4}}$ **5.** $8x(x^2-9)^3\,dx$

7. $(t^2-1)(15t^2+16t-3)\,dt$ **9.** $\dfrac{4r\,dr}{(r^2+1)^2}$ **11.** $4\cos 4\theta\,d\theta$

13. $-\tan^2\theta\,d\theta$ **15.** $-\dfrac{\csc^2 x}{2\sqrt{\cot x}}\,dx$ **17.** $4\cos^2 x\,dx$

19. $\dfrac{dx}{\sqrt{4-x^2}}$ **21.** $\dfrac{dx}{x^2+a^2}$ **23.** $-\dfrac{dx}{x\sqrt{x^2-1}}$ **25.** $\dfrac{\sqrt{x^2-a^2}}{x}\,dx$

27. $\dfrac{M}{t}\,dt$ **29.** $\dfrac{du}{u^2-a^2}$ **31.** $\csc x\,dx$ **33.** $\cosh u\,du$

35. $\tanh x\,dx$ **37.** $2ye^{y^2-1}\,dy$ **39.** $-\sin y\,dy$ **41.** $2xy\,dy + y^2\,dx$

43. $\dfrac{2xy\,dx - x^2\,dy}{y^2}$ **45.** $2x\sin y\,dx + x^2\cos y\,dy$ **47.** $\dfrac{3y(y^2-x^2)}{2x(x^2-6y^2)}$

49. $-\dfrac{y^{1/3}}{x^{1/3}}$ **51.** $-\dfrac{b}{a}\cot\theta;\ -\dfrac{b}{a^2}\csc^3\theta$

53. $\dfrac{b}{a}\coth u,\ -\dfrac{b}{a^2}\operatorname{csch}^3 u$ **55.** $\dfrac{\sqrt{1-u^2}}{1+u^2}, \dfrac{u(u^2-3)}{(1+u^2)^2}$

Pages 144–146

1. $2\pi r\,\Delta r$ = circumference × width
3. $\frac{2}{3}\pi rh\,\Delta r = \frac{1}{3}$ height × circumference of base × change in radius
5. 0.5 cu. ft. **7.** Edge < 100 cm. = 1 m. **9.** 1.5% **11.** 2%

15. (a) $-\dfrac{\Delta x}{x^2}$; (b) $-\dfrac{\Delta x}{x}$ **17.** 0.0098 **19.** 4.96 **21.** 3.009 **23.** 6240

25. $x > 193$ **27.** $x > 34$ **29.** angle > 69°50′
31. angle < 40°25′ **33.** angle > 57°25′ **35.** $\cot\theta\,\Delta\theta$

37. (*a*) 2%; (*b*) 1.7%; (*c*) 2%; (*d*) 50% **39.** $\dfrac{\Delta x}{x}, \dfrac{\Delta x}{x \ln x}$ **43.** $x > 434.29$

45. 0.002 **47.** $N\,\Delta y$ **49.** 75.940 ± 0.004

51. $\dfrac{\sin 2\theta}{2M}\,\Delta y$ **53.** $66°31'0'' \pm 2''$

Pages 154–155

1. (*a*) $\kappa = 2$; (*b*) $\kappa = \dfrac{2}{17\sqrt{17}}$ **3.** $\rho = \frac{5}{4}\sqrt{5}$ **5.** $\rho = \dfrac{7\sqrt{7}}{4}$

7. (*a*) $\rho = \frac{5}{4}\sqrt{5}$; (*b*) $\kappa = 0$ **9.** $\rho = \dfrac{(1+M^2)^{3/2}}{M}$ **11.** $\rho = \dfrac{(1+M^2)^{3/2}}{M}$

13. $\kappa = 1$ **15.** $\dfrac{(1+\cos^2 x)^{3/2}}{|\sin x|}$ **17.** $e^{-x}(1+e^{2x})^{3/2}$ **19.** $\dfrac{(2-x^2)^{3/2}}{|x|}$

21. $\dfrac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}$ **23.** $\dfrac{(4a^2+x^2)^{3/2}}{4a^2}$ **25.** $3(axy)^{1/3}$

27. $\dfrac{(x^4+y^4)^{3/2}}{|2a^2xy|}$ **29.** max. at $\pm(45)^{-1/4}$, zero curv. at $x = 0$

31. max. at $x = 1/\sqrt{2}$, no min. **33.** max. at $x = -\frac{1}{2}\ln 2$, no min.

35. max. at $x = \pm\operatorname{argsinh} 1$, zero (min.) at $x = 0$

Page 156

1. $\rho = \frac{37}{6}\sqrt{37}$ **3.** $\rho = 2\sqrt{2}$ **5.** $\rho = a$, $\kappa = 1/a$

7. $\kappa = \dfrac{2}{3a|\sin 2\varphi|}$ **9.** $\rho = \dfrac{(a^2\sinh^2 u + b^2\cosh^2 u)^{3/2}}{ab}$

11. max. at ends of major axis, min. at ends of minor axis

13. min. at $\theta = \pi$, no max.

Pages 158–159

1. (0,4) **3.** (8,8) **5.** $-143, \frac{241}{12}$ **7.** $(0,2a)$

9. $\left(3x_1 + 2a, -\dfrac{y_1}{4a^2}\right)$ **11.** $27X^2 = 4(Y-2)^3$ **13.** $(aX)^{2/3} + (bY)^{2/3} = (a^2-b^2)^{2/3}$

15. $X = a(\theta + \sin\theta)$, $Y = -a(1-\cos\theta)$, that is, a cycloid of the same size as the original one

17. $X = \pm 2a\cosh^3 u$, $Y = -2a\sinh^3 u$ **19.** $(aX)^{2/3} - (bY)^{2/3} = (a^2+b^2)^{2/3}$

Pages 161–162

41. $x = 2\cos\left(\dfrac{8}{3}\pi t + \varphi\right)$ **43.** $x = \sqrt{33}\cos\dfrac{t}{\sqrt{2}}$ **45.** amp. = 8, per. = $3\dfrac{\pi}{2}$

Pages 166–168

7. $v = \sqrt{2}, j = j_T = j_N = 0$ **9.** $v = 1, j = 1, j_T = 0, j_N = 1$

11. $v = 8, j = 12, j_T = 0, j_N = 12$ **13.** $v = 8, j = 12, j_T = 0, j_N = 12$

15. $v = j = 2, j_T = 0, j_N = 2$ **21.** $v_x = 6/\sqrt{145}, v_y = 72/\sqrt{145}$

23. $v_y = 12, j_x = 0, j_y = 8$ **25.** $x = 0, x = \pi/3$

27. at (4,3) **29.** No collision

Pages 169–170

1. $x^2 + y^2 = a^2$, $\omega = 1$, $v = a$ 3. $\omega = 9$ radians/sec. $= 9/2\pi$ r.p.s.
5. $v_x = -\frac{16}{3}\pi$ ft./sec., $v_y = 4\pi$ ft./sec. 7. $\pm\frac{60}{61}$ radians/sec.
9. $v_x = (1 - \cos\theta)\pi$, $v_y = \pi\sin\theta$, $v = 2\pi\sin\frac{1}{2}\theta$ ft./sec.
11. $v = 4\pi\sin\frac{1}{2}\theta$ ft./sec., $j = \frac{8}{3}\pi^2$ ft./sec.^{2} 13. $-\dfrac{y}{x^2+y^2}$ 15. $-\dfrac{22}{x^2+y^2}$

Pages 170–172

1. $\frac{80}{3}\sqrt{10}$ 3. $\dfrac{(81y^2 + 16x^2)^{3/2}}{1296}$ 5. $\frac{13}{18}\sqrt{13}$ 7. $\frac{5}{4}\sqrt{5}$ 9. 1
11. $\dfrac{|a|(1 + \sec^2\alpha)^{3/2}}{|\tan^3\alpha|}$ 13. $\varphi = 0, \pi$ 25. amp. $= 7$, per. $= \pi/6$
27. $v = \sqrt{5}$, $j = 2\sqrt{2}$, $j_T = 2/\sqrt{5}$, $j_N = 6/\sqrt{5}$
29. $v = \frac{3}{2}$, $j = \frac{3}{2}$, $j_T = 0$, $j_N = \frac{3}{2}$
31. $v = \sqrt{6}$, $j = \sqrt{34}$, $j_T = 14/\sqrt{6}$, $j_N = 2/\sqrt{3}$ 33. $v = \frac{1}{5}\sqrt{34}$, $j = \frac{16}{125}$

Pages 176–177

1. $\psi = \pi/3$, no slope 3. $\psi = -\dfrac{\pi}{4}$, $m = 1$ 5. $\psi = \dfrac{5}{6}\pi$, $m = 0$
7. $\psi = \arctan\frac{5}{9}\sqrt{3}$ 9. 45°, $\arctan\frac{1}{2} = 26°34'$ approximately 11. 8.32
13. $-\pi$ 15. 60° 17. 90° 19. $\arctan\frac{4}{3}$, 45° (at the pole) 21. $\arctan\frac{3}{4}$

Pages 179–180

1. $\frac{27}{5}a$, $\frac{3}{5}a$, $\frac{9}{5}\sqrt{10}\,a$, $\frac{3}{5}\sqrt{10}\,a$ 3. $-a\sqrt{2}$, $-a\sqrt{2}$, $2a$, $2a$
5. $a\theta^2$, a, $a\theta\sqrt{1 + \theta^2}$, $a\sqrt{1 + \theta^2}$
7. $a\sin^2\theta$, $a\sec^2\theta$, $a\tan\theta\sqrt{1 + \sin^2\theta\cos^2\theta}$, $a\sqrt{\tan^2\theta + \sec^4\theta}$
9. $-2a\sqrt{\theta}$, $-\dfrac{a}{2\theta^{3/2}}$, $\dfrac{a}{\sqrt{\theta}}\sqrt{1 + 4\theta^2}$, $\dfrac{a}{2\theta^{3/2}}\sqrt{1 + 4\theta^2}$

Pages 182–183

1. $\kappa = 1/a$ 3. $\rho = \frac{4}{3}a$ 5. $\kappa = 10/a$, $\kappa = 2/3a$
7. $\rho = e^{a\theta}\sqrt{1 + a^2}$ 9. $\rho = \dfrac{a}{\theta^4}(1 + \theta^2)^{3/2}$ 11. $\kappa = \dfrac{(1 - c\cos\theta)^3}{ac(1 + c^2 - 2c\cos\theta)^{3/2}}$
13. $\theta = 0, 2\pi$ 15. $\theta = 0$

Pages 184–185

1. $v_r = -\frac{8}{3}\pi a\sin\theta$ ft./sec., $v_\theta = \frac{8}{3}\pi a\cos\theta$ ft./sec., $v = \frac{8}{3}\pi a$ ft./sec.
3. $v_r = -\dfrac{a}{4}\pi\sqrt{6}$ ft./sec., $v_\theta = \dfrac{a\pi}{2\sqrt{2}}$ ft./sec., $v = \dfrac{a\pi}{\sqrt{2}}$ ft./sec.
5. $v_r = 6$ in./sec., $v_\theta = 3$ in./sec., $\omega = 3$ radians/sec.
9. $\psi = \arctan(-\frac{1}{3}\sqrt{3})$, $m = \frac{1}{5}\sqrt{3}$
11. $\psi = 120°$, $m = 0$ 13. $\psi = -\arctan\frac{1}{2}$, $m = \frac{1}{3}$
15. $\psi = -\arctan\dfrac{1}{4\sqrt{3}}$, $m = -\dfrac{13}{3\sqrt{3}}$ 17. 135°, 45° 19. $\arctan\frac{3}{4}$

21. $\theta = \pm \arctan 1/\sqrt{5}, 90°, 270°$ **23.** $\theta = 30°, 90°, 210°, 270°$

25. $\frac{1}{2}a \sin^2 \theta \tan \theta$, $2a \sin \theta \cos \theta$, $\frac{a}{2} \sin^2 \theta \sqrt{4 + \tan^2 \theta}$ **27.** $\rho = \frac{a^2}{3r}$

29. center at $\left(\frac{2}{3}a, \frac{\pi}{2}\right)$ **31.** $\Delta r = 0.009$, $\Delta s = 0.012$ **33.** $\Delta r = 0.21$, $\Delta s = 0.60$

Page 190

9. $\xi = \sqrt{\frac{7}{3}}$ **11.** $\xi = \pm\sqrt{\frac{7}{3}}$ **13.** $\xi = 1.08$ **15.** $\xi = 1.5$

Pages 195–197

1. $\frac{5}{3}$ **3.** ∞ **5.** 2 **7.** $+\infty$ **9.** -1 **11.** 0
13. 4 **15.** $\frac{1}{2}$ **17.** ∞ **19.** $-\infty$ **21.** 1 **23.** 0
25. 1 **27.** $\frac{1}{2}$ **29.** 1 **31.** ∞ **33.** 1 **35.** 0
37. 0 **39.** $-\frac{1}{2}$ **41.** ∞ **43.** If $a \leqq 0$, no limit (∞); if $a > 0$, limit is 0

Pages 198–199

1. $-\infty$ **3.** $-\frac{1}{2}$ **5.** $-\frac{1}{2}$ **7.** 0 **9.** 1
11. If $a \leqq 0$, no limit ($-\infty$), if $a > 0$, limit is 0 **13.** 1 **15.** 1 **17.** 1
19. 1 **21.** 1 **23.** e^k **25.** e^2

Page 204

1. $|R| < 10^{-7}$ **5.** $n = 3$ (that is, three terms) **7.** $|R| < 0.000001$

9. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} \sin \xi \; (0 < \xi < x)$

11. Five terms, that is, $1 - \frac{(0.2)^2}{2!} + \frac{(0.2)^4}{4!}$

Pages 207–208

1. No max. or min. **3.** min. at $x = -2$ **5.** min. at $\theta = 0$
7. No max. or min. **9.** 0 **11.** 1 **13.** 0 **15.** 1 **17.** $-\infty$ **19.** k
21. 1 **27.** $|R| < 0.000005$ **29.** $|R| < 0.0003$

Pages 214–215

1. $y = x^2 + x - 10$ **5.** $y = \frac{1}{2}x^2 - 5x + \frac{19}{2}$ **7.** $y = x^2 - 5x + 8$
9. $y = x^3 - 2x^2 - 6x + 9$ **11.** $y = -16t^2 + 128t$, 256 ft., 128 ft./sec.
13. 3 sec., 96 ft./sec. **15.** 400 ft. above the ground, 160 ft./sec.

17. The parabola $y = -\frac{g}{2v_0^2} x^2 \sec^2 \alpha + x \tan \alpha$ **19.** $\frac{v_0^2}{2g} \sin^2 \alpha$

Pages 221–222

25. $N = 100e^{0.343t}$ **27.** 29,000 **29.** In 34.6 years

Page 235

1. $\sin \theta - \theta \cos \theta + C$ **3.** $e^x(x - 1) + C$
5. $x \tan x + \ln |\cos x| + C$ **7.** $\frac{1}{2}e^{x^2} + C$
9. $(y^2 - 2) \sin y + 2y \cos y + C$ **11.** $\frac{1}{15}(3x^2 - 2)(x^2 + 1)^{3/2} + C$
13. $(x^3 + 6x) \sinh x - (3x^2 + 6) \cosh x + C$

15. $x \tanh x - \ln \cosh x + C$

17. $\dfrac{x^2+2}{\sqrt{x^2+1}} + C$

19. $x \arctan x - \frac{1}{2} \ln (1 + x^2) + C$

21. $\frac{1}{4}x^2(2 \ln x - 1) + C$

23. $\frac{1}{3} \ln^3 x + C$

25. $\dfrac{x^4}{2} \sqrt{x^4 - 16} - \frac{1}{3}(x^4 - 16)^{3/2} + C$

27. $x \operatorname{argsinh} x - \sqrt{x^2 + 1} + C$

29. $\frac{1}{2}e^x(\sin x - \cos x) + C$

31. $\frac{1}{2}e^{-\theta}(\sin \theta - \cos \theta) + C$

33. $\dfrac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$

35. $\dfrac{x^{n+1}}{n+1}\left(\ln x - \dfrac{1}{n+1}\right) + C$

37. $-\frac{1}{2}[\csc x \cot x + \ln |\csc x + \cot x|] + C$

Pages 241–242

1. $\frac{1}{3} \cos^3 \theta - \cos \theta + C$

3. $-\frac{1}{6} \cos^3 2x + C$

5. $-\frac{2}{7} \cos^{7/2} \theta + \frac{2}{11} \cos^{11/2} \theta + C$

7. $\frac{1}{3} \tan 3\theta - \theta + C$

9. $\frac{1}{5} \tan^5 \theta - \frac{1}{3} \tan^3 \theta + \tan \theta - \theta + C$

11. $-\frac{1}{10} \cot^5 2x - \frac{1}{6} \cot^3 2x + C$

13. $\frac{1}{8} \tan^8 \alpha + \frac{1}{10} \tan^{10} \alpha + C$

15. $-\frac{1}{9} \csc^9 \theta + \frac{2}{7} \csc^7 \theta - \frac{1}{5} \csc^5 \theta + C$

17. $\frac{1}{2}u + \frac{1}{4} \sin 2u + C$

19. $\frac{1}{3} \sin 3x - \frac{1}{3} \sin^3 3x + \frac{1}{5} \sin^5 3x - \frac{1}{21} \sin^7 3x + C$

21. $\frac{1}{8}x - \frac{1}{32} \sin 4x + C$

23. $\frac{3}{128}x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C$

25. $\frac{1}{15} \sin^5 3\theta + C$

27. $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + C$

29. $\frac{1}{3}x \cos^3 x - x \cos x + \frac{2}{3} \sin x + \frac{1}{9} \sin^3 x + C$

31. $-\frac{1}{6} \cot^3 2x + \frac{1}{2} \cot 2x + x + C$

33. $\frac{1}{10}e^x(5 - 2 \sin 2x - \cos 2x) + C$

35. $\dfrac{1}{2}\left[\dfrac{\sin (m-n)x}{m-n} - \dfrac{\sin (m+n)x}{m+n}\right] + C$

37. $-\dfrac{1}{2}\left[\dfrac{\cos (m+n)x}{m+n} + \dfrac{\cos (m-n)x}{m-n}\right] + C$

39. $\frac{1}{16}(4 \sin 2x - \sin 8x) + C$

Pages 247–248

1. $2\sqrt{x} - 2 \ln (1 + \sqrt{x}) + C$

3. $\frac{2}{3}x^{3/2} - 3x + 18\sqrt{x} - 54 \ln (3 + \sqrt{x}) + C$

5. $2\sqrt{9 - 4x} + 3 \ln \left|\dfrac{3 - \sqrt{9 - 4x}}{3 + \sqrt{9 - 4x}}\right| + C$

7. $\frac{3}{40}(1 + x)^{5/3}(5x - 3) + C$

9. $\frac{4}{125}(3 + 5x)^{1/4}(5x - 12) + C$

11. $\sqrt{a^2 + x^2} + C$

13. $\sqrt{a^2 - x^2} + \dfrac{a}{2} \ln \left|\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}}\right| + C$

15. $\frac{1}{15} \sqrt{x^2 - a^2}\,(3x^4 - 11a^2x^2 + 23a^4) + a^5 \arcsin \dfrac{a}{x} + C$

17. $\frac{2}{21}(1 + x^3)^{7/2} - \frac{4}{15}(1 + x^3)^{5/2} + \frac{2}{9}(1 + x^3)^{3/2} + C$

19. $\dfrac{2}{x^2 + 4} + \dfrac{1}{2} \ln (x^2 + 4) + C$

21. $\dfrac{a^2}{a^2 - x^2} - \dfrac{a^4}{4(x^2 - a^2)^2} + \dfrac{1}{2} \ln |x^2 - a^2| + C$

23. $\dfrac{(2x^2 - 1)\sqrt{1 + x^2}}{3x^3} + C$

25. $-\frac{1}{\sqrt{2}}\ln\left|\frac{x+1+\sqrt{2(5x^2+4x+1)}}{3x+1}\right|+C$ **27.** $\frac{x}{9\sqrt{x^2+9}}+C$

29. $-\frac{1}{4a^2x^4}+\frac{1}{2a^4x^2}-\frac{1}{2a^6}\ln\left(\frac{x^2+a^2}{x^2}\right)+C$

31. $2\sqrt{4+x}-8\ln(4+\sqrt{4+x})+C$

33. $\frac{4}{3}\sqrt{1+\sqrt{x}}\,(\sqrt{x}-2)+C$ **35.** $2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}+C$

37. $\ln\sec^2\sqrt{x}+C$ **39.** $\frac{2}{3}(e^x+8)\sqrt{e^x-4}+C$

41. $\frac{2}{15}(e^x+1)^{3/2}(3e^x-2)+C$ **43.** $\frac{1}{4}(1+\cos x)\cos x+\frac{1}{8}\ln|1-2\cos x|+C$

45. $\frac{1}{3}(\tan^2\theta-18)\sqrt{9+\tan^2\theta}+C$

Pages 250–251

1. $\frac{-\sqrt{4-x^2}}{4x}+C$ **3.** $\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$

5. $-\frac{\sqrt{a^2+x^2}}{a^2x}+C$ **7.** $\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})+C$

9. $\frac{1}{a^2}\ln\frac{\sqrt{x^2-a^2}}{|x|}+C$ **11.** $\frac{1}{3}(x^2-7)^{3/2}-7\sqrt{x^2-7}-7\sqrt{7}\arcsin\frac{\sqrt{7}}{x}+C$

13. $\frac{x}{25\sqrt{9x^2+25}}$ **15.** $\frac{x}{2}\sqrt{5-3x^2}+\frac{5}{2\sqrt{3}}\arcsin\sqrt{\frac{3}{5}}\,x+C$

17. $\frac{1}{250}\left[\frac{5(x+3)}{x^2+6x+34}+\arctan\frac{x+3}{5}\right]+C$

19. $\sqrt{ax-x^2}+a\arcsin\sqrt{\frac{x}{a}}+C$ **21.** $\frac{1}{a}\ln\left|\frac{x}{x+a}\right|+C$

23. $-\frac{\sqrt{x}}{2(1+2x)}+\frac{1}{2\sqrt{2}}\arctan\sqrt{2x}+C$ **25.** $x-\ln|1-e^x|+C$

27. $2\,\mathrm{arcsec}\,e^{x/2}+C$ **29.** $\frac{1}{4a^4}\ln\frac{x^4}{a^4+x^4}+C$

31. $-\frac{\sqrt{4x+x^2}}{2x}+C$ **33.** $\frac{x}{a^2\sqrt{a^2+x^2}}+C$

35. $\frac{1}{a}\,\mathrm{argtanh}\,\frac{x}{a}+C=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

Pages 257–259

1. $\frac{1}{3}\ln\left|\frac{x}{x+3}\right|+C$ **3.** $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C$ **5.** $\frac{1}{8}\ln|(x-7)^7(x+1)|+C$

7. $\frac{1}{30}\ln\left|\frac{x^5(x-3)}{(x-1)^6(x+2)}\right|+C$

9. $\frac{1}{2}x^2+4x+\frac{27}{2}\ln|x-3|-\frac{1}{2}\ln|x-1|+C$

11. $\frac{1}{10}\ln\frac{|x^2-1|}{x^2+4}+C$ **13.** $\frac{1}{5}\ln|1-5\csc\theta|+C$

15. $\frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta}+C$ **17.** $\frac{1}{3}\ln\frac{|e^x-1|}{e^x+2}+C$

19. $\frac{2x-1}{2x^2}+\ln\left|\frac{x}{1+x}\right|+C$ **21.** $\frac{x}{8(4-x^2)}+\frac{1}{32}\ln\left|\frac{x+2}{x-2}\right|+C$

23. $\frac{1}{2}x^2 + 2x - \frac{3}{x-1} - \ln|x^2 + 2x - 3| + C$ **25.** $\frac{1-2x}{x(x-1)} + 2\ln\left|\frac{x}{x-1}\right| + C$

27. $\frac{-1}{2-\sin\theta} + \ln\frac{2-\sin\theta}{1-\sin\theta} + C$ **29.** $\frac{1}{3x^3} + \ln|2x-1| + C$

31. $-\frac{2e^x + 1}{e^x(e^x+1)} + 2\ln(1 + e^{-x}) + C$

33. $\frac{1}{2(1-\sin\theta)} + \frac{1}{2}\ln|\sec\theta - \tan\theta| + C$

35. $\frac{2}{5}\arctan\frac{x}{2} + \frac{1}{10}\ln\frac{(x-1)^2}{x^2+4} + C$

37. $\frac{1}{4}\ln\frac{x^2}{x^2+2x+2} - \frac{1}{2}\arctan(x+1) + C$

39. $\ln(x^2+4)(x^2+1)^2 + \frac{3}{2}\arctan\frac{x}{2} - \arctan x + C$

41. $\frac{1}{6}\ln\frac{(x+1)^2}{(x^2-x+1)} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$

43. $\frac{1}{16}\ln\frac{x^2+2x+2}{x^2-2x+2} + \frac{1}{8}\arctan(x+1) + \frac{1}{8}\arctan(x-1) + C$

45. $\frac{8-x}{2(x^2+4)} + \frac{1}{4}\arctan\frac{x}{2} + \ln|x| + C$ **47.** $-\frac{x}{2(x^2+1)} - \frac{1}{x} - \frac{3}{2}\arctan x + C$

Pages 261–262

1. $\frac{1}{2}\ln\left|\frac{x-2+\sqrt{x^2+x+4}}{x+2+\sqrt{x^2+x+4}}\right| + C$ **3.** $\ln\left|\frac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}\right| + C$

5. $-\frac{4}{x+\sqrt{x^2+2x}} + \ln|x+1+\sqrt{x^2+2x}| + C$

7. $\frac{x-1}{\sqrt{2x-x^2}} + C$ **9.** $\frac{1}{2}\ln\left|\frac{\sqrt{4+x} - 2\sqrt{1-x}}{\sqrt{4+x}+2\sqrt{1-x}}\right| + C$

11. $\frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right| + C$ **13.** $\frac{1}{4}\ln\left|\tan\frac{x}{2} - 3\right| - \frac{1}{4}\ln\left|3\tan\frac{x}{2} - 1\right| + C$

15. $\ln\left|\tan\frac{x}{2}\right| + C$ **17.** $-\ln\left|1 + \cot\frac{x}{2}\right| + C$

19. $\frac{x}{4} - \frac{5}{18}\arctan\left(\frac{4 + 5\tan\frac{3x}{2}}{3}\right) + C$

Pages 265–266

1. $\frac{1}{12}\sin^4\theta(1 + 2\cos^2\theta) + C$ **3.** $-\frac{1}{15}\cos^3\theta(2 + 3\sin^2\theta) + C$
5. $\sin^2\theta(\sin\theta\tan\theta + \cos\theta) + 2\cos\theta + C$
7. $-\frac{1}{2}\cot\theta\csc\theta + \frac{1}{2}\ln|\csc\theta - \cot\theta| + C$
9. $\frac{1}{6}\sin^5 x\cos x - \frac{1}{24}\sin^3 x\cos x - \frac{1}{16}\sin x\cos x + \frac{1}{16}x + C$
11. $\frac{3}{8}(x - \sin x\cos x) - \frac{1}{4}\sin^3 x\cos x + C$ **13.** $-\frac{5}{171}\cos^{9/5}\theta(10 + 9\sin^2\theta) + C$

15. $\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2+x^2}) + C$

17. $\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\arctan\frac{x}{a} + C$ **19.** $-\frac{2}{9}(x^3+2)\sqrt{1-x^3} + C$

Pages 266–268

1. $2\sin\dfrac{x}{2} - \dfrac{2}{3}\sin^3\dfrac{x}{2} + C$ **3.** $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

5. $\frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x} + C$ **7.** $\frac{3}{5}\tan^{5/3} x + \frac{3}{11}\tan^{11/3} x + C$

9. $\frac{1}{3}(x^2 + 2a^2)\sqrt{x^2 - a^2} + C$ **11.** $\frac{1}{70}(1 + 2x^2)^{5/2}(5x^2 - 1) + C$

13. $-2\sqrt{a^2 - x^2} + C$ **15.** $\frac{5}{128}x + \frac{1}{48}\sin^3 2x - \frac{1}{128}\sin 4x - \frac{1}{1024}\sin 8x + C$

17. $\frac{1}{3}x\tan^3 x - x\tan x - \frac{1}{6}\tan^2 x + \frac{1}{2}x^2 + \frac{4}{3}\ln|\sec x| + C$

19. $4\sqrt{1+\sqrt{x}} + \dfrac{4}{\sqrt{1+\sqrt{x}}} + C$ **21.** $\frac{1}{3}(e^{2x} - 2)\sqrt{e^{2x} + 1} + C$

23. $\frac{1}{30}(1 + 2\sin\theta)^{3/2}(6\sin\theta - 2) + C$ **27.** $-\frac{1}{15}(3x^2 + 32)(16 - x^2)^{3/2} + C$

29. $-\dfrac{x}{2(x^2 + a^2)} + \dfrac{1}{2a}\arctan\dfrac{x}{a} + C$ **31.** $\dfrac{2}{b}\left[\sqrt{x} - \sqrt{\dfrac{a}{b}}\arctan\sqrt{\dfrac{bx}{a}}\right] + C$

33. $\frac{1}{6}\ln|x| - \frac{5}{2}\ln|x - 2| + \frac{7}{3}\ln|x - 3| + C$ **35.** $\dfrac{1}{5}\ln\left|\dfrac{x-2}{2x+1}\right| + C$

37. $-\dfrac{1}{2(x-2)} + \dfrac{1}{4}\ln\left|\dfrac{x}{x-2}\right| + C$

39. $\dfrac{1}{49}\left[\dfrac{63}{x+3} + 16\ln|x - 4| + 33\ln|x + 3|\right] + C$ **41.** $\dfrac{1}{2a^2}\ln\dfrac{u^2}{u^2 + a^2} + C$

43. $\dfrac{1}{6a^2}\ln\dfrac{(x+a)^2}{x^2 - ax + a^2} + \dfrac{1}{a^2\sqrt{3}}\arctan\dfrac{2x - a}{a\sqrt{3}} + C$

45. $\ln\dfrac{|x^2 - 1|(x-1)^2}{x^2 + 2x + 5} - \dfrac{3}{2}\arctan\dfrac{x+1}{2} + C$ **47.** $2\sqrt{x^2 + x + 1} + C$

49. $2\arctan(x + \sqrt{x^2 + 2x - 1}) + C$

51. $-\dfrac{4}{x + \sqrt{x^2 + 4x}} + \ln|x + 2 + \sqrt{x^2 + 4x}| + C$

53. $\dfrac{2}{\sqrt{15}}\arctan\left(\sqrt{\dfrac{3}{5}}\tan\dfrac{x}{2}\right) + C$ **55.** $\tan\dfrac{x}{2} + C$

Pages 274–275

1. $s(5) = 2.80$, $S(5) = 3.20$ **3.** 1.066 **5.** 0.889 **7.** 0.822

9. $s(20) = 1.209$, $S(20) = 1.229$ **11.** $s(20) = 0.9169$, $S(20) = 0.9562$

Pages 280–281

1. 3 **3.** 1.099 **5.** 0.910 **7.** 0.865 **9.** 1.219 **11.** 0.9371

13. $\ln 2 = 0.69315$ **15.** $\pi/8$ **17.** $\operatorname{arcsec} 4 - \dfrac{\pi}{3} = 0.2531$ **19.** $\pi/18$

21. $\ln 2$ **23.** 1 **25.** $\sqrt{3}\,a^2$ **27.** 0 **29.** $\frac{1}{2}$ **31.** $a(\sqrt{2} - 1)$

33. $\frac{4}{3}a^2$ **35.** 2 **37.** $\dfrac{\pi}{2}a^2$ **39.** 9

Pages 284–285

1. $\frac{1}{2}\pi a^2$ **3.** $\frac{14}{5}\sqrt{3}\,a^5$ **5.** $4(1 - \ln 2)$ **7.** $a\left(\sqrt{3} - \dfrac{\pi}{3}\right)$ **9.** 2π

11. 128 **13.** $\frac{2}{3}a^3$ **15.** $\frac{1}{8}\pi a^4$ **17.** $\pi/4$ **19.** 0 **21.** $\frac{4}{3}a^2$ **23.** $\frac{1}{2}\pi a^2$

25. $\frac{1}{3}a^2b$ **27.** $\frac{4}{3}a^2b$ **29.** $\frac{3}{2}a$

Pages 290–291

1. $\pi/2$ **3.** $2\sqrt{a}$ **5.** Divergent **7.** -1
9. Divergent **11.** $3\sqrt[3]{2a}$ **13.** 0 **15.** $\frac{1}{2}\pi a^2$
17. $\frac{1}{2}$ **19.** $\pi/8a^2$ **21.** Divergent **23.** Divergent
25. Divergent **27.** $1/a^2$ **29.** $0.347/a^2$ **31.** Divergent

Pages 292–294

1. $\frac{1}{128}$ **3.** Divergent **5.** $\frac{1}{2}(1 - \ln 2)$ **7.** $3\pi/16$ **9.** $\pi/12$ **11.** $\frac{8}{3}$
13. $\pi/2$ **15.** $6 - 2\sqrt{2}$ **17.** e **19.** 0 **21.** -4 **23.** $\frac{848}{105}$ **25.** 3.8975

Pages 298–299

1. $\frac{4}{3}$ **3.** 12 **5.** $\frac{32}{3}$ **7.** 4 **9.** $\frac{20}{3}$ **11.** $\frac{8}{3}a^2$
15. $a^2[2\sqrt{3} - \ln(2 + \sqrt{3})]$ **17.** 16 **19.** $-24 + 50 \arcsin \frac{3}{5} = 8.3$ **21.** 1
23. $2\sqrt{2}$ **25.** $\frac{2}{3}a^2(3\pi - 2)$ **27.** $3\pi a^2$ **29.** $\dfrac{3\pi ab}{4}$ **31.** $\frac{3}{8}\pi a^2$ **33.** $\frac{1}{8}a^2$
35. $\frac{1}{2}a^2\alpha$

Page 302

1. πa^2 **3.** $\frac{3}{2}\pi a^2$ **5.** $\frac{8}{3}a^2$ **7.** a^2 **9.** $\frac{1}{2}\pi a^2$ **11.** $\frac{4}{3}\pi^3 a^2$ **15.** $\frac{1}{2}\pi a^2$
17. $\frac{1}{2}a^2(2\pi - 3\sqrt{3})$ **19.** $1.4a^2$ **21.** $\frac{32}{3}$ **23.** $\dfrac{a^2}{2}\left(1 - \dfrac{\pi}{4}\right)$

Pages 306–308

3. $\frac{8}{5}\pi a^3$ **5.** $\frac{4}{3}\pi a^3$ **7.** $\frac{16}{5}\pi$ **9.** $\frac{4}{3}\pi ab^2$ **13.** $(224\pi)/3$ **15.** $4\pi\sqrt{3}\,a^3$
19. $\frac{1}{2}\pi^2$ **21.** $\frac{1}{3}\pi h^2(3a - h)$ **23.** $\frac{32}{105}\pi a^3$ **25.** $\frac{1}{2}\pi e^2(e^4 - 1) = 621.8$
27. $2\pi\left(e - \dfrac{3}{e}\right)$ **29.** $\pi/2$ **31.** $\frac{4}{3}\pi h^3$ **33.** $\frac{4}{3}\pi(8 - 3\sqrt{3})a^3$ **35.** $\frac{224}{3}\pi$
37. $2\pi^2 abc$

Pages 310–311

1. $\frac{1}{6}abc$ **3.** $\frac{4}{3}\pi abc$ **5.** $\frac{1}{3}abh$ **7.** $\frac{1}{3}a^3$ **9.** $\frac{2}{5}a^3$ **11.** $\frac{34}{3}$
13. $\frac{16}{15}a^3$ **15.** $\frac{1}{12}\pi a^3$ **17.** $\frac{16}{3}a^3$ **19.** $a^3 \ln(2 + \sqrt{3})$
21. $\frac{32}{15}a^3$ **23.** $\frac{1}{3}\pi a^2 c$ **25.** $\frac{1}{4}a^2 b$ **27.** $\frac{8}{3}ab^2$

Page 315

1. $\frac{335}{27}$ **3.** $5 - \sqrt{10} + 3\ln\dfrac{3 + \sqrt{10}}{2}$ **5.** $\ln(2 + \sqrt{3})$
7. $a[\sqrt{2} + \ln(1 + \sqrt{2})] = 2.29a$ **9.** $6a$ **11.** $8a$ **13.** $4\sqrt{3}$ **15.** $\frac{3}{2}\pi a$
17. $\dfrac{a}{2}\sqrt{2}\,[\sqrt{2} + \ln(1 + \sqrt{2})]$ **19.** $\frac{1}{2}\sqrt{13}\,(1 - e^{-2\pi})$

Page 319

1. $4\pi a^2$ **3.** $\frac{8}{3}\pi(2\sqrt{2} - 1)a^2$ **5.** $\frac{32}{5}\pi a^2$ **7.** $\frac{441}{4}\pi$ sq. in. **11.** $\frac{12}{5}\pi a^2$
13. $2\sqrt{2}\,\pi a^2(\sqrt{2} - 1)$ **15.** $2\pi a^2(1 - e^{-1})$ **17.** $\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$
19. $\dfrac{e^{a\pi} - 2a}{1 + 4a^2}$ **21.** $2\pi b^2 + \dfrac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \arccos\dfrac{b}{a}$ **23.** a^2 **25.** $\dfrac{a^2}{12}(5\sqrt{5} - 1)$

Pages 321–322

1. $2/\pi$ **3.** $(a/4)\pi$ **5.** $\frac{8}{3}a^2$ **7.** $\frac{4}{3}a^2$ **9.** $\frac{4}{3}ab$ **11.** $(\pi/4)a^2$ **13.** $(4/\pi)a$
15. $(2/\pi)a$ **17.** a

Pages 322–323

1. $\pi + 2$ **3.** $\pi/2$ **5.** $\frac{4}{3}a^2$ **7.** $\frac{5}{8}\pi^2 r^2$ **9.** $\frac{9}{2}\pi a^2$
11. $\frac{1}{2}\pi a^2$ **13.** $2\pi^2 a^2$ **17.** $2.89a^3$ **19.** $\frac{1}{4}a^2$ **21.** $\pi a^2 \tan \alpha$
23. $\ln (1 + \sqrt{2})$ **25.** $-\frac{1}{2} + \ln 3$ **31.** $\frac{\sqrt{1 + a^2}}{a}(r_2 - r_1)$
35. $4a^2$ **27.** $\frac{2}{9}\pi a^3$ **39.** $\frac{1}{2}\pi v_0 a^2$ cu. ft.

Page 326

1. 30 ft.-lb. **3.** $\frac{1}{2}n^2hw$ ft.-lb. **5.** $\frac{250}{3}$ ft.-lb.
7. $43{,}750\pi w$ ft.-lb. **9.** $540\pi w$ ft.-lb. **11.** $\frac{783}{4}\pi w$ ft.-lb.
13. 3,529 ft.-lb. **15.** $24\pi w$ ft.-lb. **17.** $\frac{2752}{3}\pi w$ ft.-lb.

Page 330

1. $(-\frac{5}{8}, \frac{31}{16})$ **5.** $(\frac{9}{5}, \frac{4}{5})$ **7.** $(\frac{4}{3}, 2)$
9. $\bar{z} = \frac{29}{12}$ in. **11.** $\bar{z} = 6.9$ in. **13.** $\frac{18}{7}$ in. above base

Pages 332–333

1. $(\frac{6}{5}, \frac{3}{2})$ **3.** $(0, \frac{2}{5}a)$ **5.** $(\frac{8}{15}, \frac{8}{21})$ **7.** $\bar{y} = \frac{4a}{3\pi}$ **9.** Intersection of medians

11. $(\frac{8}{5}, -1)$ **13.** $(2, \frac{1}{5})$ **15.** $(1.61a, 0)$ **17.** $\left(\frac{256a}{315\pi}, \frac{256a}{315\pi}\right)$ **21.** $\bar{x} = \frac{5}{8}a$

Pages 334–335

1. $\bar{x} = \frac{2}{5}a$ **3.** $\bar{y} = \frac{5}{8}a$ **5.** $\bar{x} = \frac{3}{8}a$ **7.** $\bar{y} = \frac{3}{8}\sqrt{3}\,a$ **9.** $\bar{x} = 0$
11. $\bar{z} = 2a$ **13.** $\bar{x} = \frac{3}{8}a$ **15.** $(\frac{3}{16}\pi a, \frac{3}{8}a, \frac{3}{32}\pi a)$ **17.** $\frac{3}{8}\sqrt{3}\,a$ units above base
19. 0.38 in. below base of cone

Page 337

1. $\left(\frac{2a}{\pi}, \frac{2a}{\pi}\right)$ **3.** $(\frac{2}{5}a, \frac{2}{5}a)$ **5.** $\bar{x} = \frac{a \sin \alpha}{\alpha}$ **7.** $\bar{x} = \frac{4}{5}a, \bar{y} = 0$
9. $\frac{1}{2}(a + b)$ units from the center
17. 6 ft. (approximately) above vertex of lower cone

Pages 340–341

1. $\frac{9}{4}w$ lb. **3.** $\frac{140}{3}w$ lb. **5.** 12.0 lb. **7.** $27\pi w$ lb. **9.** 600π lb.
11. $\frac{16}{5}w$ lb. **13.** 40.9 ft. below surface of water **15.** $\frac{5}{2}$ ft. below top of tank
17. $\frac{8}{7}$ ft. below surface of water

Pages 345–346

1. $I_x = 79, I_y = 87$ **3.** $I_{xy} = 118, I_{yz} = 65, I_{xz} = 120$
5. $I_x = 133, I_y = 248, I_z = 311$ **7.** a^2M **9.** $\frac{1}{3}l^2M$ **11.** $\frac{1}{4}a^2M$
13. $\frac{1}{8}a^2M$ **15.** $\frac{8}{35}a^2M$ **17.** $I_y = \pi^2 - 4$ **19.** $I_x = \frac{1}{9}, I_y = 2$
21. $\frac{1}{2}a^2M$ **23.** $\frac{1}{2}a^2M$ **27.** $\frac{5}{9}a^2M$ **29.** $\frac{49}{5}\pi$
31. $\frac{2}{5}a^2M$ **33.** $\frac{3}{2}a^2M$ **35.** $\frac{3}{8}a^2M$ **37.** $\frac{2}{3}a^2M$ **39.** $\frac{1}{2}a^2M$

Pages 351–353

1. 42.5 **3.** 20.4 **5.** 63.3 **7.** 26.4 **9.** 0.93 **11.** 0.9974 **15.** 12.5
17. 26.7 **19.** 1.19 **21.** 2.48 **23.** 0.877

Pages 353–354

1. 4 ft.-lb. **3.** $\dfrac{\pi a^2h^2}{12}w$ ft.-lb. **5.** $(\frac{9}{16}, -\frac{3}{4})$ **7.** $\bar{z} = \frac{57}{25}$

9. $\bar{z} = \frac{51}{11}$ in. **11.** $(\frac{9}{5}a, \frac{9}{5}a)$ **13.** $(\pi/2, \frac{3}{8})$ **17.** $\left(\dfrac{256a}{315\pi}, \dfrac{256a}{315\pi}\right)$

19. $\bar{y} = \dfrac{4(a^2 + ab + b^2)}{3\pi(a + b)}$ **21.** $\bar{z} = \frac{3}{4}a$ **23.** $\bar{z} = \frac{1}{4}h$

25. $2\frac{7}{10}$ in. above vertex of cone **27.** $\bar{x} = \frac{1}{3}(1 + \sqrt{2})a$ **29.** $\frac{32}{15}w$ lb.

31. $\frac{1}{4}a^2M$, $\frac{1}{4}b^2M$ **33.** $\dfrac{21\pi a^4}{2048}$ **35.** $\frac{512}{15015}\pi a^5$ **37.** $\frac{1}{8}\pi a^2M$

Pages 359–360

1. $\dfrac{\partial z}{\partial x} = 3x^2 + 8xy, \dfrac{\partial z}{\partial y} = 4x^2$ **3.** $\dfrac{\partial z}{\partial x} = 2 - \dfrac{y}{x^2}, \dfrac{\partial z}{\partial y} = \dfrac{1}{x}$

5. $\dfrac{\partial z}{\partial x} = \dfrac{3x^2}{x^3 + y^2}, \dfrac{\partial z}{\partial y} = \dfrac{2y}{x^3 + y^2}$ **7.** $\dfrac{\partial u}{\partial x} = 2xe^{x^2+3t}, \dfrac{\partial u}{\partial t} = 3e^{x^2+3t}$

9. $\dfrac{\partial u}{\partial x} = (2x + y)10^{x^2+xy+z^2}\ln 10, \dfrac{\partial u}{\partial y} = x \cdot 10^{x^2+xy+z^2}\ln 10, \dfrac{\partial u}{\partial z} = 2z \cdot 10^{x^2+xy+z^2}\ln 10$

11. $\dfrac{\partial w}{\partial x} = \sin y + y \sec x \tan x, \dfrac{\partial w}{\partial y} = x \cos y + \sec x$

13. $\dfrac{\partial x}{\partial s} = 3s(s^2 - t^2)^{½}, \dfrac{\partial x}{\partial t} = -3t(s^2 - t^2)^{½}$

15. $\dfrac{\partial v}{\partial x} = yz + 2xy - z^3, \dfrac{\partial v}{\partial y} = xz + x^2 + 4y^3, \dfrac{\partial v}{\partial z} = xy - 3xz^2$

17. $\dfrac{\partial w}{\partial x} = \dfrac{2x}{a^2}, \dfrac{\partial w}{\partial y} = \dfrac{2y}{b^2}, \dfrac{\partial w}{\partial z} = \dfrac{2z}{c^2}$

19. $\dfrac{\partial z}{\partial x} = 2x \sinh (x^2 + 3y), \dfrac{\partial z}{\partial y} = 3 \sinh (x^2 + 3y)$

21. $\dfrac{\partial s}{\partial x} = -\dfrac{y}{x^2 + y^2}, \dfrac{\partial s}{\partial y} = \dfrac{x}{x^2 + y^2}$ **23.** $\dfrac{\partial Q}{\partial r} = -\dfrac{\sin\theta}{r^2}, \dfrac{\partial Q}{\partial \theta} = \dfrac{\cos\theta}{r}$

25. $\dfrac{\partial z}{\partial x} = \dfrac{x}{z}, \dfrac{\partial z}{\partial y} = \dfrac{y}{z}$ **27.** $\dfrac{\partial z}{\partial x} = -\dfrac{c^2x}{a^2z}, \dfrac{\partial z}{\partial y} = -\dfrac{c^2y}{b^2z}$

29. $\dfrac{\partial z}{\partial x} = -\dfrac{z}{x}, \dfrac{\partial z}{\partial y} = -\dfrac{z}{y}$ **31.** $\dfrac{\partial z}{\partial x} = \dfrac{8xy - 9x^2y^5}{68z^3}, \dfrac{\partial z}{\partial y} = \dfrac{4x^2 - 15x^3y^4}{68z^3}$

33. $\dfrac{\partial r}{\partial x} = \cos\theta, \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r}$ **35.** $\dfrac{\partial r}{\partial x} = \dfrac{1}{2}e^{-\theta}, \dfrac{\partial \theta}{\partial y} = -\dfrac{e^{\theta}}{2r}$

37. 4, -8 **39.** $3y + z - 10 = 0, x = 2$ **41.** $y + 3z + 6 = 0, x = 2$

Pages 361–362

1. $6xy^2, 6x^2y, 2x^3 + 20y^3$ **3.** $0, \cos y, -x \sin y$

5. $e^{x+y^2}, 2ye^{x+y^2}, 2(2y^2 + 1)e^{x+y^2}$ **7.** $\dfrac{x^2 + 16z^2}{256z^3}, \dfrac{xy}{64z^3}, -\dfrac{y^2 + 4z^2}{16z^3}$

Pages 366–367

1. $dz = (3x^2y + 2xy^2)\,dx + (x^3 + 2x^2y)\,dy$
3. $dz = (8x^3 - 9x^2y + 2xy^2 + 4y^3)\,dx + (-3x^3 + 2x^2y + 12xy^2 + 4y^3)\,dy$
5. $du = e^{x^2-y^2-2z}(2x\,dx - 2y\,dy - 2\,dz)$
7. $dw = e^{\frac{r}{s}}\left(\frac{dr}{s} - \frac{r\,ds}{s^2}\right)$
9. $ds = \dfrac{3(r\,dr + t\,dt)}{r^2 + t^2}$
11. $dz = \dfrac{y^2\,dx - xy\,dy}{(x^2 + y^2)^{3/2}}$
13. $dw = \dfrac{1}{3}\sec^2\left(2t + \dfrac{r}{3}\right)[6\,dt + dr]$
15. $dz = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$
17. $dz = \dfrac{y\,dx - x\,dy}{y^2}\cosh\dfrac{x}{y}$
19. 1.6 sq. ft.
21. 4%
23. 1006.8 cu. in.
25. 0.020, 3%
27. 2 ft.
29. 0.29 ft.
31. 0.09
33. 21.7 sq. ft.

Pages 370–371

1. $5 + \sqrt{3}$ deg. per linear unit
3. max. $= 10$ for $\alpha = \arctan \frac{4}{3}$; min. $= 0$ for $\alpha = \arctan\left(-\frac{3}{4}\right)$
5. $\alpha = \arctan \frac{1}{2}$, 0.152
7. $-\dfrac{1}{\sqrt{x_1^2 + y_1^2}}$, 0
9. $x + 2y - 4 = 0$
11. $\alpha = \arctan \dfrac{y_1}{x_1}$, that is, in the direction of OP_1; value $= 1$
13. $\dfrac{e}{4}\sqrt{2}\,(\sqrt{3} - 1)$

Pages 374–375

1. $2x \cos t - 8y \sin t$
3. $\dfrac{1}{t} e^x \sin y + 2te^x \cos y$
5. $\dfrac{1}{r^2}\sinh\dfrac{y}{x}\,(xe^{\varphi} - 2y\varphi)$
13. $\dfrac{8x}{s} - 36ty,\ -\dfrac{8tx}{s^2} - 36sy$
15. $\dfrac{s}{x^2 + y^2}(xe^t - ye^{-t}),\ \dfrac{xe^t + ye^{-t}}{x^2 + y^2}$
17. $2\left(uye^x + ve^{-y} - \dfrac{wy}{x^2}\right),\ 2\left(ue^x - vxe^{-y} + \dfrac{w}{x}\right)$

Pages 380–381

1. $2x - y - 3z + 14 = 0,\ \dfrac{x + 2}{2} = \dfrac{y - 1}{-1} = \dfrac{z - 3}{-3}$
3. $2x + y - 9z - 28 = 0,\ \dfrac{x - 4}{2} = \dfrac{y - 2}{1} = \dfrac{z + 2}{-9}$
5. $x + 6y + 8z - 21 = 0,\ \dfrac{x + 5}{1} = \dfrac{y - 3}{6} = \dfrac{z - 1}{8}$
7. $x - 10y + 12z + 15 = 0,\ \dfrac{x + 1}{1} = \dfrac{y - 5}{-10} = \dfrac{z - 3}{12}$
9. $3x + 4y = 25,\ \dfrac{x - 3}{3} = \dfrac{y - 4}{4} = \dfrac{z - 6}{0}$

11. $6x + 3y - 2z - 84 = 0,\ \dfrac{x-1}{6} = \dfrac{y-8}{3} = \dfrac{z+27}{-2}$

15. $2x - y - 2z + 12 = 0$ **17.** $11x + 10y + 2z - 66 = 0$

Pages 385–386

1. min. $z = -6$ **3.** No max. or min. **5.** No max. or min.
7. max. at $(0,0,1)$ **9.** $\frac{50}{3} \times \frac{50}{3} \times \frac{100}{3}$ **11.** Square base, depth $= \frac{1}{2}$ width
13. $a/\sqrt{3}$ **15.** $2x + y + 2z = 6$ **19.** $(\frac{4}{5}, -\frac{8}{5})$

Pages 388–390

11. $\csc^2 \dfrac{x}{y}\left(\dfrac{x}{y^2}\,dy - \dfrac{dx}{y}\right)$ **13.** $\dfrac{dx}{x} + \tan y\,dy$ **15.** 169.2 cu. in., 1.3%

17. 0.97 ft. **19.** 0 **21.** $\dfrac{a}{x^2}(x \sin \theta + y \cos \theta - y)$

29. $(2x + y) \cos \theta + x \sin \theta,\ xr \cos \theta - (2x + y)r \sin \theta$

31. $\dfrac{\partial z}{\partial x} = \dfrac{1}{y} \ln v + 2x\left(\dfrac{u}{v} + 2vw\right) + yv^2 \sec^2 x$ **35.** $x_1x + y_1y + z_1z = a^2$

39. max. $z = 33$ **41.** $(0,0,1)$

Pages 397–398

1. 16 **3.** $\frac{15}{8}$ **5.** $\frac{5}{6}$ **7.** $\frac{1}{2}$ **9.** $\frac{4}{3}a^3$ **11.** $\frac{32}{45}a^3$
13. $(\pi/4)a$ **15.** 8π **17.** $\frac{5}{3}$ **19.** $\frac{2}{3}a^3$ **21.** 4π **23.** $\frac{3}{10}a^3$
25. $\frac{31}{60}a^3$ **27.** $\pi/32$ **29.** $\frac{2}{15}a^3$ **31.** $\frac{4}{3}\pi a^3$ **35.** $2a^2(\pi - \frac{2}{3})$
37. $a^2[2\sqrt{3} - \log(2 + \sqrt{3})]$ **39.** $\frac{1}{3}a^2$

Pages 401–402

1. 4π **3.** $\frac{4}{3}\pi a^3$ **5.** $\frac{32}{9}a^3$ **7.** $\frac{5}{3}\pi a^3$ **9.** 6π **11.** $\frac{1}{18}a^3(3\pi - 4)$
13. πa^3 **17.** $\frac{61}{64}\pi a^3$ **21.** $\frac{1}{2}a^2(\pi - 2)$

Pages 404–405

1. 36 **3.** $\frac{2}{15}a^5$ **5.** $\frac{1}{24}\pi(2\pi^2 - 3)$ **7.** $\frac{3}{2}\pi a^4$ **9.** $\frac{1}{18}a^3(9\pi + 44)$

Pages 406–407

1. $\frac{4}{3}\pi a^3$ **3.** $2\pi^2a^2b$ **5.** $\frac{2}{21}\pi a^3$ **7.** $\frac{11}{6}\pi a^3$ **9.** $\frac{40}{3}\pi$

Pages 410–411

1. $4\pi a^2$ **3.** $\pi a^2\sqrt{1 + m^2}$ **5.** $\pi a^2\sqrt{2}$ **7.** $\frac{1}{8}\pi a^2\sqrt{2}$ **9.** $\dfrac{a^2}{n}(\pi - 2)$

11. $a^2\left(\dfrac{\pi}{12} + \dfrac{1}{2}\sqrt{3} - 1\right)$ **15.** πa^2

Pages 416–417

1. $\frac{11}{24}$ **3.** $\pi/12$ **5.** $\frac{1}{20}$ **7.** $\frac{11}{84}$
9. $a^4\left(\dfrac{1}{3} - \dfrac{\pi}{16}\right)$ **11.** ka^5 **15.** $\frac{1}{10}\pi kha^2(a^2 + 2h^2)$ **17.** $\frac{1}{4}\pi^2ka^4$

Pages 420–421

1. $\frac{4}{3}\pi a^3$ 3. $\frac{1}{2}\pi a^4 k$ 5. $2\pi(b^2 - a^2)k$ 7. $\frac{56}{5}\pi\sqrt{3}\,a^5 k$
9. $\frac{2}{3}a^5$ 11. $\frac{1}{4}\pi h a^4$ 13. $\frac{5}{4}\pi a^4$ 15. $\frac{8}{15}\pi a^5$

Pages 423–425

1. $\frac{2}{3}l$ units from A 3. $\bar{y} = 4a/3\pi$ 5. $\bar{y} = 3a/2\pi$
7. $\frac{275}{432}a$, $\frac{275}{432}a$ 9. $\bar{x} = \frac{21}{20}a$ 11. $\dfrac{a(3a^2 + 2b^2)}{4(a^2 + b^2)}, \dfrac{b(2a^2 + 3b^2)}{4(a^2 + b^2)}$
13. $\frac{5}{8}a$, $\frac{5}{8}a$, $\frac{1}{2}a$ 15. $\bar{z} = \frac{8}{15}a$ 17. $\bar{z} = \frac{3}{8}a$
19. $\frac{3}{10}a^2 M$ 21. $\frac{14}{15}a^2 M$ 23. $\frac{12}{25}a^2 M$
25. $\dfrac{3(b^5 - a^5)}{5(b^3 - a^3)}M$ 27. $\frac{14}{15}a^2 M$ 29. $\frac{1}{2}a^2 M$
31. $\frac{11}{5}a^2 M$

Pages 425–427

1. 8 3. $\frac{1}{3}(\pi - 3)$ 5. $\frac{1}{4}\pi a^4$ 7. $\frac{32}{9}ha^3$ 9. $\frac{3}{2}a^3$
11. $\frac{4}{35}\pi a^3$ 13. $\frac{1}{90}abc$ 15. $\dfrac{\pi}{32}$ 17. $\frac{124}{3}\pi a^3$ 19. $\frac{8}{3}\pi a^3$
21. $\frac{24}{5}a^2$ 23. $\frac{2}{3}\pi a^3 k$ 25. $\frac{2}{3}\pi k h a^3$ 27. $2\pi k \ln \dfrac{b}{a}$ 29. $\bar{y} = 16a/15\pi$
31. $\bar{x} = \dfrac{3\sqrt{3}\,a}{2(3\sqrt{3} - \pi)}$ 33. $\frac{7}{12}a$, $\frac{7}{12}a$, $\frac{7}{12}a$ 37. $\frac{1}{5}a^2 M$ 39. $\frac{427}{8}\pi\delta$

Page 429

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty} x^n$ 5. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{y^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{(n+1)(n+2)}$ 9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n^2}$

11. $\dfrac{1}{2} + \dfrac{\sqrt{2}}{5} + \dfrac{\sqrt{3}}{10} + \dfrac{\sqrt{4}}{17} + \dfrac{\sqrt{5}}{26} + \cdots$

13. $x - \dfrac{x^2}{4} + \dfrac{x^3}{9} - \dfrac{x^4}{16} + \dfrac{x^5}{25} - \cdots$ 15. $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots$

17. $(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$

Page 433

1. Divergent 3. Convergent 5. Divergent 7. Convergent
9. Divergent 11. Divergent 13. Convergent 15. Convergent

Page 436

1. Divergent 3. Convergent 5. Convergent
7. Convergent 9. Convergent 11. Convergent

Pages 438–439

1. Convergent 3. Convergent 5. Convergent 7. Divergent
9. Convergent 11. Divergent 13. Convergent 15. Divergent

Pages 441–442

Convergent: numbers 1, 3, 7, 9, 11, 13, 15, 19; divergent: numbers 5, 17

Page 445

1. Convergent 3. Divergent 5. Convergent 7. Convergent
9. Divergent 11. $\frac{1}{121}$ 13. 0.63212

Page 448

1. Conditionally convergent 3. Absolutely convergent
5. Divergent 7. Absolutely convergent
9. Absolutely convergent 11. Divergent
13. Absolutely convergent

Pages 451–452

1. $2\sum_{n=2}^{\infty}\frac{n^2}{n^4-1}$ 3. $\sum_{n=1}^{\infty}\frac{1}{n^4+n^2}$ 5. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n+1}{n^2}$
7. $1+\frac{5}{8}+\frac{19}{38}+\frac{65}{216}+\cdots$ 9. $1+0+\frac{1}{3}-\frac{1}{12}+\frac{19}{120}-\cdots$
11. $1+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{48}+\cdots$ 13. $1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

Page 455

1. $-1 < x < 1$ 3. $-1 \leqq x < 1$ 5. $-\infty < x < \infty$
7. $-1 \leqq x \leqq 1$ 9. $-1 < x < 1$ 11. $-1 \leqq x \leqq 1$
13. $x = 0$ 15. $-\sqrt{2} < x < \sqrt{2}$ 17. $0 < x < 2$ 19. $-4 \leqq x \leqq 0$

Pages 458–459

1. $1 + x^2 + x^4 + x^6 + \cdots \quad (-1 < x < 1)$
3. Sum $= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ for $-1 < x < 1$
5. $2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} \cdots$ (all values of x)
7. $1 + 2x + 2x^2 + 2x^3 + \cdots \quad (-1 < x < 1)$
9. $f'(x) = 2 + 8x + 24x^2 + 64x^3 + \cdots + 2^n \cdot nx^{n-1} + \cdots \quad (-\frac{1}{2} < x < \frac{1}{2})$

Pages 462–464

29. 1.6487 31. 19.774 33. 10.583 35. 8.0260 37. 0.19997

Page 468

13. $-\sinh x$, $-\cosh x$, $\tanh x$

Pages 468–470

1. $\ln 6 + \sum_{n=1}^{\infty}\frac{2^n}{n!}$ 5. Convergent 7. Convergent
9. Convergent 11. Convergent 13. Convergent

15. Divergent
17. Divergent
19. Convergent
21. $s_{10} = 0.103638$, error negative and numerically not greater than 0.0000006
23. Conditionally convergent
25. Absolutely convergent
27. Absolutely convergent
29. Absolutely convergent
31. $-1 \leq x < 1$
33. $x = 0$
35. $\frac{3}{2} < x < \frac{9}{2}$
37. $a^m(1 + x)^m$
39. $x - \dfrac{x^3}{3} + \dfrac{x^5}{2!5} - \dfrac{x^7}{3!7} + \cdots$ for all values of x
41. $x - \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \dfrac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$

Page 474

1. Second order, first degree
3. First order, second degree
5. Third order, first degree
7. Second order, first degree
9. Second order

Page 475

1. $\dfrac{d^2y}{dx^2} + y = 0$
3. $\dfrac{dy}{dx} = y - x + 1$
5. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$
7. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$
9. $y^2 + \left(\dfrac{dy}{dx}\right)^2 = 1$
11. $\dfrac{dy}{dx} = \dfrac{y}{x}$
13. $\dfrac{dy}{dx} = \dfrac{y}{2x}$
15. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Pages 477–478

1. $2xy + y^2 = C$
3. $xy + \ln|x| = C$
5. $x^2y^3 - 2xy^2 + x^4 - y^3 = C$
7. $xy^2 + e^{\frac{y}{x}} = C$
9. $\sin(x + y) + \cos xy = C$

Pages 479–480

1. $x^2 + y^2 = C$
3. $(x - 3)(y + 1) = C$
5. $\arcsin x + \sqrt{y^2 - 1} = C$
7. $u = C \cos v$
9. $r = C(\sec\theta + \tan\theta)$
11. $e^{-2x} + \arctan\dfrac{y}{2} = C$
13. $e^{2x} + 2e^y = C$
15. $\dfrac{4}{y} + y + x + \ln|x| = C$
17. $y^2 = \sin(x^2 + C)$
19. $(x^2 + 1)(y - 1)^2 = C$

Page 481

1. $x^2 + 2xy - y^2 = C$
3. $x^4 + 2x^2y^2 = C$
5. $x^2 - y^2 = Cx$
7. $\ln|Cx| = \arcsin\dfrac{y}{x}$
9. $-2\sqrt{\dfrac{x}{y}} = \ln|Cy|$
11. $2x^3 + 3xy^2 + 3y^3 = C$
13. $\ln(2t^2 + 10st + 25s^2) + \dfrac{2}{5}\arctan\dfrac{t + 5s}{t} = C$
15. $\sqrt{x^2 + y^2} + x\ln|x| + y\arcsin\dfrac{y}{x} = Cx$
17. $e^{\frac{x}{y}} + \ln|x| = C$
19. $\dfrac{x}{y} + \ln|y| = 4$

Pages 483–484

1. $y = x^2 + Cx$ 3. $2y = e^x + Ce^{-x}$
5. $(x - 2y)\sqrt{x^2+1} = \ln(x + \sqrt{x^2+1}) + C$
7. $2y + \sin x + \cos x = Ce^x$ 9. $y + 2 = 2\sin x + Ce^{-\sin x}$
11. $s\cos t = 2t + \sin 2t - \cos 2t + C$
13. $s = 4 + C\sqrt{1-t^2}$ 15. $\dfrac{1}{x} + y\ln|x| = C$
17. $Cx^2y^2 + 2xy^2 = 1$ 19. $s^2 = t^2\ln t^2 + Ct^2$

Pages 485–486

1. $x^2 + 2xy + 2\ln|x| = C$ 3. $\arctan\dfrac{y}{x} + \ln|\sec y| = C$
5. $\frac{1}{3}(x^2+y^2)^{3/2} + \arctan(y/x) = C$ 7. $2xy + e^{-2y} = C$
9. $x^2 + y^2 - \dfrac{2y}{x} = C$

Page 487

1. $e^y = 1 + Ce^{x+y}$ 3. $xy = \sin x - x\cos x + C$
5. $\arctan\dfrac{y}{x} + \ln|xy| = C$ 7. $y = 1 + C\cos x$
9. $y^2 - \dfrac{1}{x^2} + 2\ln|y| = C$ 11. $xy = \sinh x + C$
13. $xy + \frac{1}{3}y^3 = C$ 15. $Cx^2y^n + xy^n - 1 = 0$
17. $x^2 + 4xy - 3y^2 - 4x + 20y = C$ 19. $e^{\frac{y}{x}} + \ln|x| = C$
21. $x^6 + 2x^3y^3 = C$ 23. $y\sec x + \ln|\sec x + \tan x| = C$
25. $y = Ce^{-\frac{y}{x}}$ 27. $(\sec x + \tan x)(y - 1) = -x + C$ 29. $x - y - 1 = Ce^{-x}$
31. $(x^2 - 16)\dfrac{dy}{dx} - xy = 0$ 33. $25\left(\dfrac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3$

Pages 490–491

1. $x^2 + y^2 = C$ 3. $y = Ce^{-\frac{1}{2}x^2}$ 5. $y = Ce^{\frac{1}{2}x^2}$
7. $r = Ce^{\frac{\theta}{k}}$ 9. $r^2 = C\sin 2\theta$ 11. $y = Cx$
13. $2x^2 + y^2 = C$ 15. $x - y + 2 + Ce^x = 0$
17. $v = v_0e^{-kt}$ 19. $v = \sqrt{g/k}\tanh(\sqrt{gk}\,t)$

Pages 495–496

1. $y = c_1e^{ax} + c_2e^{-ax}$ 3. $y = c_1e^x + c_2e^{2x} + c_3e^{-3x}$
5. $y = (c_1 + c_2x)e^{-2x} + c_3e^x$ 7. $y = c_1 + c_2e^{4x} + c_3e^{-5x}$
9. $y = c_1 + c_2x + c_3e^x + c_4e^{-x}$ 11. $y = e^{-3x}(c_1e^{\sqrt{15}x} + c_2e^{-\sqrt{15}x})$
13. $y = c_1 + c_2x + c_3x^2 + (c_4 + c_5x + c_6x^2 + c_7x^3)e^{2x}$

Page 497

1. $y = A\cos 5x + B\sin 5x$ 3. $y = c_1e^{2x} + c_2e^{-2x} + c_3\sin 2x + c_4\cos 2x$
5. $x = (c_1 + c_2t)\cos 2t + (c_3 + c_4t)\sin 2t$

7. $y = c_1e^{2x} + e^{\frac{1}{2}x}(c_2 \sin \frac{1}{2}\sqrt{7}\,x + c_3 \cos \frac{1}{2}\sqrt{7}\,x)$
9. $y = e^{-2x}[(c_1 + c_2x) \sin \sqrt{2}\,x + (c_3 + c_4x) \cos \sqrt{2}\,x]$

Pages 501–502

1. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x - \frac{1}{4}x$
3. $y = A \cos x + B \sin x + \frac{1}{2}e^x + x^2 + 2x - 2$
5. $y = c_1 + c_2e^{5x} + C_3e^{-x} + \frac{1}{6}xe^{-x} + \frac{1}{13} \sin x + \frac{3}{26} \cos x$
7. $y = (c_1 + c_2x)e^x + \frac{1}{12}x^4e^x + x + 1$
9. $y = e^{-x}(c_1 + c_2x + c_3x^2) - e^{-x} \sin x$ 11. $s = c_1e^{3t} + c_2e^{2t} + 3te^{3t}$
13. $y = c_1 + c_2e^{-2x} + c_3e^{2x} + \frac{1}{4}e^{-x} - \frac{3}{8} \sinh x - \frac{1}{8} \cosh x$
15. $s = c_1e^{2t} + c_2e^t + \frac{1}{20}e^{2t}(3 \sin 2t - \cos 2t)$
17. $y = (c_1 + c_2x)e^x + (\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{8})e^{3x}$

Page 506

1. $y = c_1x^2 + c_2$
3. $y = e^x + \sinh x + c_1 + c_2x + c_3x^2 + c_4x^3$
5. $y = c_1 + \ln |\sec (x + c_2)|$
7. $\ln \left|\dfrac{c_1 + y}{c_1 - y}\right| = c_1x + c_2$
9. $\sqrt{2e^y + C} - \sqrt{C} = ke^{\frac{y + \sqrt{C}x}{2}}$
11. $c_1y = c_2e^{c_1x} - 2$

Pages 506–507

1. $y = c_1 + c_2e^{2x} + c_3e^{-x}$
3. $y = c_1 + c_2e^{2x} + c_3e^{-2x} + \frac{1}{5}x \cos x - \frac{12}{25} \sin x$
5. $y = c_1e^{ax} + c_2e^{bx}$ 7. $y = c_1 + c_2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{2}x$
9. $y = c_1e^{2x} + e^{-x}(c_2 \sin \sqrt{3}\,x + c_3 \cos \sqrt{3}\,x)$ 11. $y = c_1 + c_2 \ln |x| + \frac{1}{2} \ln^2 |x|$
13. $y = c_1e^{2x} + c_2e^{3x} + xe^{2x} - \frac{1}{10} \sin x + \frac{1}{10} \cos x$
15. $c_1y = \cosh (c_1x + c_2)$ 17. $y = c_1 + c_2x + c_3 \sin 4x + c_4 \cos 4x$

INDEX